185. Note on a Point in the Demonstration of the Binomial Theorem

Author(s): V. Ramaswami Aiyar

Source: *The Mathematical Gazette*, Vol. 3, No. 55 (Jan., 1906), pp. 276-277

Published by: The Mathematical Association

Stable URL: <http://www.jstor.org/stable/3602087>

Accessed: 07-06-2016 21:01 UTC

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at

<http://about.jstor.org/terms>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

The Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*

Since $(1+px)/(1+qx)$ is greater than 1, it now follows that *

$$0 < (1+x)^{\frac{1}{n}} - (1+px)/(1+qx) < y/n;$$

so that the error in the approximation (for positive values of x) is less than

$$\frac{1}{12}(n^2-1)x^3/n^3,$$

provided n has any rational value greater than 1.

Mr. Jackson gave this result as an approximation to the error, but without noticing that the estimated error was really in excess of the true error, for positive values of x .

In the special cases $n=2, 3$ (which are those of chief interest, perhaps), it is easy to verify by direct multiplication that

$$\begin{aligned} (1+x)(1+\frac{1}{4}x)^2 - (1+\frac{3}{4}x)^2 &= \frac{1}{16}x^3 < \frac{1}{16}x^3(1+\frac{1}{4}x)^2, \\ (1+x)(1+\frac{1}{3}x)^3 - (1+\frac{2}{3}x)^3 &= \frac{1}{27}(2x^3+x^4) < \frac{1}{27}x^3(1+\frac{1}{3}x)^3, \end{aligned}$$

leading to

$$\begin{aligned} (1+x)^{\frac{1}{2}} - (1+\frac{3}{4}x)/(1+\frac{1}{4}x) &< \frac{1}{32}x^3, \\ (1+x)^{\frac{1}{3}} - (1+\frac{2}{3}x)/(1+\frac{1}{3}x) &< \frac{1}{81}x^3. \end{aligned}$$

These are the same results as were found above by the aid of the calculus; but to extend the algebraic method to any integral value of n seems rather laborious, although some of the transformations are interesting.

It may be worth while to recall the fact that Dedekind makes use of this approximation (for $n=2$) in his theory of irrational numbers.

T. J. F'a. BROMWICH.

185. [A. 1. c.] *Note on a point in the demonstration of the Binomial Theorem.*

When x is numerically less than 1, the series $1+mx+\frac{m(m-1)}{1 \cdot 2}x^2+\dots$ is absolutely convergent for all values of m , and denoting it by $f(m)$ we establish the addition theorem that $f(m) \cdot f(n) = f(m+n)$.

The Binomial Theorem is, that $f(m)$ is, for all values of m , equal to $(1+x)^m$; that is, to the positive quantity which, under the theory of Indices, is represented by the expression $(1+x)^m$.

Usually, the theorem is first shown to be true for positive integral values of m , and then it is considered for positive fractional values, and finally for negative values of m .

In dealing with the case when m is a positive fraction $=\frac{p}{q}$, what is usually done is, after proving that the q th power of $f(\frac{p}{q})$ is equal to $(1+x)^p$, it is quietly assumed that $f(\frac{p}{q}) = (1+x)^{\frac{p}{q}}$. But if q be an even integer, might not $f(\frac{p}{q})$ be rather equal to $-(1+x)^{\frac{p}{q}}$?

This difficulty (so far as I know) is raised and discussed only in *Chrystal*; but the discussion there is by no means elementary. The point, however, admits of an elementary treatment, by taking the cases as follows.

CASE I. If m is a positive integer, we have $f(m) = \{f(1)\}^m = (1+x)^m$, and there is no ambiguity.

CASE II. If m is a negative integer, $=-n$ say, we have

$$f(-n) \cdot f(n) = f(0) = 1;$$

* If a, b, n are all greater than 1, $a^n - b^n > n(a-b)$. This is obvious on division if n is an integer; if n is not an integer we have

$$\frac{d}{dx}(x^n - 1) = nx^{n-1} > n$$

if $x > 1$. Thus $x^n - 1 > n(x-1)$. Write $x=a/b$, and the result follows.

$$\begin{aligned}\therefore f(m) &= f(-n) \\ &= \frac{1}{f(n)} = \frac{1}{(1+x)^n}, \text{ by Case I,} \\ &= (1+x)^{-n}, \text{ by the theory of Indices,} \\ &= (1+x)^m.\end{aligned}$$

CASE III. Let m be a negative quantity numerically less than unity, $= -m'$ say.

$$\begin{aligned}\text{Then } f(m) &= 1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \dots \\ &= 1 - m'x + \frac{m'(m'+1)}{1.2}x^2 - \frac{m'(m'+1)(m'+2)}{1.2.3}x^3 + \dots\end{aligned}$$

If x be negative, each term in the series is positive, and therefore $f(m)$ is positive. If x be positive, the terms are alternately positive and negative, and each term is numerically greater than the preceding term. Hence in this case also the value of $f(m)$ is positive. Thus, whether x be positive or negative, $f(m)$ is, *a priori*, seen to be a positive quantity.

Let now $m = -p/q$ where p and q are positive integers, $p < q$.

$$\begin{aligned}\text{Then } \{f(m)\}^q &= f(mq) = f(-p) \\ &= (1+x)^{-p}, \text{ by Case II. ;}\end{aligned}$$

$$\therefore f(m) = (1+x)^{-\frac{p}{q}}, \text{ that is, the positive value denoted by this expression,} \\ = (1+x)^m.$$

CASE IV. Lastly, let m be a quantity not falling under any of the above cases. Then we can put $m = n - m'$ where n is a positive or negative integer, and m' is positive and less than 1. Accordingly,

$$\begin{aligned}f(m) &= f(n - m') = f(n) \cdot f(-m') \\ &= (1+x)^n \cdot (1+x)^{-m'}, \text{ by Cases I. or II., and III.,} \\ &= (1+x)^{n-m'}, \text{ by the theory of Indices,} \\ &= (1+x)^m.\end{aligned}$$

V. RAMASWAMI AYYAR.

186. [C. 1. e. a.] Indeterminate Forms.

A geometrical illustration.

Let $y = F(x)$ and $y = f(x)$ be the equations of two curves, each intersecting the axis of x at the point $x = a$.