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THE PHYSICAL DECOMPOSITION OF THE EARTH'S PERMANENT MAGNETIC FIELD.¹ No. III.

- a.* The Earth's Total Magnetic Energy.
- b.* The Seat of the Principal Cause of the Secular Variation.
- c.* On the Improvement of the Gaussian Theory.

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a. THE EARTH'S TOTAL MAGNETIC ENERGY.²

The purpose of this paper is to determine whether the accumulation of magnetic data is already sufficient to ascertain if the total magnetic energy of the Earth's so-called "permanent" magnetic field has remained constant for the period of the accumulated data, or has been subject to a secular variation, as in the case of all the magnetic elements. A secular variation of the magnetic elements may result from: (a) secular changes in the direction or directions of magnetization of the Earth; (b) secular changes in the intensity of magnetization of the Earth, and (c) from operation of both causes (a) and (b). From previous investigations of the secular variation, cause (a) appeared to me the predominant factor.

¹For No. I of these researches, see this Journal, Vol. IV, 1899, p. 35, and for No. II, see Vol. VI, 1901, p. 13.

²Presented, in abstract, for the author, by Professor A. Schuster, before Sec. A of the British Association for the Advancement of Science, Sept., 1903.

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The expression to be established, defining what I term the "Earth's total magnetic energy" is independent of the *directions* of the magnetization of the Earth, and hence should remain constant for all time, if the secular variations be due solely to cause (a). If, however, the values of the magnetic energy for two periods, separated by a suitable interval, differ by an amount which can not be accounted for by uncertainties in the values themselves, arising from inadequacy of the data or from imperfections in the method of calculation, then we shall have evidence of great importance to theories of the Earth's magnetism, viz: that the Earth's intensity of magnetization, owing to some cause, is subject to secular change. And if so, it is very important to know whether the Earth's magnetic energy is increasing or decreasing. Manifestly, therefore, the present investigation is one well worth our attention, whatever the outcome may be, even though it may but lead to the conclusion that our present knowledge of the actual distribution of the Earth's magnetism, in spite of the extensive data already accumulated, is in reality a very unsatisfactory one.

How are we to define the "Earth's total magnetic energy"?

The potential of the Earth's total magnetism can be expressed, with great approximation, as Gauss first showed, in solid spherical harmonics to terms of the fourth order inclusive; the addition of terms of the fifth and sixth order, as revealed by the investigations of Adams, Fritsche, and Schmidt effect some improvement. I have shown in my previous papers, that the spherical harmonic of the first order can be physically interpreted as arising from a homogeneous magnetization of the Earth about a diameter inclined to the axis of rotation;¹ the inclination of this axis is about $11.^\circ 4$, and this first harmonic term represents, approximately, $\frac{1}{10}$ of the Earth's total magnetization. The well-known expression of the "Earth's magnetic moment," as introduced by Gauss, represents only that portion of the total magnetization which arises from the first term, *i. e.*, from the homogeneous magnetization. Now, if we tabulate all the values of the magnetic moment obtained thus far by the various computers of the Gaussian potential, whose detailed results will be found given in a subsequent table, it would appear as though the magnetic moment has suffered a perceptible diminution—1.6%—during the period 1838–1884.

¹I have recently found that I was anticipated in this interpretation of the first harmonic by Professor J. D. EVERETT, in his "Units and Physical Constants." London. Macmillan & Co., 1879, p. 125, paragraph 151.

VALUES OF THE EARTH'S MAGNETIC MOMENT IN C. G. S. UNITS.

No.	Computer of Potential.	Epoch.	Value.	
1	Erman-Petersen	1829	0.32690 R ³	
2	Adams	1842.5	32820	Mean 1, 2 & 3, 1838, 0.32812 R ³ 4, 5, 6 & 7, 1884, 32288
3	Fritsche	1842.5	32927	
4	Adams	1880	32344	Change in 46 years, 0.00524 R ³ Hence, 1.6%
5	Fritsche	1885	32272	
6	Neumayer-Petersen . .	1885	32237	
7	Schmidt	1885	32298	

Since the quotient of the magnetic moment by the Earth's total volume, represents the intensity of magnetization per unit of volume, the conclusion would, therefore, apparently be that the intensity has decreased at an alarming rate in forty-six years. The same conclusion would be reached with regard to the Earth's magnetic energy, since the latter, for the homogeneous magnetization portion, can be expressed in terms of the magnetic moment.

The question arises: Is this comparatively great loss in the magnetic moment during an interval of but half a century, to be ascribed to insufficiency or imperfection of data for the periods for which the calculations are made, or, if we were to take into account the remaining portion of the Earth's magnetization, represented by the second, third, and fourth harmonics, would the loss of energy of the first term be compensated for, wholly or partially, by a possible gain in energy of the remaining terms, or would the remaining terms likewise exhibit a decrease in energy?

To answer these questions, I shall introduce the well-known function in physics, expressing the energy of a distribution of forces in terms of the field, and I shall designate it by the letter *W*, the initial letter of "work" or the German "werk."¹

It is defined as the sum total of the magnetic energy of all the elements of volume in the space bounded by the Earth's surface and a sphere of infinite radius, *i. e.*, of radius large enough that the energy of the excluded portion may be neglected,² as due to the field of force arising from the Earth's magnetization. The magnetic energy of an element of volume in a field of force of the kind

¹The Earth's total potential is not used for these investigations, as the integral of this potential for any closed space is zero for every period, just as the Earth's total quantity of magnetism is supposed always zero.

²At a height above the Earth, fifteen times that of the Earth's radius, or about one-fourth the distance to the Moon, the Earth's total magnetic force will have dwindled to a quantity equal to the error of observation of the best-class instruments used in making field observations of the magnetic elements.

here discussed, may be taken as proportional to the square of the force of the field at the point considered; it represents the amount of work done to bring, for example, a unit magnetic pole repelled by the field from an infinite distance to the point. The factor of proportionality μ , is the inductivity or the magnetic permeability of the medium; for the space regarded we may take this, I presume, as uniform, and set it equal to unity. Were the portion of space occupied by the Earth itself, also embraced, then μ could no longer be regarded as uniform. Owing to our ignorance of the precise composition of the Earth's interior, the law of variation of μ is not known, and there would, hence, result expressions in the integral, representing W , which could not be evaluated. *W has, therefore, been confined to the space outside.*

It is, doubtless, unnecessary to remark that the magnetic energy of any element of volume, outside the Earth, due to fields giving rise to periodic or non-periodic magnetic variations, *e. g.*, the diurnal variation, magnetic storms, etc., is not here considered, since in the field of force, usually referred to as the "permanent or quasi-permanent field," all variations at any particular time are supposed eliminated.

The mathematical expression for the energy of a field of force obeying Newton's law, as will be found from various text-books,¹ is:

$$W = \frac{1}{8\pi} \int \mu F^2 d\tau \quad (1)$$

μ = magnetic permeability at any point in space; F = the total field intensity at any point, x, y, z ; $d\tau = dx dy dz$ = element of volume.

Confining the integral to the space outside of the Earth, μ will be regarded, as already stated, constant and set equal to unity. Introducing the polar co-ordinates, r (radius vector), u (co-latitude), and λ (the longitude counted positive to the eastward), and letting X , be the component directed positive to the north, Y , the component directed positive to the east, and Z , the component, directed positive downwards (towards nadir), and if R be the Earth's mean radius we have:

$$W = \frac{1}{8\pi} \int_0^{2\pi} \int_0^\pi \int_R^\infty (X^2 + Y^2 + Z^2) r^2 \sin u d\lambda du dr \quad (2)$$

In this equation, as in all subsequent ones, r is supposed counted positive outwards from the center of the Earth.

¹ An admirable work for American students, is: WEBSTER'S Electricity and Magnetism. Macmillan & Co. 1897.

If V be the magnetic potential at any point in the external space, then

$$X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y} \quad \text{and} \quad Z = -\frac{\partial V}{\partial z}$$

Substituting in equation (2), applying Green's theorem to the right hand member, and remembering that for the space considered, $\Delta V = 0$, in accordance with Laplace's equation¹ and also that both V and Z vanish for $r = \infty$, we obtain as another expression for the energy:

$$W = -\frac{K^2}{8\pi} \int_0^{2\pi} \int_0^\pi V Z \sin u \, d\lambda \, du. \quad (3)$$

This integral, as the one in (2), is positive, since, in accordance with the notation used in this paper, V and Z are, at every point of opposite sign.

Substituting the dimensions of field intensity and of potential in equation (2) or (3), it will be found that W has correctly the dimensions of energy or work, $[M L^2 T^{-2}]$.

The Earth's total magnetic energy can thus be calculated in two ways:

1. *By obtaining the sum total of the products of the square of the total intensity of the field into the element of volume, at every point in the entire space outside the Earth, and dividing the sum by 8π .*

2. *By obtaining the sum total of the products of the magnetic potential and the vertical component of the field intensity into the element of surface, for the entire surface of the Earth, and dividing the sum by 8π .*

FIRST METHOD OF COMPUTING THE EARTH'S MAGNETIC ENERGY.

We must know the magnetic potential V at every point for the space considered. Following the notation as already used in Nos. I and II of these researches,² which is the same as that adopted by Schmidt, we have:

$$X = \frac{1}{r} \cdot \frac{\partial V}{\partial u}, \quad Y = -\frac{1}{r \sin u} \cdot \frac{\partial V}{\partial \lambda} \quad \text{and} \quad Z = \frac{\partial V}{\partial r}. \quad (4)$$

¹ $\Delta V = 0$, is also the condition that W shall have a minimum value for the space considered.

² In No. I, p. 38, equation (2) should read

$$X = +\frac{1}{R} \frac{\partial V}{\partial u}.$$

$F^2 = X^2 + Y^2 + Z^2$ is then known at every point, and the integrations are easily effected.

The potential at any point in the space considered, due to a magnetic system inside the Earth, may be expressed as follows:

$$V = \frac{R^0}{r^2} \cdot P_1 + \frac{R^4}{r^3} \cdot P_2 + \frac{R^5}{r^4} \cdot P_3 + \dots \quad (5)$$

$P_1, P_2, P_3, \dots$ are, respectively, surface spherical harmonics of 1st, 2d, 3d, and etc., orders.

For example, $P_1 = [g_{10} \cos u + (g_{11} \cos \lambda + h_{11} \sin \lambda) \sin u]$.

$$X = \frac{R^0}{r^3} \left(\frac{\partial P_1}{\partial u} + \dots \right), \quad Y = -\frac{1}{\sin u} \cdot \frac{R^0}{r^3} \left(\frac{\partial P_1}{\partial \lambda} + \dots \right),$$

$$\text{and } Z = -\frac{R^0}{r^3} \left(2 P_1 + 3 \frac{R}{r} \cdot P_2 + 4 \frac{R^2}{r^2} \cdot P_3 + \dots \right). \quad (6)$$

As the expression W will contain the squares of the Gaussian coefficients (the g 's and h 's), I shall employ entirely subscripts to designate the coefficients of the various terms, and for convenience in writing and printing, omit furthermore the commas between the subscripts. The g 's and h 's, of which the collective number is 24 for terms to the fourth order, are coefficients determined for any particular period from the known values X and Y and Z .¹

The values of g and h as obtained by various computers of the Earth's magnetic potential, are given later.

Forming the quantity $X^2 + Y^2 + Z^2$, omitting at once terms which by well-known formulæ in integral calculus, or in spherical harmonics, are known to vanish when integrated over the entire Earth,² and performing the various integrations we have:

$$\begin{aligned} W &= W_1 + W_2 + W_3 + W_4 + \dots \\ &= R^0 \left\{ \frac{1}{2} (g_{10}^2 + g_{11}^2 + h_{11}^2) + \frac{1}{2} [g_{20}^2 + \frac{1}{2} (g_{21}^2 + h_{21}^2) \right. \\ &\quad \left. + 2 (g_{22}^2 + h_{22}^2)] + \frac{1}{8} [g_{30}^2 + \frac{1}{2} (g_{31}^2 + h_{31}^2) + \frac{1}{2} (g_{32}^2 + h_{32}^2) + 16 (g_{33}^2 + h_{33}^2)] \right. \\ &\quad \left. + \frac{1}{16} [g_{40}^2 + \frac{1}{2} (g_{41}^2 + h_{41}^2) + \frac{1}{2} (g_{42}^2 + h_{42}^2) + 4 (g_{43}^2 + h_{43}^2) \right. \\ &\quad \left. + 32 (g_{44}^2 + h_{44}^2)] + \dots \right\} \quad (7) \end{aligned}$$

¹ Like the magnetic elements, the coefficients are subject to a secular variation, and presumably, in accordance with the same law as applies to the elements themselves. Thus, since the secular variation investigations of the observed elements show that for an interval of a century or more the variation can be expressed in the form of a Bessel series, of which the first sine term represents a very close approximation, we may expect that the first approximation to the law of secular variation of the Gaussian coefficients is also that of a simple sine function of the element t , time.

² Thus, for example, it can be shown that only terms involving the squares of the individual coefficients to contribute to W , all the product terms disappearing when integrated over the entire surface.

SECOND METHOD OF COMPUTING THE EARTH'S MAGNETIC ENERGY.

Substituting the values of V and Z in equation (3), and omitting all products of the spherical harmonic terms, which when integrated over the entire Earth's surface are known to vanish, in accordance with well-known theorems in spherical harmonics, we have:

$$W = \frac{R^3}{8\pi} \int_0^{2\pi} \int_0^\pi (2 P_1^2 + 3 P_2^2 + 4 P_3^2 + \dots) \sin u \, d\lambda \, du \quad (8)$$

This is a decidedly simpler expression than the one from the first method; it was found to lead to the same result as already given, and thus equation (7) was checked.

With the aid of Schmidt's " R " functions, which, to avoid confusion with R , the Earth's mean radius, I shall designate by " S ," and which are equal to the P functions multiplied by a factor " s ," determined so that:

$$\int_0^{2\pi} \int_0^\pi (S^2 = s^2 P^2) \cos^2 m \lambda \text{ (or } \sin^2 m \lambda) \sin u \, d\lambda \, du = 4\pi \quad ,$$

the expression for W can be put in its most elegant form.

Instead of the g and h coefficients of Gauss, we shall now have the γ and η coefficients of Schmidt, the two sets of coefficients being related thus:

$$\frac{s}{nm} \cdot \gamma = \frac{g}{nm} \quad , \quad \frac{s}{nm} \cdot \eta = \frac{h}{nm} \quad .$$

The values of s_{nm} are given later. Continuing, we have as our final expression:

$$\begin{aligned} W = R^3 [& (\gamma_{10}^2 + \gamma_{11}^2 + \eta_{11}^2) + \frac{3}{2} (\gamma_{20}^2 + \gamma_{21}^2 + \eta_{21}^2 + \gamma_{22}^2 + \eta_{22}^2) \\ & + 2 (\gamma_{30}^2 + \gamma_{31}^2 + \eta_{31}^2 + \gamma_{32}^2 + \eta_{32}^2 + \gamma_{33}^2 + \eta_{33}^2) + \frac{5}{2} (\gamma_{40}^2 + \gamma_{41}^2 \\ & + \eta_{41}^2 + \gamma_{42}^2 + \eta_{42}^2 + \gamma_{43}^2 + \eta_{43}^2 + \gamma_{44}^2 + \eta_{44}^2) + \dots] \quad (9) \end{aligned}$$

I have made independent computations, using both formulæ (7) and (9), getting identical results, and have thus checked the correctness of these two formulæ.

Table II gives a tabulation of the Gaussian coefficients as obtained by various computers for various epochs. The references to the publications from which the figures were taken are appended. The figures in column 10 have been obtained from those in column 13, by multiplying by the conversion factor, as given in Table I. It should be noted that the signs in Table II, in conformity with the

notation used by Schmidt and myself, are the reverse of those given by the computers who followed Gauss's notation. Strictly speaking it is not quite correct, because of the greatly varying magnitudes of the P coefficients, as was first pointed out by Schmidt, to express all the g 's and h 's to the same decimal; their relative importance can not always be gauged by their magnitudes. In this respect the coefficients introduced by Schmidt, and as given in Table III, have a decided advantage.

Columns 1 to 5 inclusive, and 11 and 12, give the values of the coefficients, concluding the harmonic series with terms of the fourth order, whereas columns 6-10 and 13 give the results with terms of the fifth and sixth order included. The effect on the coefficients, resulting from extension of the series, can thus be seen by comparing the respective columns.

The results from the potential computation of Quintus Icilius for 1880, are too defective to warrant including them in the table, and have accordingly, been omitted.

Column 14 gives the coefficients for the small external potential deduced by Schmidt for 1885, and 15, the same, as deduced by Fritsche from the combined results for 1842 and 1885.

TABLE I. FACTORS FOR CONVERTING SCHMIDT'S COEFFICIENTS INTO GAUSSIAN COEFFICIENTS.

	Factor	Loga- rithm		Factor	Loga- rithm		Factor	Loga- rithm		Factor	Logarithm
S_{00}	1	0.00000	S_{30}	$\frac{5}{8} \sqrt{\frac{1}{7}}$	0.82049	S_{60}	$\frac{83}{8} \sqrt{\frac{1}{11}}$	1.41695	S_{90}	$\frac{231}{8} \sqrt{\frac{1}{13}}$	1.71646
S_{10}	$\sqrt{\frac{1}{8}}$	0.23856	S_{31}	$\frac{5}{8} \sqrt{\frac{1}{42}}$	0.90853	S_{61}	$\frac{81}{8} \sqrt{\frac{1}{105}}$	1.52787	S_{91}	$\frac{231}{8} \sqrt{\frac{1}{278}}$	1.83351
S_{11}	$\sqrt{\frac{1}{8}}$	0.23856	S_{32}	$\frac{1}{2} \sqrt{\frac{1}{105}}$	0.70956	S_{62}	$\frac{83}{8} \sqrt{\frac{1}{1155}}$	1.40635	S_{92}	$\frac{231}{8} \sqrt{\frac{1}{2750}}$	1.73145
S_{20}	$\frac{3}{2} \sqrt{\frac{1}{8}}$	0.52558	S_{33}	$\frac{1}{2} \sqrt{\frac{1}{70}}$	0.32049	S_{63}	$\frac{1}{8} \sqrt{\frac{1}{770}}$	1.19337	S_{93}	$\frac{1}{8} \sqrt{\frac{1}{2790}}$	1.55535
S_{21}	$\sqrt{\frac{1}{16}}$	0.58805	S_{40}	$\frac{1}{8} \sqrt{\frac{1}{5}}$	1.11810	S_{64}	$\frac{83}{8} \sqrt{\frac{1}{885}}$	0.86676	S_{94}	$\frac{1}{8} \sqrt{\frac{1}{61}}$	1.29391
S_{22}	$\frac{1}{2} \sqrt{\frac{1}{16}}$	0.28702	S_{41}	$\frac{3}{4} \sqrt{\frac{1}{10}}$	1.22016	S_{65}	$\frac{1}{8} \sqrt{\frac{1}{154}}$	0.36676	S_{95}	$\frac{1}{8} \sqrt{\frac{1}{2372}}$	0.92373
			S_{42}	$\frac{3}{4} \sqrt{\frac{1}{6}}$	1.06964				S_{96}	$\frac{1}{8} \sqrt{\frac{1}{606}}$	0.38414
			S_{43}	$\frac{3}{8} \sqrt{\frac{1}{70}}$	0.79761						
			S_{44}	$\frac{3}{8} \sqrt{\frac{1}{36}}$	0.34607						

With the aid of the figures in Tables I and II, we can now compute the magnetic energy. To show the effect of the various terms, the following specimen, using formula (9) is given.

TABLE II. Values of the Gaussian Coefficients, in units of the fifth decimal.

	1. Rr-Pet. 1839	2. Gauss 1830	3. Adams 1840-5	4. Adams 1880	5. Neu-Pet 1885	6. Adams 1840-5	7. Fritzsche 1840-5	8. Adams 1880	9. Fritzsche 1885	10. Schmidt 1885		11. Neu-Pet. 1885	12. Schmidt 1885	13. Schmidt 1885	14. V S. 1885	15. V Pr 42-85
G ₅₀	—	32007	32248	32173	31572	32187	32271	31684	31635	31733	710	—18228	—18243	—18321	—107	—137
G ₅₁	+	2835	3111	2833	2470	2778	2751	2427	2414	2356	711	—1433	—1471	—1361	—139	—140
G ₅₂	+	6011	6246	5820	6026	5783	5931	6030	5914	5984	711	—3479	—3496	—3455	—2	—22
G ₅₃	+	121	771	57	791	127	28	731	526	785	720	—236	—243	—234	—1	—28
G ₅₄	+	4454	5064	4887	4980	4933	4746	5136	4962	4895	721	—1286	—1267	—1264	—32	—13
G ₅₅	—	72	211	194	1300	181	245	1293	1307	1243	721	—336	—340	—321	—4	—7
G ₅₆	+	125	17	38	507	35	66	530	589	587	722	—293	—300	—303	—28	—12
G ₅₇	+	1264	1353	1174	1260	1165	1148	1289	1230	1294	722	—651	—653	—668	—2	—32
G ₅₈	+	1876	659	2448	2856	2416	2148	2680	2356	2342	720	—368	—403	—354	—0	—15
G ₅₉	+	2986	4296	3366	2435	3130	3262	4197	3507	3775	721	—488	—538	—466	—14	—41
G ₆₀	+	1645	1670	1213	3956	1372	1263	748	1005	907	721	—91	—57	—112	—0	—25
G ₆₁	+	3073	2558	2769	2786	2802	2660	2822	2667	2767	722	—544	—527	—540	—14	—34
G ₆₂	+	621	796	408	44	409	400	40	13	51	722	—9	—8	—10	—5	—9
G ₆₃	+	127	49	207	327	207	189	289	368	316	722	—156	—155	—151	—6	—14
G ₆₄	+	725	655	678	549	678	699	493	555	540	722	—263	—263	—258	—21	—18
G ₆₅	+	2738	3804	2909	3440	3105	2944	3375	4014	3478	720	—262	—259	—265	—12	—8
G ₆₆	+	3840	5329	3183	3060	3386	3426	3683	3104	2839	721	—184	—158	—171	—14	—20
G ₆₇	+	320	2240	498	1188	551	541	1062	1381	1567	721	—72	—68	—94	—3	—7
G ₆₈	+	1559	1600	1557	1975	1528	1647	1933	2128	2019	722	—168	—172	—172	—15	—14
G ₆₉	+	1085	1488	996	715	989	956	522	652	669	722	—61	—63	—57	—19	—11
G ₇₀	+	671	691	352	539	353	304	559	601	646	722	—109	—106	—103	—1	—4
G ₇₁	+	301	6	391	684	391	429	433	525	471	722	—82	—79	—75	—14	—15
G ₇₂	+	90	144	14	512	14	000	108	63	118	722	—38	—40	—53	—18	—14
G ₇₃	—	111	111	101	97	100	103	94	103	55	722	—44	—42	—25	—2	—3
G ₇₄	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₇₅	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₇₆	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₇₇	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₇₈	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₇₉	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₈₀	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₈₁	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₈₂	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₈₃	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₈₄	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₈₅	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₈₆	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₈₇	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₈₈	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₈₉	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₉₀	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₉₁	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₉₂	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₉₃	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₉₄	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₉₅	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₉₆	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₉₇	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₉₈	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₉₉	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—
G ₁₀₀	—	—	—	—	—	—	—	—	—	—	722	—	—	—	—	—

TABLE III. Values of Schmidt's Coefficients, in units of the fifth decimal.

	11. Neu-Pet. 1885	12. Schmidt 1885	13. Schmidt 1885	14. V S. 1885	15. V Pr 42-85
G ₅₀	—	18228	18243	18321	—107
G ₅₁	—	1433	1471	1361	+139
G ₅₂	+	3479	3496	3455	+132
G ₅₃	+	236	243	234	+1
G ₅₄	+	1286	1267	1264	+32
G ₅₅	+	336	340	321	+4
G ₅₆	+	293	300	303	+28
G ₅₇	+	651	653	668	+2
G ₅₈	+	368	403	354	+0
G ₅₉	+	488	538	466	+14
G ₆₀	+	91	57	112	+0
G ₆₁	+	544	527	540	+14
G ₆₂	+	9	8	10	+5
G ₆₃	+	156	155	151	+6
G ₆₄	+	263	263	258	+21
G ₆₅	+	184	184	171	+12
G ₆₆	+	262	259	265	+14
G ₆₇	+	168	168	94	+3
G ₆₈	+	109	172	172	+15
G ₆₉	+	61	63	57	+19
G ₇₀	+	82	79	75	+1
G ₇₁	+	38	40	53	+14
G ₇₂	+	44	42	25	+2
G ₇₃	—	—	—	—	—
G ₇₄	—	—	—	—	—
G ₇₅	—	—	—	—	—
G ₇₆	—	—	—	—	—
G ₇₇	—	—	—	—	—
G ₇₈	—	—	—	—	—
G ₇₉	—	—	—	—	—
G ₈₀	—	—	—	—	—
G ₈₁	—	—	—	—	—
G ₈₂	—	—	—	—	—
G ₈₃	—	—	—	—	—
G ₈₄	—	—	—	—	—
G ₈₅	—	—	—	—	—
G ₈₆	—	—	—	—	—
G ₈₇	—	—	—	—	—
G ₈₈	—	—	—	—	—
G ₈₉	—	—	—	—	—
G ₉₀	—	—	—	—	—
G ₉₁	—	—	—	—	—
G ₉₂	—	—	—	—	—
G ₉₃	—	—	—	—	—
G ₉₄	—	—	—	—	—
G ₉₅	—	—	—	—	—
G ₉₆	—	—	—	—	—
G ₉₇	—	—	—	—	—
G ₉₈	—	—	—	—	—
G ₉₉	—	—	—	—	—
G ₁₀₀	—	—	—	—	—

TABLE IV. Specimen of calculation of $\frac{W}{R^3}$.

	11. Neu.-Pet. 1885	12. Schmidt 1885	13. Schmidt 1885		11. Neu.-Pet. 1885	12. Schmidt 1885	13. Schmidt 1885
I	γ_{10}^2	0033226	0'033281	II	γ_{20}^2	.000009	.000009
	γ_{11}^2	205	216		γ_{21}^2	248	241
	η_{11}^2	1210	1222		γ_{22}^2	16	18
					γ_{23}^2	13	14
					η_{22}^2	63	64
		.034641	34719			349	346
			34945				344
II	γ_{20}^2	28	33	III	γ_{40}^2	18	17
	γ_{21}^2	48	58		γ_{41}^2	8	6
	γ_{22}^2	1	1		η_{41}^2	2	1
	γ_{23}^2	59	56		γ_{42}^2	8	7
	γ_{24}^2	0	0		η_{42}^2	1	1
	γ_{25}^2	5	5		γ_{43}^2	2	3
	γ_{26}^2	14	14		η_{43}^2	2	1
					γ_{44}^2	0	0
					η_{44}^2	0	0
III		155	167	IV		41	37
			147				40
II + III + IV, .000545 .000550 .000531 Total,035186 .035269 .035476							

TABLE V. VALUES OF THE EARTH'S TOTAL MAGNETIC ENERGY.
[The tabular numbers are to be multiplied by the cube of the Earth's mean radius.]

No.	Computer of Potential	Epoch	I	II	III	IV	II + III + IV	Total
1	Ermann-							
3 and 6	Petersen .	1829	.03562	.00026	.00015	.00004	.00045	.03607
4	Adams . . .	1842.5	3590	29	15	3	47	3637
8	Fritsche . . .	1842.5	3614	28	14	3	45	3659
9	Adams . . .	1880	3481	36	17	4	57	3538
	Fritsche . . .	1885	3472	34	15	4	53	3525
	Neumayer							
5 and 11	Petersen .	1885	3464	35	16	4	55	3519
10 and 13	Schmidt . .	1885	3494	34	15	4	53	3548
	Mean of first three . . .	1838	.03589				.00046	.03635
	Mean of last four . . .	1884	3478				54	3532
	Change in . .	46 years	-.00111				+.00008	-.00103

Table V, giving the values of the Earth's total magnetic energy, will be readily understood; thus, for 1829, the energy from the first harmonic is $0.03562 R^8$; from the second, $0.00026 R^8$; from the third, $0.00015 R^8$, from the fourth, $0.00004 R^8$, the total of the four terms being $0.03607 R^8$. The results from the Gaussian computation are omitted, as it was not deemed fair to Gauss to present his results—preliminary, as he desired them considered—alongside of those dependent upon superior material.

A close scrutiny of the material and method of computation, employed by the various computers of the Gaussian coefficients, decided me to give equal weight in the combination of the results for the magnetic energy. Thus we have:

Earth's Total Magnetic Energy	
1838,	$0.03635 R^8$
1884,	$0.03532 R^8$
Loss in 46 years, $0.00103 R^8$ or 2.88 per cent, or about $\frac{1}{35}$ part.	

This loss is almost entirely in the term representing the portion of the Earth's magnetism to be referred to a homogeneous magnetism, it amounting to 3.15 per cent.

The result is so startling—if it be true, and the loss were to continue at the same rate as during the period 1838–1884, the Earth would lose its entire magnetic energy in about 1,600 years—that extreme caution should be used before coming to a definite conclusion.

Glancing over the results in the last column of Table V, it will be seen that the various values of the magnetic energy for the same epoch and dependent upon practically the same material, differ on the average, about 0.3 per cent, or about $\frac{1}{3}$ of the observed decrease in energy between 1838–1884.

The question as to the effect on the value of the energy due to the completeness or incompleteness of the material employed, is somewhat difficult to determine, as modern constructors of magnetic charts, have not published in detail the material on which the charts are based. Some evidence as to the amount of the effect may be gathered by comparing the 1829 and 1842 values. In both cases the material used extended over a considerable period before and after the epoch selected, in the latter one, however, the material was somewhat more extensive than for the former. Referring the 1829 value to 1842.5, using the annual rate of decrease

$$\frac{.00103}{46} R^8 = 0.0000224 R^8,$$

we obtain for the magnetic energy $.03577 R^3$, against the mean of the two 1842 results, $0.03648 R^3$, hence a difference of $0.00071 R^3$ or 1.97 per cent. Whether the difference in the completeness of the material between the 1842 charts, and the 1880 or 1885 charts, is as pronounced as for the comparison just made can not be said off-hand. In certain regions of the Earth, there has been either no gain at all, or very little in the material since the Sabine charts of 1840-1845.

While then the observed decrease in magnetic energy in the interval of 46 years, between 1838 and 1884, is larger than can apparently be ascribed to method of computation of the Gaussian coefficients, or to the character of the material employed, it will be safest, at the present stage, until other related investigations have been completed, to leave open the inquiry whether the Earth's total magnetic energy remains constant, or is suffering an increase or a decrease with the lapse of time.

This investigation, however, most emphatically indicates the great need of a fresh potential computation as dependent upon strictly homogeneous data, distributed fairly uniformly, say between the parallels 60° North and 60° South, the entire data having been obtained within a sufficiently brief interval, to eliminate the effect of secular variation. Thus, in view of the fact, that for very large areas of the globe, as already stated, practically no additional material has been obtained since the epoch for which Sabine drew his magnetic charts of 1840-1845, upon which the results given above for 1840-1845 depend, it may easily happen that the potential expression for 1885 may not represent any very great absolute improvement over that for 1840-1845, no matter how carefully and scientifically the computer may have performed his task. I take this occasion, also, to emphasize the need of complete publication of the original material on which a magnetic chart or a potential computation depends, so that an outsider may obtain some idea as to the amount of reliance to be put upon a chart or upon the results of a potential computation, based thereon.

For the sake of comparison, it may be of interest to give the value of the magnetic energy for the external potential determined by Schmidt and Fritsche. Supposing an internal system, producing equivalent effects on the Earth's surface, to those produced by the external potential, the magnetic energy of this portion would be $0.0000449 R^3$, or 0.0125 per cent of the internal portion above given.

USES OF THE ENERGY FUNCTION.

Could we assume that the Earth's total magnetic energy remains constant, either in its entirety, or only as far as some of its constituents are concerned, we should have a valuable aid in the energy function for expressing the Gaussian coefficients as functions of the time. In a paper presented to the American Association for the Advancement of Science, in 1894 (See *Proc.* Vol. XLIII, 1894), I showed how past declination and inclination data could be utilized for this purpose. The solution there proposed will be effectively advanced by the use of the energy function. I shall enter into this matter more fully in a paper entitled "On the Time Extension of the Gaussian Analysis."

A few examples will serve to illustrate the practical application of the energy function.

1. *What work is required at Washington to move a unit magnetic pole, repelled by the Earth's field, from a height where the Earth's magnetic force has vanished down to the Earth's surface?*

According to formula three

$$w = dW = -\frac{1}{8\pi} VZ dS$$

For the case considered, we have, approximately for 1885, in C. G. S. units, $V = -0.26 R$, $Z = 0.57$, $dS =$ element of surface, taken as one square centimeter. [For many purposes it will be sufficient to take $V/R = -\frac{1}{2} Z$, as will be the case for a homogeneous magnetization.] Taking R in cms. $= 6.37 \times 10^8$,

$$w = \frac{1}{8\pi} [(0.26) \times (0.57) \times 6.37 \times 10^8] = 3.76 \times 10^8 \text{ ergs} = 0.0383$$

kilogrammeters $= 0.277$ foot pounds $= 3\frac{1}{3}$ inch pounds; that is, the work performed would be equal to raising a pound or about a half kilogram, $3\frac{1}{3}$ inches high against the Earth's gravitational force.

For the vicinity of the north magnetic pole, where we may take $Z = 0.61$, $V/R = -0.32$ or about $\frac{1}{2} Z$, $dS = 1$ sq. cm., $w = 4.95 \times 10^8$ ergs $= 0.0504$ kilogrammeters $= 0.365$ foot pounds $= 4.4$ inch pounds. *For points vertically above the magnetic equator*, since here $Z = 0$, $w = 0$. *For the entire Earth*, the mean of the seven values in the last column of Table V gives $W = 0.03576 \times R^3 = 9.243 \times 10^{24}$ ergs $= 9.422 \times 10^{16}$ kgms. $= 68.15 \times 10^{16}$ foot

pounds. Per square centimeter of surface we have, $w = 1.812 \times 10^8$ ergs $= 0.0185$ kgm. $= 0.134$ foot pounds $= 1.6$ inch pounds.

By turning back to table V, we find that the average values of the magnetic energy arising from the various constituent portions of the Earth's permanent magnetism are:

$$W/R^3 = \begin{array}{c|c|c|c|c} \text{I} & \text{II} & \text{III} & \text{IV} & \text{Total} \\ \hline 0.03525 & 0.000317 & 0.000153 & 0.000037 & 0.03576 \end{array}$$

Hence W_{II} is about $\frac{1}{100} W_I$, W_{III} is about $\frac{1}{200} W_I$, and W_{IV} , about

$\frac{1}{1000} W_I$. The magnetic energy of that portion of the Earth's

permanent magnetic field represented by the residual field in No. I of these researches, namely, the terms II, III, and IV, is 0.000507

$R^3/0.03525 R^3$ or about $\frac{1}{70}$ that arising from the homogenous

magnetization taken as the normal field. If simply the homogenous magnetization symmetrical about the Earth's rotation axis be taken as the normal field, then the magnetic energy of the residual portion

will be $0.001915 R^3/0.03385 R^3$, or $\frac{1}{17.7}$ that of the symmetrical field.

This latter ratio is about 100 times that existing between the magnetic energies of the external portion of the diurnal variation field and the external portion of the permanent field.

2. *What is the total work required to remove a gram mass from every point of the Earth's surface to a height where the Earth's gravitational force has vanished?*

If M be the Earth's total mass, r the radius vector to any point in the space outside, then the gravitation potential will be $V = M/r$ and the vertical force, $Z = M/R^2$

$$\text{hence, } dW = \frac{1}{8\pi} VZ dS = \frac{1}{8\pi} \frac{M^2}{R^3} dS.$$

Integrating over the entire Earth's surface, in accordance with formula three, we get for the Earth's gravitational energy in the space outside:

$$W = \frac{M^2}{2R} = \frac{(6.14 \times 10^{27})^2}{2 \times 6.37 \times 10^8} = 2.96 \times 10^{46} \text{ ergs} =$$

0.302×10^{39} kgms. $= 2.18 \times 10^{39}$ foot pounds. Per square centimeter, we have, $w = 0.582 \times 10^{28}$ ergs $= 0.0593 \times 10^{21}$ kgms. $= 0.429 \times 10^{21}$ foot pounds. For a square element of surface with sides 0.4 millimeter long, $w = 9.31 \times 10^{24}$ ergs $= 9.49 \times 10^{16}$

kgms. = 68.64×10^{16} foot pounds. Hence the gravitational energy over a square 0.4 mm. of surface is equal to the Earth's total magnetic energy over the entire surface.

3. An application of the energy function to the *diurnal variation forces* has likewise been made, and it would appear, that if we take into consideration the external portion of the Earth's permanent magnetic field, in addition to the diurnal variation field, we are dealing with a distribution of forces in the upper regions, very similar to that of the permanent field, and that the difference is merely one of degree, and, as to the seat of the forces,—whether internal or external. These calculations will furnish the subject of a separate paper.

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b. THE SEAT OF THE PRINCIPAL CAUSE OF THE SECULAR VARIATION OF THE EARTH'S MAGNETISM.

Innumerable hypotheses have been advanced for the cause of the secular variation, but to my knowledge, the only attempt made to settle the question, which must first be answered, before a successful determination of the cause can be made, as to the *seat* of the cause of the secular variation—whether within the Earth's crust or outside of it,—was my preliminary one, in this Journal, Vol. IV, 1899, pp. 54–58.

I there solved the problem for that portion of the Earth's magnetism, which can be referred to a homogeneous magnetization, with the aid of the following proposition:

“If the observed changes in the compass and the dip needle, can be completely accounted for by hypothetical causes within the Earth's crust, then they can not be produced by causes outside of the crust.”

The secular changes in the *compass* needle can equally well be explained by internal or external causes. By taking, however, into account also the secular changes in *dip*, it can be determined whether the principal cause of the secular variation is an internal or an external one.

The general conclusion drawn was, that the seat of the principal cause, was within the Earth's crust.

I shall now reinvestigate this matter, considering this time the changes in the three magnetic elements, declination, dip, and intensity, during the period from 1840–1845 (Sabine's charts), to 1880–1885 (Creak and Neumayer's charts), and use the same analytical method employed by Gauss, in determining whether the Earth's permanent magnetic field arose from an internal or an external system, and later by Schuster, when solving the same question, with respect to the diurnal variation.

The potential due to an internal system of magnetic forces may be expressed as follows:

$$V_i = \frac{R^0}{r^2} \cdot P_1 + \frac{R^2}{r^3} \cdot P_2 + \frac{R^4}{r^4} \cdot P_3 + \dots, \quad (1)$$

and the potential due to an external system of magnetic forces:

$$V_e = r \cdot P_1 + \frac{r^2}{R} \cdot P_2 + \frac{r^3}{R^2} \cdot P_3 + \dots \quad (2)$$

r , is the radius vector from the center of the Earth to the point considered; R , the Earth's mean radius, and P_1, P_2, P_3 , etc., surface spherical harmonics.

On the surface of the Earth, $r = R$, we have

$$V = V_e, \quad X = X_i \text{ and } Y = Y_e, \quad (3)$$

$$Z_i = -2 P_1 - 3 P_2 - 4 P_3 - \dots, \quad (4)$$

$$Z_e = + P_1 + 2 P_2 + 3 P_3 + \dots. \quad (5)$$

Hence, with the aid of the vertical force, it can be ascertained whether the system producing a given magnetic distribution, is an internal or an external one, or both. For a system of coefficients which will satisfy the X and Y values on the Earth's surface, can not fulfill at the same time, both conditions expressed by the Z_i and Z_e equations. *If the Z_i condition is fulfilled, then the system is an internal one, and if, on the other hand, the Z_e condition is satisfied, then the system is an external one, at least as far as the principal effects are concerned.* Again, by the degree of divergence in the fulfillment of either condition, it can be determined as to how much of the observed effects are to be ascribed to a system external to the principal one.

This is the method used by Gauss and Schuster, when solving their respective problems, as already stated.

Precisely similar equations, to the above hold for any magnetic variation possessing a potential. Let dV, dX, dY , and dZ , represent, respectively, the average annual change in V, X, Y , and Z , during the interval 1840-1845 to 1880-1885, i. e., during a period about forty-two years. Furthermore, X_1, Y_1, Z_1 , and X_2, Y_2, Z_2 , shall represent, respectively, the mean magnetic components along a complete parallel of latitude, for the early and the later epoch.

$$\Delta X = X_1 - X_2, \quad \Delta Y = Y_1 - Y_2, \quad \Delta Z = Z_1 - Z_2,$$

$$dX = \frac{1}{42} \Delta X, \quad dY = \frac{1}{42} \Delta Y, \quad dZ = \frac{1}{42} \Delta Z.$$

We have to terms of the fourth order:

$$X = -g_{10} \sin u - 2 g_{20} \sin u \cos u - 3 g_{30} (\cos^2 u - \frac{1}{2}) \sin u - 4 g_{40} (\cos^2 u - \frac{7}{8}) \sin u \cos u, \quad (6)$$

$$Y = 0, \quad (7)$$

$$Z_i = -2 g_{10} \cos u - 3 g_{20} (\cos^2 u - \frac{1}{2}) - 4 g_{30} (\cos^2 u - \frac{3}{2}) \cos u - 5 g_{40} (\cos^4 u - \frac{7}{2} \cos^2 u + \frac{3}{8}), \quad (8)$$

$$Z_e = +g_{10} \cos u + 2 g_{20} (\cos^2 u - \frac{1}{2}) + 3 g_{30} (\cos^2 u - \frac{3}{2}) \cos u + 4 g_{40} (\cos^4 u - \frac{7}{2} \cos^2 u + \frac{3}{8}). \quad (9)$$

These equations being linear in the coefficients g , precisely similar equations pertain to the differentials; we have simply to substitute dX for X , dg for g , etc.

If, as is usually done in the initial steps of a potential computation, the X , Y , Z , components for points along the same parallel of latitude, are expressed in the form of a series:

$$X = m_0 + m_1 \cos \lambda + M_1 \sin \lambda + m_2 \cos 2\lambda + M_2 \sin 2\lambda + \dots \quad (10)$$

$$Y = l_0 + l_1 \cos \lambda + L_1 \sin \lambda + l_2 \cos 2\lambda + L_2 \sin 2\lambda + \dots \quad (11)$$

$$Z = k_0 + k_1 \cos \lambda + K_1 \sin \lambda + k_2 \cos 2\lambda + K_2 \sin 2\lambda + \dots \quad (12)$$

$$\text{then will } X = m_0, Y = l_0 \text{ and } Z = k_0. \quad (13)$$

$$\Delta X = \Delta m_0, \Delta Y = \Delta l_0 \text{ and } \Delta Z = \Delta k_0. \quad (14)$$

The values of m_0 , l_0 , k_0 , have been determined by Adams and Fritsche, from the magnetic elements on Sabine's charts for 1840-1845, by Adams for 1880, from Creak's chart of 1880, and by Fritsche, Neumayer and Schmidt, from Neumayer's chart for 1885. As Fritsche's values for 1842 are dependent upon scalings for points 30° apart in longitude, and his later values for 1885 for points 15° apart in longitude, whereas Adams' values for both epochs depend upon scalings 10° apart, I have confined myself for the m_0 values entirely to the latter's values, as representing fuller and more homogeneous data.

Theoretically, according to the above formulæ, $Y = l_0 = 0$. As a matter of fact, however, l_0 is not quite equal to zero,¹ thus indicating the possible existence of effects arising from a system not possessing a potential, *e. g.*, electric currents passing vertically through the Earth's crust, either from within or without, or from both sources. However, the departures from zero are of such an amount, that for the present investigation, we can leave them out of consideration. We can thus confine ourselves entirely to determining Δg_{10} , Δg_{20} , Δg_{30} , Δg_{40} , from the $\Delta X = \Delta k_0$ values. If the values thus found, satisfy the ΔZ_i equation, then it has been proven that the seat of the principal source of the secular variation is within the Earth, and if, on the other hand, the ΔZ_e condition is better fulfilled, then the seat is outside of the Earth.

If Δk_0 and $\Delta k_0'$, represent the average ΔX values on two parallels, u and $180-u$ (*i. e.*, on two parallels equi-distant from the equator), we have the conditional equations:

¹ *T. M.* Vol. II, p. 17.

$$\frac{1}{2}(\Delta k_0 + \Delta k'_0) = -\Delta g_{10} \sin u - 3 \Delta g_{30} (\cos^2 u - \frac{1}{2}) \sin u, \quad (15)$$

$$\frac{1}{2}(\Delta k_0 - \Delta k'_0) = -2 \Delta g_{20} \sin u \cos u - 4 \Delta g_{40} (\cos^2 u - \frac{3}{2}) \sin u \cos u, \quad (16)$$

for each pair of parallels employed, viz: 60° N. and S., 50° N. and S., 40° N. and S., 30° N. and S., 20° N. and S., and 10° N. and S., the equator not being used, so as to preserve the uniformity of the system of equations.

Parallels	$\frac{1}{2}(k_0 + k'_0)$ Brit. Units.		$\frac{1}{2}(k_0 + k'_0)$ C. G. S. Units.		$\frac{1}{2}(\Delta k_0 + \Delta k'_0)$ C. G. S. Units.		$\frac{1}{2}(\Delta k_0 - \Delta k'_0)$ C. G. S. Units.	
	1842.5	1880	1842.5	1880	Observ'd	Comp'd	Observed	Comp'd
60 N. and S.	3.1453	3.0990	0.1450	0.1429	+ 0.0021	+ 0.0039	— 0.0020	— 0.0012
50 " " "	4.1814	4.0501	0.1928	0.1867	+ 61	+ 48	— 42	— 46
40 " " "	5.1196	4.9913	0.2361	0.2301	+ 60	+ 55	— 78	— 81
30 " " "	5.9998	5.8674	0.2766	0.2705	+ 61	+ 60	— 96	— 101
20 " " "	6.7362	6.6060	0.3106	0.3046	+ 60	+ 63	— 82	— 92
10 " " "	7.2384	7.1009	0.3337	0.3274	+ 63	+ 64	— 70	— 55

The values $\frac{1}{2}(k_0 + k')$ and $\frac{1}{2}(k_0 - k')$, as used in above table, have been taken from Vol. II, pp. 520-523, of "The Scientific Papers of John Couch Adams;" they are given there in British units, and have been converted by me into C. G. S. units. Columns 6 and 8, give the quantities used in the determination of the values Δg by least squares. The first number in column 6, for instance, represents the average secular change in the X component along the parallels 60° N. and S., for the entire interval 1842.5 to 1880, the plus sign signifying that X , on the average, decreased during that period. To obtain the average annual change, dX , the numbers would have to be divided by the elapsed interval of 37.5 years.

The resulting values of Δg , expressed in C. G. S. units, are: $\Delta g_{10} = -0.00685$, $\Delta g_{20} = +0.00795$, $\Delta g_{30} = -0.00047$, and $\Delta g_{40} = -0.01025$. Hence the annual increments from division by 37.5: $d g_{10} = -0.000183$, $d g_{20} = +0.000212$, $d g_{30} = -0.000013$, $d g_{40} = -0.000273$.

How closely the computed Δg 's¹ will reproduce the observed values, can be seen from columns 7 and 9.

¹ We have a check upon our values from Adams' own computations. He computed the g 's for both epochs, once from the X and Y components alone, and next from the Z components, and thus got two sets of values for each epoch (see p. 607 of volume above cited); his values are slightly different from mine as he included terms to the 6th order. Converting into C. G. S. units we obtain from his X , Y , figures: $\Delta g_{10} = -0.0066$, $\Delta g_{20} = +0.0077$, $\Delta g_{30} = -0.0006$, $\Delta g_{40} = -0.0094$, $\Delta g_{50} = +0.0033$, and $\Delta g_{60} = +0.0042$.

If now the principal cause of the secular variation is an internal one, then the above $\Delta g'$ values must satisfy this condition:

$$\Delta Z_i = -2 \Delta g_{10} \cos u - 3 \Delta g_{20} (\cos^2 u - \frac{1}{3}) - 4 \Delta g_{30} (\cos^2 u - \frac{3}{5}) \cos u - 5 \Delta g_{40} (\cos^4 u - \frac{6}{7} \cos^2 u + \frac{3}{35}) \quad (17)$$

If, on the other hand, the principal cause is an external one, then the other condition must be fulfilled:

$$\Delta Z_e = + \Delta g_{10} \cos u + 2 \Delta g_{20} (\cos^2 u - \frac{1}{3}) + 3 \Delta g_{30} (\cos^2 u - \frac{3}{5}) \cos u + 4 \Delta g_{40} (\cos^4 u - \frac{6}{7} \cos^2 u + \frac{3}{35}) \quad (18)$$

The figures in the following table show the result of the test of both hypotheses; the value of each term in each expression is given, so as to show its effect and magnitude. It is most instructive to exhibit the law of variation in ΔZ from parallel to parallel, instead of a pair of equi-distant parallels, and as Adams only gives the Z^m (or m_o) values for pairs of parallels, I have made use of de Tillo's tabulations of Z , for points 10° apart in longitude. All the quantities in the table are in terms of the fourth decimal, C. G. S., units, hence, in terms of 107.

	Observation.			I. Computation: cause internal								II. Computation: cause external							
Lat.	$Z^m = m_o$		ΔZ_o^m	Term				I	Obs'd. ΔZ_c^m	Co'pd	Term				E	Obs'd. ΔZ_c^m	Co'pd		
	1842.5	1885	42-'85	1	2	3	4	1			2	3	4	ΔZ_c^m					
N60°	+5488	+5430	+ 58	+119	-99	+2	+0	+22	+36	-59	+66	-2	+0	+5	+53				
50°	+5032	+5000	+ 32	+105	-60	0	-4	+41	-9	-52	+40	0	+3	-9	+41				
40°	+4420	+4303	+117	+88	-19	-2	-5	+62	+55	-44	+13	+2	+4	-25	+142				
30°	+3449	+3464	-15	+68	+20	-3	-3	+82	-97	-34	-13	+2	+3	-42	+27				
20°	+2340	+2295	+55	+47	+51	-3	0	+95	-40	-23	-34	-2	0	-55	+110				
10°	+1124	+1022	+102	+24	+72	-2	+3	+91	-11	-12	-48	+2	-3	-61	+163				
Equ'r	96	-221	+125	0	+79	0	+4	+83	+42	0	-53	0	-4	+57	+182				
10°	-1262	-1359	+97	-24	+72	+2	+3	+53	+44	+12	-48	-2	-3	-41	+138				
20°	-2286	-2398	+112	-47	+51	+3	0	+7	+105	+23	-34	-2	0	-13	+125				
30°	-3200	-3226	+26	-68	+20	+3	-3	-48	+74	+34	-13	-2	+3	+22	+4				
40°	-4087	-3927	-160	-88	-19	+2	-5	-110	-50	+44	+13	-2	+4	+59	-219				
50°	-4823	-4600	-223	-105	-60	0	-4	-169	-54	+52	+40	0	+3	+95	-318				
S60°	-5437	-5209	-228	-119	-99	-2	0	-220	-8	+59	+66	-2	0	+127	-355				

The fourth column contains the secular change in the vertical component during the interval 1842.5 to 1885, on the average, for the respective parallel opposite to which it stands, a plus sign signifying that the vertical force has decreased during the said period. The columns marked I and E, respectively, give the computed secular changes for the two hypotheses: internal and external; the residuals contained in the columns adjacent (on the right) to I and E,

enable one to judge as to which of the two hypotheses appears the more probable one. It is readily seen that the hypothesis of an internal cause gives the smaller residuals. The diagram (Fig. 1) tells the story best.

The curves shown, express the law of variation in the average secular change of the vertical component, from parallel to parallel, the full curve as dependent on observations between 1842 and 1885, the broken curve being the computed variation upon the hypothesis of an internal cause, whereas the dotted curve being the computed variation as on the hypothesis of an external one. It will be noticed, that while the observed curve is irregular, it agrees in general well with the internal-cause curve, and disagrees markedly from the external-cause curve.

FIG. 1. Comparison of observed (full curve) with computed secular variation (broken curve supposing cause internal, and dotted curve supposing cause external.)

In order to find out whether the irregularities in the observed curve are due to errors in de Tillo's tabulated quantities, I have also plotted the curve as dependent on Fritsche's m_0 values for 1842.5 and 1885; his first values, as already stated, depend upon 30° scalings, and the latter upon scalings made 15° apart in longitude. The curve thus obtained, agreed so well with the one already given, that, to avoid confusion, it was omitted in the reproduction. The irregularities are, therefore, inherent in the primary quantities, and

from other investigations, I am conducting, I am inclined to think that they are, to a large extent, facts of nature. These irregularities are most pronounced near parallels 20-30, N. and S.—parallels of such interest to the meteorologist.

It is quite possible, of course—in fact probable—that in addition to the main internal cause of the secular variation, there is, in addition, a subordinate, external cause. The investigation of this is reserved for a future paper.

The conclusion drawn from this paper, is that :

The main cause of the secular variation of the Earth's magnetism is due to a system of forces imbedded in the Earth, and not to an external system. This does not exclude there being, in addition, an external system producing minor effects. The main cause is due, furthermore, to a system of forces possessing a potential.

c. ON THE IMPROVEMENT OF THE GAUSSIAN THEORY OF THE EARTH'S MAGNETISM.

As it is my intention to make some time a new mathematical analysis of the Earth's magnetic field, on somewhat different principles than hitherto employed, I am making a critical examination and comparison of the results obtained thus far by the various analysts, with the view of ascertaining precisely wherein improvement is required. The conclusions reached, thus far, follow from the tables herein presented.

Table A, gives the residuals obtained by subtracting the computed X components from the so-called observed values or values based upon the magnetic charts used in the potential computation. Thus, beginning with the second column, marked F , we have the residuals for every 30° in longitude along the parallel 60° N, the computed quantities, X_c , resulting from Fritsche's analysis for 1840-1845, as dependent upon the Sabine's charts for that period. Fritsche scaled from these maps, the declination, inclination, and horizontal intensity for every 10th degree parallel, 80° N. to 80° S., and for every 30th degree meridian, and from these scaled quantities, computed the X_o , Y_o , Z_o 's. Having deduced the Gaussian coefficients to terms of the sixth order, as given in the table in section a of this paper, he next computed the X_c , Y_c , Z_c 's, and formed the residuals, which are here reproduced with the signs changed to conform with the notation adopted in this paper. [X + north, Y or D , + east, Z or I , + down, $\Delta X = X_o - X_c$ etc.]

TABLE A. Residuals of X components for 1842.5 and 1885, according to *Fritzsche and Schmidt*. ($X_0 - X_c$).
[Expressed in units of fourth decimal, C. G. S.]

λE of G°	60° N. 1885					40° N. 1885					20° N. 1885					Equator. 1885					20° S. 1885					40° S. 1885					60° S. 1885					λE of G°
	1842	N F	N S	S F	F	N F	N S	S F	F	N F	N S	S F	F	N F	N S	S F	F	N F	N S	S F	F	N F	N S	S F	F	N F	N S	S F	F							
0	+13	+71	—5	+76	—61	—3	+45	—48	+10	—60	—35	—25	+92	+29	+14	+15	—98	—99	—49	—41	—70	—17	+21	—38	—61	—121	—43	—78	—82	15						
15	+7	+99	+12	+105	—78	+14	—16	—31	+232	—50	—24	—34	—144	—9	+13	—22	—99	—6	+10	—16	—35	+48	+29	+19	—78	—81	—16	—65	—30	—65	30					
30	+7	+131	+32	+106	—38	+25	—40	—05	—30	+33	+13	+22	+81	—70	+8	—78	+51	+11	+10	—8	+32	+91	+10	+105	—31	—70	—35	—35	—44	45						
45	+70	+138	+51	+79	+38	+37	—38	+54	+122	+69	+30	+43	+194	—73	—26	—47	+51	+11	+10	—8	+32	+91	+10	+105	—31	—70	—35	—35	—44	45						
60	+24	+135	+52	+83	+81	+45	—39	+54	+122	+69	+30	+43	+194	—73	—26	—47	+51	+11	+10	—8	+32	+91	+10	+105	—31	—70	—35	—35	—44	45						
75	+24	+135	+52	+83	+81	+45	—39	+54	+122	+69	+30	+43	+194	—73	—26	—47	+51	+11	+10	—8	+32	+91	+10	+105	—31	—70	—35	—35	—44	45						
90	+24	+135	+52	+83	+81	+45	—39	+54	+122	+69	+30	+43	+194	—73	—26	—47	+51	+11	+10	—8	+32	+91	+10	+105	—31	—70	—35	—35	—44	45						
105	+52	+159	+54	+86	+11	+71	—8	+79	+85	+19	+32	+41	+118	+38	+31	—28	+8	+33	+5	+11	—3	+37	+95	+10	+85	—24	+31	—72	+103	135	105					
120	+52	+159	+54	+86	+11	+71	—8	+79	+85	+19	+32	+41	+118	+38	+31	—28	+8	+33	+5	+11	—3	+37	+95	+10	+85	—24	+31	—72	+103	135	105					
135	+27	+76	—49	+70	—81	+34	+7	+51	+73	+28	+7	+42	+93	+50	—61	+111	+210	+81	+31	+63	—32	+12	+11	—33	+8	+13	—18	+10	195	150						
150	+27	+76	—49	+70	—81	+34	+7	+51	+73	+28	+7	+42	+93	+50	—61	+111	+210	+81	+31	+63	—32	+12	+11	—33	+8	+13	—18	+10	195	150						
165	+5	+41	—107	+150	+40	+34	+71	+109	—4	+113	+46	+101	+80	+87	+15	+72	+181	+66	+33	+63	—32	+12	+11	—33	+8	+13	—18	+10	195	150						
180	+5	+41	—107	+150	+40	+34	+71	+109	—4	+113	+46	+101	+80	+87	+15	+72	+181	+66	+33	+63	—32	+12	+11	—33	+8	+13	—18	+10	195	150						
195	—66	+87	—49	+160	+19	+82	+36	+50	—40	+113	+46	+101	+80	+87	+15	+72	+181	+66	+33	+63	—32	+12	+11	—33	+8	+13	—18	+10	195	150						
210	—66	+87	—49	+160	+19	+82	+36	+50	—40	+113	+46	+101	+80	+87	+15	+72	+181	+66	+33	+63	—32	+12	+11	—33	+8	+13	—18	+10	195	150						
225	—	+88	—14	+169	—24	+92	+37	+61	—14	—11	—28	+17	+147	+52	+7	+32	+92	+45	+56	—4	—2	—57	—44	—13	+111	+75	+68	+21	225	210						
240	—	+88	—14	+169	—24	+92	+37	+61	—14	—11	—28	+17	+147	+52	+7	+32	+92	+45	+56	—4	—2	—57	—44	—13	+111	+75	+68	+21	225	210						
255	+6	+83	—15	+174	—38	+97	+16	+89	+69	+18	—53	+30	+17	+45	+13	+30	+60	+45	+56	—4	—2	—57	—44	—13	+111	+75	+68	+21	225	210						
270	+6	+83	—15	+174	—38	+97	+16	+89	+69	+18	—53	+30	+17	+45	+13	+30	+60	+45	+56	—4	—2	—57	—44	—13	+111	+75	+68	+21	225	210						
285	—22	+33	—54	+174	—112	+108	—17	+45	+40	+68	+17	+34	+67	+37	+16	+15	—57	+41	+46	—4	+42	—17	—26	+9	—11	—9	+66	—75	315	270						
300	—22	+33	—54	+174	—112	+108	—17	+45	+40	+68	+17	+34	+67	+37	+16	+15	—57	+41	+46	—4	+42	—17	—26	+9	—11	—9	+66	—75	315	270						
315	+32	+3	—9	+19	+52	—48	—23	—21	—60	—36	—53	+27	+97	+30	+43	—18	+41	+24	+28	—52	—4	—32	—36	—20	+3	+47	—89	—38	—61	330	300					
330	+32	+3	—9	+19	+52	—48	—23	—21	—60	—36	—53	+27	+97	+30	+43	—18	+41	+24	+28	—52	—4	—32	—36	—20	+3	+47	—89	—38	—61	330	300					
345	+32	+3	—9	+19	+52	—48	—23	—21	—60	—36	—53	+27	+97	+30	+43	—18	+41	+24	+28	—52	—4	—32	—36	—20	+3	+47	—89	—38	—61	330	300					
Mean	27.1	77.1	38.5	88.6	52.3	47.2	30.6	49.6	64.9	47.2	22.8	40.8	10.6	42.5	26.6	40.9	73.1	38.0	23.8	29.5	36.1	41.2	24.1	35.7	43.8	57.1	46.2	50.5	Mean	Mean						

TABLE B. Residuals of *Y* components for 1842.5 and 1885, according to *Fritzsche and Schmidt*. ($Y_0 - Y_c$).
[Expressed in units of fourth decimal, C. G. S.]

A.E. of Gr	60° N. 1885				40° N. 1885				Equator. 1885				20° S. 1885				40° S. 1885				60° S. 1885								
	1842	F	NF	NS	SP	1842	F	NF	NS	SP	1842	F	NF	NS	SP	1842	F	NF	NS	SP	1842	F	NF	NS	SP	A.E. of Gr			
0	-40	-22	+44	-66	+22	+47	+34	+7	+13	-23	+111	+23	+15	+11	+23	+88	-41	+44	-20	+64	-12	-18	-5	-13	+33	+14	29	0	
15	-15	-20	-44	-28	-54	-16	-23	+9	-25	+40	-58	+23	-5	-11	-12	-6	+12	-25	+40	-70	+7	-5	-6	+1	-15	-21	+16	15	
30	-45	+25	-34	+59	-13	-43	-23	+22	-23	-38	-58	+19	-106	-106	-106	-6	+12	-25	+40	-53	+29	+17	+37	-21	+57	+53	+37	30	
45	+54	+62	+2	+36	-3	+32	+5	+32	-23	-53	-29	-48	-56	-56	-56	+19	-53	-23	-53	-33	+29	+17	+37	-21	+57	+53	+37	45	
60	+40	+35	+13	+36	-83	+60	+5	+32	-23	-23	-29	-48	-35	-35	-35	-2	-21	-10	-10	-43	-14	+4	+17	-41	-71	+28	+10	60	
75	-50	-7	-2	-15	+17	-83	+87	+106	-10	+16	+10	+21	-35	-35	-35	-2	-21	-10	-10	-43	-14	+4	+17	-41	-71	+28	+10	75	
90	+30	+12	-22	+10	-51	+43	+90	-47	-12	+16	+10	+21	-35	-35	-35	-2	-21	-10	-10	-43	-14	+4	+17	-41	-71	+28	+10	90	
105	+109	+41	+26	+15	0	+46	+87	-41	-12	+16	+10	+21	-35	-35	-35	-2	-21	-10	-10	-43	-14	+4	+17	-41	-71	+28	+10	105	
120	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	120	
135	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	135	
150	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	150	
165	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	165	
180	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	180	
195	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	195	
210	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	210	
225	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	225	
240	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	240	
255	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	255	
270	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	270	
285	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	285	
300	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	300	
315	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	315	
330	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	330	
345	-10	-10	-6	-86	+86	+41	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	-9	345	
Mean	34.4	23.0	30.9	34.9	35.1	28.7	51.7	33.5	32.9	38.9	37.3	38.7	35.4	42.5	38.7	46.8	27.1	22.6	20.4	28.3	31.6	46.1	37.4	46.9	48.7	47.0	45.6	50.6	Mean

TABLE C. *Residuals of Z Components for 1842.5 and 1885, according to Fritzsche and Schmidt. (Z₀ — Z_c).*
[Expressed in units of fourth decimal, C. G. S.]

λ of Gr	66° N. 1885				46° N. 1885				20° N. 1885				Equator. 1885				20° S. 1885				46° S. 1885				66° S. 1885				λ of Gr
	1842		1842		1842		1842		1842		1842		1842		1842		1842		1842		1842		1842						
	F	NS	SP	F	NS	SP	F	NS	SP	F	NS	SP	F	NS	SP	F	NS	SP	F	NS	SP	F	NS	SP					
0	+8	+32	+61	+93	+38	+35	+33	+38	+35	+33	+38	+35	+33	+38	+35	+33	+38	+35	+33	+38	+35	+33	+38	+35	0				
15	—	74	21	81	33	60	44	15	34	58	22	51	89	90	79	83	3	3	9	54	107	18	19	16	15				
30	—	52	29	31	104	8	105	4	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	30				
45	—	117	151	202	127	102	24	4	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	45				
60	—	91	113	104	94	128	103	25	7	16	33	64	21	34	8	105	4	34	53	19	34	65	17	137	60				
75	—	91	7	98	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	75				
90	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	90				
105	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	105				
120	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	120				
135	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	135				
150	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	150				
165	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	165				
180	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	180				
195	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	195				
210	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	210				
225	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	225				
240	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	240				
255	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	255				
270	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	270				
285	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	285				
300	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	300				
315	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	315				
330	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	330				
345	—	91	31	79	81	74	7	33	34	53	19	34	65	17	137	67	14	14	—	—	39	34	74	83	345				
Mean	50.8	127	105	57.2	47.0	86.1	53.3	54.5	43.8	65.1	50.9	62.2	43.1	70.3	54.5	35.2	32.7	46.8	54.5	82.7	37.6	62.0	106	108	37.2	Mean			

The third column, marked $N F$, gives the residuals for 1885, resulting from Fritsche's analysis for that date; the residuals are given this time for every 15° in longitude, since in this instance, Fritsche based his computation on quantities scaled from Neumayer's charts for every 15th degree meridian.

The fourth column, marked $N S$, gives the residuals resulting from Schmidt's analysis for 1885, and the fifth column, marked $S F$, the residuals between Schmidt's X_c values and Fritsche's ($S-F$).

At the bottom of the columns we have the mean residuals, supposing all residuals to have the same sign.

Tables B and C , give similarly, the Y and Z residuals.

All of the 1885 residuals in the three tables have been formed by myself, starting with the "observed" components given in Schmidt's publications, and have been checked with the aid of such tables as have been published. Various typographical errors appear to have crept into Schmidt's own tables of residuals, as given in his publications II and III. Fritsche appears to have published the residuals ΔX , ΔY , ΔZ , only for 1842.

I present the residuals simply for parallels 20° apart, from 60° N. to 60° S., as sufficient for the present purpose.

Table D gives the probable errors, r , for each parallel, as based upon the residuals in the foregoing tables, supposing the chart values to be standard ones. If η is the mean residual (regardless of sign), as given at the bottom of the foregoing tables, then, with sufficient approximation for our immediate object, $r = \pm 0.845 \eta$. At the bottom of this table, the probable errors are given for a mean parallel.

Table E gives the probable errors for the Gaussian analysis (1830), and the Erman-Petersen analysis for 1829; this table is taken directly from the latter's publication (p. 24), the quantities having been converted, however, from conventional Gaussian units into C. G. S., units ($1 \text{ C. G. } U = 0.0003494 \text{ C. G. S.}$). Here the comparison of computed and "observed" quantities is made by the authors ($E. P.$) for meridians 40° apart.

By combining the probable errors of X_c , Y_c , and Z_c , the probable errors of the total force F_c are obtained.

The probable errors of H are determined from those of X and Y , whereas those for D (declination) and I (inclination), are formed directly from Fritsche's tables of residuals, of which Tables G give extracts.

We are now prepared to draw our conclusions. Beginning with

TABLE D. Mean probable errors of the computed components for 1842.5 and 1885.

[Expressed in units of fourth decimal, C. G. S.]

Parallel °	I 1842.5			1885 X			II 1885 Y			1885 Z			III 1885		
	X F	Y F	Z F	N F	N S	F S	N F	N S	F S	N F	N S	F S	X Mean	Y of F	Z and S
N. 60	±23	±29	±43	±65	±33	±75	±19	±26	±29	±107	±89	±48	±28	±2	±2
40	45	30	40	40	26	42	24	44	28	73	45	46	14	8	46
20	55	28	37	40	19	34	33	32	33	55	43	53	7	0	9
Equ'tr	90	30	39	36	22	35	36	33	40	31	40	36	26	8	10
20	62	23	59	32	20	25	19	17	24	46	30	28	10	0	1
40	30	27	40	35	20	30	39	32	40	46	70	76	8	15	20
S. 60	37	41	52	48	39	43	40	39	43	90	91	31	11	18	12
For Mean Parallel	±53	±30	±45	±44	±26	±43	±31	±33	±34	±69	±63	±48	±17	±10	±20

TABLE E. Mean probable errors of the computed components for 1829 and 1830.

[Expressed in units of fourth decimal, C. G. S.]

Parallel °	X		Y		Z	
	1830 G.	1829 E. P.	1830 G.	1829 E. P.	1830 G.	1829 E. P.
N. 67	± 41	± 50	± 60	± 45	± 103	± 106
60	42	63	20	30	124	99
50	69	83	57	49	99	101
40	103	79	66	53	128	113
15	67	75	77	62	140	101
Equator	113	79	65	35	138	119
15	91	67	45	57	110	116
40	187	89	57	36	119	101
60	188	80	157	80	297	108
S. 75	148	74	211	75	368	57
For Mean Parallel	± 117	± 75	± 99	± 56	± 183	± 102

TABLE F. Average probable errors of the computed elements for zone 60° N. to 60° S.

[The intensity quantities are given in units of fourth decimal, C. G. S.]

No.	Analyst.	Epoch	X	Y	Z	F	H	D	I
1	Gauss	1830	±117	±99	±183	±239	±153	°	°
2	Erman-Petersen	1829	75	56	102	138	94		
3	Fritsche	1842.5	53	30	45	76	61	±0.88	±0.52
4	"	1885	44	31	69	87	54	0.92	0.66
5	Schmidt	1885	26	33	63	75.5	42		
6	Schmidt-Fritsche	1885	43	34	48	73	55		
7	Schmidt and Fritsche	1885	±17	±10	±20	±28	±20		

Table *F*, and leaving out of consideration, for the present, the entries Nos. 6 and 7, it will be noticed that the average probable error in the computed *X* component, for the entire zone of the Earth, 60° N. to 60 S., supposing the observed or chart quantities to be correct, has been steadily reduced until Schmidt's analysis (No. 5) gives a probable error of but $\frac{1}{8}$ of that of Gauss's, $\frac{1}{3}$ that of Erman-Petersen's, and $\frac{1}{2}$ that of Fritsche's for 1842.5. Looking at the probable errors for *Y*, we find for 1842.5, a value about $\frac{1}{3}$ that of Gauss, and $\frac{1}{2}$ that of Erman-Petersen, but since then there has been no reduction. The probable error for *Z*, for 1842.5 is about $\frac{1}{4}$ that of Gauss, and a little less than $\frac{1}{2}$ of Erman-Petersen's. Since 1842.5, there is an increase of about 50 per cent.

Examining next the probable errors for the total force, *F*, Fritsche's probable error for 1842.5, is about $\frac{1}{3}$ of Gauss's and $\frac{1}{2}$ of Erman-Petersen's. His 1885 probable error, however, in spite of the fact that this analysis was based on twice as many points, is about 15 per cent higher than the one for 1842.5. And what is certainly surprising, Schmidt's probable error, although his analysis was the most elaborate, painstaking, and comprehensive one up to date, dependent upon $4\frac{1}{2}$ times the number of points used by Fritsche for 1885, and 9 times the number used by Fritche for 1842.5, is the same as Fritsche's for the earlier epoch! As is well known, Schmidt not only took into account the spheroidal figure of the Earth, but also, for the first time, treated the three components *X*, *Y*, *Z*, separately, so as to avoid, *ab initio*, the assumption that the entire effect was to be ascribed to a potential due solely to internal forces. He obtained a small potential due to external forces and a system of vertical electric currents. His computed components are the combined effects of the three system of forces thus found.

We are apparently forced to conclude that the 1885 charts may not represent any improvement on those for 1840-1845. If this be true, then the explanation is to be found in the following statement: The additional material obtained since the construction of Sabine's charts, has been mostly in regions of which the *general* magnetic distribution, as given on a chart, was already fairly well-known. For the most important areas, such as ocean areas, but comparatively little material has been obtained. Sabine's charts were based on data distributed over about seven decades, the epoch 1840-1845 being about midway in this interval, so that the reductions to epoch, because of secular variation were partly positive and

TABLE G. Declination and inclination residuals for 1842.5 and 1885, according to Fritzsche. ($D_0 - D_c$ or $I_0 - I_c$).

A.E. of G_{γ}		DECLINATION.												INCLINATION.												A.E. of G_{γ}				
		60° N.		40° N.		20° S.		40° S.		60° S.		60° N.		40° N.		20° N.		Equator.		20° S.		40° S.		60° S.						
°	'	1842.5	1885	1842.5	1885	1842.5	1885	1842.5	1885	1842.5	1885	1842.5	1885	1842.5	1885	1842.5	1885	1842.5	1885	1842.5	1885	1842.5	1885	1842.5	1885	1842.5	1885	°	'	
0	0	-1.3	+0.2	0.0	+0.1	-0.3	+1.8	-0.2	+1.1	-1.2	0.4	-0.1	0.4	-0.3	-0.9	+0.5	0.0	+1.7	-1.1	-0.2	+2.2	+1.2	+0.7	+0.4	-0.4	-0.7	0	0		
15	15	-0.9	-0.9	-0.6	-0.6	+0.2	+0.6	+0.6	+0.6	-0.2	-0.4	-0.4	-0.4	-0.4	-1.6	-1.6	-1.6	-1.6	-1.6	-2.2	-2.2	+1.2	+1.5	+1.5	-1.2	15	15			
30	30	-0.5	-0.5	-1.4	-1.1	-1.0	-1.1	-0.4	-0.8	-0.4	-0.9	-1.1	-1.8	-0.7	-0.4	+1.2	-0.6	-0.8	-1.1	-0.8	-1.9	-1.1	-0.1	+1.5	-0.5	-0.8	30	30		
45	45	0.0	0.0	-0.4	-0.4	-1.1	-1.1	-0.8	-0.8	-0.5	-0.5	-1.3	-0.3	-0.3	-1.2	-0.9	-0.7	-0.7	-0.9	-0.7	-0.9	-0.8	-0.8	+1.3	-0.5	-0.7	45	45		
60	60	+1.4	+1.0	-0.1	+0.6	-0.9	-0.7	-0.8	-0.3	+0.9	+0.2	+2.2	-0.7	+0.6	-0.8	-1.4	+0.5	-0.8	-1.1	+1.0	+1.6	+1.6	+0.3	+0.3	-0.3	-0.5	60	60		
75	75	+1.4	+1.0	-0.1	+0.6	-0.9	-0.7	-0.8	-0.3	+0.9	+0.2	+2.2	-0.7	+0.6	-0.8	-1.4	+0.5	-0.8	-1.1	+1.0	+1.6	+1.6	+0.3	+0.3	-0.3	-0.5	75	75		
90	90	+1.3	+0.3	-1.7	+1.7	-0.6	+0.2	-0.4	+0.2	-0.2	-0.3	-1.8	-0.4	-0.4	-1.8	-0.4	-1.2	-0.7	-0.8	-1.1	+1.2	+1.2	+0.2	+1.2	-0.2	-0.2	90	90		
105	105	+0.4	-0.4	-1.7	+1.7	-0.6	+0.2	-0.4	+0.2	-0.2	-0.3	-1.8	-0.4	-0.4	-1.8	-0.4	-1.2	-0.7	-0.8	-1.1	+1.2	+1.2	+0.2	+1.2	-0.2	-0.2	105	105		
120	120	-1.7	+0.5	-1.5	-0.2	+0.1	+0.5	+0.3	+0.9	-0.5	+1.0	+0.5	+1.8	+9.1	+6.6	0.0	+0.6	-0.6	-0.7	+0.8	+1.5	+1.5	-0.4	+1.0	-0.7	-0.2	120	120		
135	135	+0.4	-0.2	-0.9	-0.2	+0.2	+0.2	+0.2	+0.2	+0.2	+0.5	+1.8	+9.1	+6.6	+6.6	-0.1	+0.6	-0.6	-0.7	+0.8	+1.5	+1.5	-0.4	+1.0	-0.7	-0.2	135	135		
150	150	+0.9	+0.4	-1.0	+1.0	-0.1	+0.7	-0.2	+0.3	+0.3	+0.7	+0.1	+0.4	-1.2	-0.2	+0.2	+0.6	-0.5	0.0	+0.6	+0.6	+0.6	+0.3	+1.0	-0.4	+0.4	150	150		
165	165	+0.7	+0.7	+0.7	+0.7	+0.2	+0.2	+0.4	+0.4	+0.4	+0.4	+0.4	-1.7	-0.2	0.2	-0.2	-1.0	-1.4	-1.4	-0.2	-0.2	-0.2	+0.3	+1.0	-0.4	+0.4	165	165		
180	180	+3.3	+1.3	-0.2	+1.0	-0.2	-0.2	-0.9	+0.1	-0.7	-0.9	-0.3	-1.8	-2.0	1.6	-0.2	-1.2	-0.5	-1.5	-1.4	-2.8	-2.7	+0.3	+1.0	-0.4	+0.4	180	180		
195	195	+0.4	+0.4	-0.7	+1.0	-0.2	-0.2	-0.2	+0.1	-0.7	-0.9	-0.3	-1.8	-2.0	1.6	-0.2	-1.2	-0.5	-1.5	-1.4	-2.8	-2.7	+0.3	+1.0	-0.4	+0.4	195	195		
210	210	+0.8	-1.5	+0.8	-0.9	+0.5	-0.4	+1.3	-1.2	-0.3	-0.3	-0.5	-0.3	-2.9	3.4	+0.4	-0.5	-0.8	-1.5	-1.4	-2.8	-2.7	+0.3	+1.0	-0.4	+0.4	210	210		
225	225	-0.7	-2.5	0.0	-0.6	-0.9	-0.8	-0.9	-2.1	+0.4	-0.3	-0.3	-0.9	-3.0	3.4	+0.2	-0.4	0.0	-0.4	-0.3	-0.3	-0.3	+0.5	+0.5	-0.1	-0.2	225	225		
240	240	-0.7	-2.5	0.0	-0.6	-0.9	-0.8	-0.9	-2.1	+0.4	-0.3	-0.3	-0.9	-3.0	3.4	+0.2	-0.4	0.0	-0.4	-0.3	-0.3	-0.3	+0.5	+0.5	-0.1	-0.2	240	240		
255	255	-3.4	+0.5	+0.5	+0.5	+0.1	-0.7	-0.3	+0.1	+0.9	-0.3	+1.1	-0.5	+0.4	+1.2	-0.1	-0.7	+0.4	-0.9	-0.7	-0.7	-0.7	+1.0	+1.0	-0.8	-0.5	255	255		
270	270	-3.0	+3.9	+0.5	+0.5	+0.1	-0.7	-0.3	+0.1	+0.9	-0.3	+1.1	-0.5	+0.4	+1.2	-0.1	-0.7	+0.4	-0.9	-0.7	-0.7	-0.7	+1.0	+1.0	-0.8	-0.5	270	270		
285	285	-3.0	+3.9	+0.5	+0.5	+0.1	-0.7	-0.3	+0.1	+0.9	-0.3	+1.1	-0.5	+0.4	+1.2	-0.1	-0.7	+0.4	-0.9	-0.7	-0.7	-0.7	+1.0	+1.0	-0.8	-0.5	285	285		
300	300	-1.3	+1.3	-1.4	+0.1	-0.4	+0.7	-0.8	-0.8	-1.0	-0.6	+0.2	-0.9	-1.3	+0.7	0.0	-0.2	+1.0	-0.6	-0.6	-0.6	+0.5	+2.6	-0.4	-0.4	+0.5	300	300		
315	315	0.0	0.0	-1.4	+0.1	-0.4	+0.7	-0.8	-0.8	-1.0	-0.6	+0.2	-0.9	-1.3	+0.7	0.0	-0.2	+1.0	-0.6	-0.6	-0.6	+0.5	+2.6	-0.4	-0.4	+0.5	315	315		
330	330	+0.4	0.0	+2.2	+0.3	+0.4	+0.7	+0.2	+0.2	-0.6	-0.6	+0.3	+0.3	+0.3	+0.3	-0.4	-0.2	-0.2	-0.1	+0.8	0.0	+0.3	0.0	+1.4	-0.1	-1.2	+0.9	330	330	
Mean	Mean	±0.5	±0.5	±1.1	±0.3	±0.4	±1.1	-0.8	-0.5	±0.5	±0.5	±1.1	±0.5	±1.1	±0.4	±0.3	±0.4	±0.8	±0.8	±0.5	±0.5	±0.5	±1.1	±1.4	±0.3	±0.7	±0.4	±0.4	345	345
Prob. Error	Prob. Error	±1.2	±1.0	±0.8	±0.5	±0.4	±0.5	±0.6	±0.7	±0.5	±0.6	±0.6	±0.7	±1.5	±1.8	±0.2	±0.8	±0.5	±0.7	±0.5	±0.8	±0.7	±0.5	±0.9	±0.6	±0.2	±0.6	±0.5	Prob. Error	Prob. Error

partly negative, and thus the errors in reductions, to a certain extent eliminated. Neumayer, on the other hand, was obliged to depend entirely upon data accumulated up to the epoch 1885, his charts having been constructed in 1887. Hence, for large areas, he was forced to utilize data observed one-half century prior to his date and refer them to 1885, with more or less uncertain values of the secular change.

Another fact which leads one to conclude that in some manner an arbitrary or systematic error crept into the 1885 charts, due, possibly, to the reason stated, is the systematic variation in the probable errors with latitude. This will be noticed by looking over the *X* and *Z* figures in part II of Table *D*; it will be seen that the probable errors systematically increase with increasing latitude, whether we proceed north or south of the equator. This peculiar variation with latitude is not shown by the figures for either 1829, 1830, or 1842.5.

Let us pass next to the two bottom rows (Nos. 6 and 7) of Table *F*. Row No. 6 gives the probable errors resulting from the residuals between Schmidt's computed components and those of Fritsche. By glancing over the *SF* residuals in Tables *A*, *B*, *C*, it will be seen that the results of the two computations often diverge more from each other than the results of either differs from the chart values, taken as standard of comparison. The average probable error of the total force resulting from these residuals is ± 73 , hence, about the same as that of Schmidt's for 1885, and Fritsche's for 1842.5.

If we take as the computed components, the mean values of Schmidt's and Fritsche's computations, and compare the quantities thus derived with the chart values, then the residuals and probable errors will be materially reduced, as the two computations frequently diverge from the chart in opposite directions. See Part III, Table *D*, and row 7 of Table *F*. We now have probable errors about ten times those of field observations with modern instruments, and which probably represents the limit of accuracy attainable by a mathematical analysis based on general magnetic charts.

It is extremely unfortunate that we can not complete Table *F*, by including the probable errors resulting from the analyses of Adams for 1842.5, and for 1880. It is curious to note, by turning back to section *a*, Table V, that the value of the Earth's magnetic energy, as obtained from Adams' analysis for 1880, based upon Creak's charts, falls about midway between the values obtained

from the Fritsche and Schmidt's analyses. The mean value of the principal coefficient, g_{10} is 0.31684 as derived from the Fritsche-Schmidt analyses, or precisely equal to that of Adams's for 1880. So that the impression is awakened that Adams' computed components might be found to lie between those of Fritsche and Schmidt, and hence that the probable errors for Adams' analysis might be on the order of those in row 7, of Table *F*, or in any event, be smaller than either those of Fritsche or Schmidt. I trust to be able to look into this matter more fully at some future date. Adams did not compute the components.

Looking at the probable errors in declination and in inclination (rows 3 and 4 of Table *F*), for 1842 and 1885, we again note that there has been no improvement for the later date, although Fritsche based his later analysis upon twice the number of points used in the earlier one.¹

Having now obtained a good idea of the magnitude of the residuals to be expected between the computed quantities and the values based upon general, or terrestrial magnetic charts, let us next make a comparison of the declinations derived from the Schmidt analysis for 1885, and those scaled from the recently published isogonic chart of the United States for 1902. On this chart the isogonic lines, while still to some extent generalized, were drawn to conform as nearly as possible, with the large mass of data utilized. Restricted local disturbances, could of course, not be represented, however, areal and regional disturbances manifest themselves in manifold sinuosities in the curves. The scaled values for 1902.0 were referred to 1885.0, with the aid of fairly well-known secular changes.

Table *H* gives the results of the comparison. The residuals range from 2' to 3°03'; the average residual, regardless of sign is 86' from which the probable error of $\pm 72'$ is derived. Table *I* gives a comparison of the Schmidt and Fritsche residuals, at eighteen points in the United States. The *F* residuals are in general smaller than the *S* ones, and of opposite sign, except for latitude 50° N., where the *F* residuals are very large.

Tables *G*, *H*, *I*, show sufficiently, that in refined work, as, for

¹Judging from a remark attributed to Professor Schuster, when my paper on the Earth's total magnetic energy, was presented before the recent meeting of the British Association, he, apparently, has likewise made the discovery that the uncertainty in the potential computation has not been decreased for 1885; he is said to have found the uncertainty to be less for 1842.

example, secular variation investigations, we can not, as has been proposed, make much use of the results obtained from a mathematical analysis. I shall have occasion to enter into this more fully in No. IV of these contributions.

TABLE H. Comparison of computed declinations according to Schmidt, and observed declinations in the United States for 1885.

λ W. of Gr.	30° N.			35° N.			40° N.			45° N.			50° N.			λ W. of Gr.
	D o	D c	D-D o c	D o	D c	D-D o c	D o	D c	D-D o c	D o	D c	D-D o c	D o	D c	D-D o c	
65	o /	o /	o /	o /	o /	o /	o /	o /	o /	o /	o /	o /	o /	o /	o /	65
70																70
75																75
80																80
85	+ 1 25	+ 2 24	- 0 59	- 3 20	- 1 58	- 1 22	- 10 50	- 8 28	- 2 22	- 20 00	- 17 39	- 2 21				85
90	+ 4 05	+ 4 23	- 0 18	- 0 10	+ 0 51	- 1 01	- 6 35	- 4 47	- 1 48	- 15 50	- 13 19	- 2 31				90
95	+ 6 05	+ 6 03	+ 0 02	+ 6 15	+ 5 34	+ 0 41	- 2 05	- 1 15	- 0 50	- 11 30	- 8 45	- 2 45				95
100	+ 7 50	+ 7 25	+ 0 25	+ 8 20	+ 7 30	+ 0 50	+ 2 20	+ 2 02	+ 0 18	- 4 30	- 4 12	- 0 18				100
105	+ 10 00	+ 8 33	+ 1 27	+ 11 00	+ 9 09	+ 1 51	+ 5 35	+ 5 01	+ 0 34	+ 0 35	+ 0 12	+ 0 23	- 1 20	- 2 35	+ 1 15	105
110	+ 11 00	+ 9 29	+ 1 31	+ 12 50	+ 10 35	+ 2 15	+ 9 45	+ 7 41	+ 2 04	+ 10 20	+ 8 00	+ 2 20	+ 6 15	+ 3 12	+ 3 03	110
115	+ 12 10	+ 10 15	+ 1 55	+ 14 00	+ 11 49	+ 2 11	+ 12 30	+ 12 05	+ 2 25	+ 13 40	+ 11 17	+ 2 23	+ 11 30	+ 8 28	+ 3 02	115
120	+ 11 45	+ 10 54	+ 0 51	+ 14 30	+ 12 53	+ 1 37	+ 15 55	+ 13 52	+ 2 03	+ 16 05	+ 14 07	+ 1 58	+ 15 40	+ 13 04	+ 2 36	120
125	+ 12 20	+ 11 28	+ 0 52	+ 15 20	+ 13 48	+ 1 32	+ 16 55	+ 15 23	+ 1 27	+ 16 25	+ 14 29	+ 1 01	+ 16 54	+ 14 07	+ 2 47	125
							+ 17 45	+ 17 37	+ 0 08	+ 21 05	+ 21 09	+ 0 04	+ 24 10	+ 25 13	+ 1 03	

TABLE I. Comparison of declination residuals resulting from subtracting computed declinations (Schmidt's and Fritzsche's) from observed values in the United States for 1885.

λ W. of Gr.	30° N.		35° N.		40° N.		45° N.		50°	
	S	F	S	F	S	F	S	F	S	F
o	o /	o /	o /	o /	o /	o /	o /	o /	o /	o /
75			- 1 22	- 0 01	- 1 48	- 0 10	- 2 45	- 0 45	+ 3 02	- 5 56
90	+ 0 02	- 0 29	+ 0 41	- 0 08	+ 0 34	- 0 31	+ 1 33	+ 0 11	+ 1 54	+ 17 13
105	+ 1 31	- 0 30	+ 2 15	- 0 02	+ 2 25	- 0 07	+ 1 58	- 0 49	- 1 03	+ 4 12
120	+ 0 52	- 0 50	+ 1 32	- 0 04	+ 0 17	- 1 08	+ 0 08	- 1 09		

The general conclusion to be drawn from the foregoing paragraphs, emphasizes once more,—as in section *a*—the absolute necessity of a new mathematical analysis, dependent upon a critical discussion and reduction of strictly homogeneous material.

TWO POINTS OF THEORETICAL INTEREST.

There may be two directions in which the Gaussian theory¹ will warrant modifications, in addition to those introduced by Schmidt.

First. The principal terms in the Gaussian analysis, arise from sufficiently deep-seated systems of forces, so that the magnetic per-

¹What is meant by the Gaussian theory, and what physical facts it implies, are set forth in my article, *T. M.* Vol. V, p. 85.

meability of the medium or material intervening between the seat of the systems and the points of observation on the Earth's surface, may have to be taken into account if further refinement in the analysis is to be attempted. We shall then have the generalized Laplacian equation :

$$\frac{\partial}{\partial x} \left(\mu \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y} \left(\mu \frac{\partial V}{\partial y} \right) + \frac{\partial}{\partial z} \left(\mu \frac{\partial V}{\partial z} \right) = 0$$

in which μ is the magnetic permeability ; it is no longer a constant as for the case treated in section *a*.

The magnetic distribution will now not be entirely harmonic.

Second. There is a possibility that there may be a minor system of forces so near the surface, that we can not any longer treat the points of observation on the Earth's surface, as though they were entirely outside the system of effective forces, so that the strict fulfillment of even the ordinary Laplacian equation—the fundamental property of a harmonic distribution—may not take place.

The detailed examination of these points must be reserved for the future.