XI.—*Researches on Malfatti's Problem.* By H. F. TALBOT, Esq.

(Read 20th March 1865.)

The problem which bears the name of the Italian geometer MALFATTI, by whom it was first proposed and solved, has long attracted the attention and exercised the ingenuity of mathematicians, and has been made the subject of many careful and elaborate researches.

The great attention which has been bestowed upon this problem has arisen partly from its intrinsic difficulty, but chiefly on account of the extreme simplicity of the solution finally obtained by MALFATTI, which seemed to open new views of geometrical research, and gave reason to hope that simple solutions might in like manner be found of many other geometrical problems usually accounted very difficult or insoluble.

The problem of MALFATTI offers another singularity. Although it is a question of elementary geometry which can be solved by a simple and elegant geometrical construction, yet no *geometrical* proof has ever been given, as far as I am aware, of the truth of this construction. It has been established hitherto only by a very elaborate use of algebraic analysis, in the course of which, however indisputable the result may be, all *geometrical* perception of its truth is lost. And yet there can be little doubt, it should seem, that a *geometrical* reason must exist for any simple series of facts belonging to elementary geometry.

The necessity of calling in the aid of analysis can only arise from the true connection of the geometrical principles involved in the problem being imperfectly understood.

I now offer to the Royal Society a purely geometrical solution of the problem ; and, for the sake of clearness, I have divided it into several parts, which I have called Lemmas. Some of these are well deserving of attention for their own sake, and irrespective of MALFATTI's problem. When these theorems have been established, their combination affords a lucid proof of the truth of the solutions which mathematicians have hitherto only obtained by the help of analysis.

History of the Problem.

In the year 1803, a distinguished Italian geometer, Signor MALFATTI, proposed the following problem in the Memoirs of the Italian Society of Sciences, vol. x. part 1 :—*

* See GERGONNE, vol. i. p. 347.

“ In a given triangle to inscribe three circles touching each other, and each of them touching two sides of the triangle.”

He gave at the same time a remarkably simple geometrical solution which he had discovered, but unaccompanied by any *geometrical* demonstration of its truth. He contented himself with showing that, if we calculate the algebraic values of the three radii which result from the above-mentioned geometrical construction, these three values, when substituted in the analytical equations deduced from the original conditions of the problem, do in fact satisfy them, and are therefore demonstrated to be true. But he gives no indication of any process of reasoning by which he arrived at the knowledge of these values.

In the year 1810, GERGONNE proposed this problem for solution in the “*Annales des Mathématiques*,” vol. i. p. 196, without knowing that it had been previously solved by MALFATTI; and no solution of it being sent to him by his correspondents, he took up the inquiry himself.* He makes the following preliminary statement:—

“ Il y a plus de 10 ans que ce difficile problème s'est offert pour la première fois aux rédacteurs de ce recueil, mais bien qu'ils l'aient attaqué un grand nombre de fois ils n'ont pu pendant longtemps parvenir à le résoudre ni même à s'assurer s'il était résolvable par la ligne droite et le cercle.

“ Ils ont cru devoir faire encore de nouvelles tentatives, et plus heureux cette fois que les précédentes ils sont parvenus sinon à trouver une construction du problème, du moins à l'abaisser au premier degré.”

Then follows an analytical investigation, which finally gives an algebraic value for the radius of any one of the circles in terms of known quantities. But this value does not lead to any *simple* geometrical construction, nor is it easy to show that it agrees with that previously found by MALFATTI, which, however, must necessarily be the case.

Having succeeded to this extent, M. GERGONNE did not believe the problem to be susceptible of much further simplification, when he first became acquainted with the previous researches of MALFATTI. Having procured and perused the memoir of that author, he found that it threw no light upon the point of chief interest, viz., the mode of investigation by which a result so unexpectedly simple had been obtained.†

Nothing further appears to have been done till the year 1820, when M. LECHMÜTZ, of Berlin, published a memoir in GERGONNE'S *Annales*, vol. x. p. 289, in which he succeeded for the first time in solving the problem, by a course of *à priori* reasoning. His investigation is algebraical, and his results coincide with those of MALFATTI. In the year 1826, STEINER, a distinguished geometer of Berlin, threw an entirely new light upon the problem, by giving, in CRELLE'S *Annals* (vol. i. p.

* See GERGONNE, vol. i. p. 343.

† Ibid. vol. ii. p. 60.

178), a geometrical construction of singular simplicity, but entirely unaccompanied with proof. His solution is as follows:—

Let ABC be the triangle in which it is required to inscribe three circles touching each other, and each of them touching two sides of the triangle. Bisect the angles A, B, C , by three lines AO, BO, CO , which will meet in the point O . In the three triangles AOB, BOC, COA , inscribe three circles, touching the sides of the given triangle in the points D, E, F , which letters will serve to denote those circles respectively. From the point of contact D , draw a line DG , touching the circle E on its inner side. Similarly from the point of contact E , draw a line EG touching the circle D , on its inner side. And let DG, EG , intersect in G . Then we have a trapezium $BDEG$, and STEINER affirms, *first*, that a circle can always be inscribed in this trapezium; and, *secondly*, that the circle so inscribed will be one of the three required circles.

But no doubt a difficulty will be observed immediately. STEINER directs that the line DG shall be tangent to the circle E ; and plainly there is no reason why the circle E should be selected rather than the circle F . But the reply to this is, according to STEINER, that if the line DG touches the circle E , it will also *necessarily* touch the circle F .

But of this most remarkable theorem he gives no demonstration whatever, although there is assuredly no theorem in the whole of geometry which has less claim to be considered as an axiom. Moreover, he affirms that the same line, DG , touches two of the required circles of the problem, at the point where they touch each other. This being admitted, the construction of the problem follows at once, as it is only requisite to describe a circle touching AB, BC , two sides of the given triangle, and also the known line DG , and this circle will be one of the three circles required, the others being found with equal facility.

STEINER'S solution, therefore, would have left nothing more to be wished for, if it had been accompanied with a demonstration. But of such his memoir contains not a single syllable. He says, indeed (page 178), that this solution of a difficult problem shows "the fruitfulness of the preceding theory;" but the critical researches of subsequent inquirers have not failed to discover the singular circumstance, that there is no connection whatever between this solution of MALFATTI'S problem and the theories set forth in the preceding part of STEINER'S memoir.

The great simplicity and elegance of this solution discovered by STEINER rendered a demonstration of it very desirable, which was at length accomplished by ZORNOW of Königsberg, in CRELLE'S Annals for 1833 (vol. x. p. 300). The demonstration of ZORNOW is remarkably elegant, but it chiefly depends upon some very dexterous algebraic transformations, in the course of which, however, all perception of a geometric proof of the construction necessarily disappears.

In the following year, PLÜCKER of Bonn resumed the subject in CRELLE'S "Annals," xi. p. 121. His memoir, which bears date October 1831, throws a great deal of new light upon the subject. His object was, like that of ZORNOW, to demonstrate the truth of STEINER'S construction, in which attempt he succeeds up to a certain point, by a well-conducted train of geometrical reasoning; but beyond that point he cannot proceed without the help of analysis. In fact, he shows geometrically that there exists a certain point O within any given triangle ABC (see former figure), which possesses the property, that if AO , BO , CO , are joined, and three circles are inscribed in the three triangles AOB , BOC , COA , as in that figure, then the line DG will touch both the circles E , F , and also two of the required circles. It remained to discover what point of the triangle the point O was, and to verify STEINER'S assertion that it was the centre of the inscribed circle, or that the lines AO , BO , CO , respectively bisect the three angles of the given triangle. But of this capital point PLÜCKER was unable to find any geometrical proof. He has recourse, therefore, to a very free and prolix use of trigonometry and algebra, through which I doubt whether any of his readers have had the courage to follow him, but which finally conducts him to the conclusion that STEINER'S assertion is true. It will be observed that PLÜCKER'S memoir, though published subsequently to that of ZORNOW, preceded it in point of date; that of ZORNOW being dated in October 1832. He was, therefore, the first who succeeded in demonstrating STEINER'S theorem. PLÜCKER concludes his memoir with the following remarks upon the mode in which STEINER has treated the question:—*

"The construction which I have given is essentially the same with that proposed by STEINER in vol. i. of this journal, p. 178. There is, however, at that place no indication of a demonstration. The introductory words of the author—*'to show the fruitfulness of the theorems set forth in paragraphs 1, 2, 3, by a suitable example, we add the geometrical solution, and also a greater generalisation of Malfatti's problem, omitting the proof,'*—might cause a person who (as I must confess to be my own case) has no idea how the construction of that problem can depend upon the well-known theorems explained in the above quoted paragraphs concerning points of similitude, &c., &c., to think that the given construction is not proved."†

I have said that PLÜCKER'S recourse to a difficult and very prolix analysis in order to justify the assertion that O is the centre of the inscribed circle, is the weak point of his able investigation. He has admitted this himself, for he says (p. 126), "So soon as this theorem is brought into its proper connection there

* Vol. xi. p. 126.

† Die einleitenden Worte des Verfassers; "*Um die Fruchtbarkeit,*" &c. &c., könnten demjenigen, der, wie ich von mir bekennen muss, keine Idee davon hat, wie die Construction, &c., &c. . . . den Gedanken aufdrängen, dass die gegebene Construction nicht bewiesen sei.

can be no doubt that an easier proof will be found of it." And he adds a wish for "a simple geometrical proof."

In the demonstration which I now submit, I shall follow PLÜCKER's geometrical proof in a general way, up to the point where he breaks away from geometry into the regions of analysis, and then give a proof, by geometry alone, of the remaining portion of the investigation.

Lemma 1.—If a circle is inscribed in a triangle, the difference of the sides equals the difference of the segments of the base.

Fig. 1.

This is evident. D, E, F, being the points of contact, $AB - AC = BE - CF = BD - DC$.

Lemma 2.—Let two circles touch each other at O, and let BFGC be their common external tangent, and AOD their common internal tangent. Let A be any point in the tangent AOD, and let AEB, AHC, be drawn touching the circles; then if a circle be inscribed in the triangle ABC, it will touch the base BC at D.

Fig. 2.

Demonstration.—We have manifestly the equal tangents $AE = AH$, $DF = DG$, $BE = BF$, and $CG = CH$. Therefore $AB - AC = BE - HC = BF - GC = BD - DC$.

Therefore by Lemma 1 the inscribed circle touches the base in D.

Lemma 3.—This is only another case of Lemma 2, when the point A is taken so near to O that the tangents AE, AH, diverge from the base BC, but their prolongations AB, AC, intersect the base at B, C.

In this case also we have the equal tangents $AE = AH$, $DF = DG$, $BE = BF$, and $CG = CH$. Therefore $AB - AC = BE - HC = BF - GC = BD - DC$.

Fig. 3.

Lemma 3 is the case which occurs in the solution of Malfatti's problem, but as the same demonstration applies to Lemma 2, I have given both of them.

Lemma 4.—If tangents of equal length are drawn to a circle, the locus of their extremities is a circle concentric to the first.

Lemma 5.—Let there be two circles A, B, and let RS and its equal $\acute{R}\acute{S}$ be their two common internal tangents; then if OP, OQ are two tangents drawn from

any point O not situated either in RS produced, or in $\hat{R}\hat{S}$ produced, $OQ - OP$ is not equal to RS .

Fig. 4.

Demonstration.—Let the points A, B , be the centres of the given circles. From centre A with radius AO describe a circle cutting RS produced in Z . Then by Lemma 4, $OQ = ZS$, whence $OQ - OP = ZS - OP$. But OP is not equal to ZR , because O lies on the circumference ZO , which is not concentric to the circle B . Therefore $OQ - OP$ is not equal to $ZS - ZR$; therefore it is not equal to RS . *Q. E. D.*

Corollary.—If $OQ - OP$ is equal to RS , O must either lie in RS produced or in $\hat{R}\hat{S}$ produced.

Lemma 6.—Let A, B, C , be three circles. Let DE be the internal common

Fig. 5.

tangent of A and B ; FG of B and C ; and HI of C and A . Then if these three tangents, when produced, meet in a single point O ,

$$DE = FG + IH$$

or the greatest common tangent equals the sum of the two others.

For,

$$IH = OI - OH = OD - OG = DE - FG.$$

Lemma 7.—Let A, B, C , be three circles. Let DE be the external tangent of A and B . Let FG be the external tangent of A and C ; and let HI be the internal tangent of B and C . Then if these three tangents, when produced, meet in a single point O ,

$$FG - DE = HI$$

For,

$$FG - DE = OG - OE = OI - OH = HI.$$

Lemma 8.—If three internal common tangents of three circles meet in a point O , their other three internal common tangents meet in another point P .

Demonstration.—In figure 5, suppose the common tangents meeting in O to be effaced, and replaced by the three other internal common tangents, it is required to show that these also meet in a point.

Let two of them, viz. those touching the circle A , meet in the point P . Denoting the new tangents by the same letters as before, but accentuated, we have of course

$$DE = \acute{D}\acute{E}, FG = \acute{F}\acute{G}, HI = \acute{H}\acute{I}$$

Now we proved in Lemma 6 that

$$DE = FG + HI, \text{ or } ED - HI = FG$$

therefore we have

$$\acute{E}\acute{D} - \acute{H}\acute{I} = \acute{F}\acute{G}$$

The new common tangents meeting in P will be $P\acute{E}\acute{D}$, $P\acute{H}\acute{I}$; but these have a part which is equal in each, namely $PE' = PH'$, which are tangents to the same circle A . Therefore subtracting this part, we have

$$P\acute{E}\acute{D} - P\acute{H}\acute{I} = \acute{E}\acute{D} - \acute{H}\acute{I}$$

which we proved to be equal to $\acute{F}\acute{G}$.

Therefore P is a point from which tangents PD' , PI' have been drawn to the circles B and C , and their difference has been found equal to $\acute{F}\acute{G}$ the internal common tangent of B and C . Therefore by the corollary to Lemma 5, P is a point in $\acute{F}\acute{G}$ produced. *Q. E. D.*

Lemma 9.—If two external and one internal common tangents of three circles meet in a point O , the three other *corresponding* tangents meet in another point P .

Fig. 6.

The demonstration of this is the same as the last, employing Lemma 7 instead of Lemma 6. These theorems may be called PLÜCKER's tangents, from the name of their discoverer (CRELLE's *Annals*, tom. xi.). It is evident that there are several more cases besides those considered in Lemmas 6, 7, 8, 9; but I omit them, because they are not required for the solution of MALFATTI's problem. The demonstration of each would be nearly in the same words.

Lemma 10.—If three lines issue from a point A, and contain the angles DAX, EAX, which may be called θ and ϕ ; and if two circles, with centres B and C, are

Fig. 7.

inscribed anywhere in these angles, touching the outer sides at D and E; then if DE is joined, the intercepted chords DF, GE are in a constant ratio; namely, in the ratio of $\tan \frac{1}{2} \theta$ to $\tan \frac{1}{2} \phi$.

Demonstration.—Join BD, and draw the perpendicular BH, dividing the chord DF into two equal parts. Draw AI perpendicular to DE.

Then, since the triangles ADI, BDH are similar,

$$AD : AI :: BD : DH$$

$$\therefore DH = AI \cdot \frac{BD}{AD} = AI \cdot \tan \frac{1}{2} \theta$$

$$\therefore \text{chord } DF = 2 AI \cdot \tan \frac{1}{2} \theta$$

By similar reasoning it may be shown that chord $GE = 2 AI \tan \frac{1}{2} \phi$

$$\therefore DF : GE :: \tan \frac{1}{2} \theta : \tan \frac{1}{2} \phi$$

Corollary 1.—If $\theta = \phi$, $DF = GE$. That is:—"If the circles subtend equal angles at A, the intercepted chords are equal."

Corollary 2.—If θ is greater than ϕ , then DF is greater than GE .

Lemma 11.—If any angle DAE is bisected by the line AX , and two circles are inscribed anywhere in the semi-angles DAX , EAX touching the sides at D and E ; then the tangent DK is equal to the tangent EL .

Demonstration.—Join DE . We have shown in Lemma 10 that in this case the

Fig. 8.

intercepted chords DF , GE are equal. Subtract them from the whole line DE , and the remainders DG , EF will be equal. Therefore

$$DG \cdot DE = EF \cdot ED \quad \therefore DK^2 = EL^2$$

and

$$\therefore DK = EL$$

Corollary.—Conversely, if $DK = EL$ it follows that the angle $DAX =$ angle EAX .

For, if those angles are not equal, let DAX be the greater. Then because the angle DAX is greater than the angle EAX , the chord DF is greater than the chord EG (by Lemma 10, corollary 2). Subtract them successively from the line DE , and the remainder EF will be smaller than the remainder DG , therefore $EF \cdot ED$ is less than $DG \cdot DE$; therefore EL^2 is less than DK^2 ; and EL is less than DK . But on the contrary $EL = DK$ by hypothesis. Consequently it is not true that the angle DAX is greater than the angle EAX . In the same way it is shown that it is not less; consequently it is equal to EAX . *Q.E.D.*

Lemmas 10 and 11 are particular cases of a much more general theorem which I propose to give on another occasion. In the first nine Lemmas I have chiefly followed PLÜCKER, but have endeavoured to make his demonstrations more rigorous by going more into detail than he has done. But Lemmas 10 and 11,

and the more general theorem of which they are particular cases, are original; at least I am not aware of their having been published elsewhere. They supply the link that was missing in PLÜCKER'S investigation, and singularly facilitate the demonstration of STEINER'S elegant construction.

By the help of the preceding Lemmas, we can show the truth of that construction in the following manner:—

Malfatti's Problem.

It is required to inscribe in the triangle ABC three circles touching each other, and each of them touching two sides of the triangle.

Solution.—Suppose the thing done, and the three circles A' , B' , C' found, it is plain that their three internal common tangents meet in a point N , and bisect the external tangents GH , IJ , KL .

Produce the lines EN , FN beyond the point N until they meet the base BC in two points Q , R . Then if in the triangle QRN so formed, we in-

scribe a circle which may be* called α , it will touch the base BC in the point D (by Lemma 3).

By an exactly similar process we obtain a circle β , touching DN , EN produced, and the side AC in F ; and a circle γ touching DN , FN produced, and the side AB in E .

Fig. 9.

Fig. 10.

* I have called it α , because it stands opposite to the angle A of the original triangle. Similarly for the names of β and γ . I have called α , β , γ the *secondary circles*.

This will be seen better by referring to fig. 10, in which, to avoid confusion, I have only represented one of the required circles B' , and the two secondary circles α , γ , which belong to it. I have represented the prolongations of the three tangents DN , EN , FN by dotted lines. The secondary circles touch these dotted lines, and also touch the sides of the triangle at D and E , where the tangents intersect them. A simple inspection of the figure suffices to show that the tangent DY drawn from D , which touches both the circles B' and γ , is the sum of two parts, which equal the external tangents DG and EL respectively. And that the tangent EZ , drawn from E , which touches both the circles B' and α , is the sum of two parts, which equal the external tangents EL and DG respectively. Therefore, these two tangents DY , EZ are equal to each other, since each of them equals $DG + EL$.

But the three lines DN , EN , FN meet in one point at N ; that is, the three internal common tangents of the circles α , β , γ meet in one point. Therefore, by Lemma 8, their *other* three common tangents must also meet in one point, which point may be called O .

Moreover, the line DN produced is the external tangent of the circles B' and γ ; and EN produced is the external tangent of the same circle B' and circle α ; and (as we said before), FN produced is the *internal* tangent of the circles α and γ . But DN , EN , FN concur in a point; that is, two external and one internal common tangents of the circles α , γ , and B' concur in a point. Therefore, by Lemma 9 the other three *corresponding* tangents of those circles meet in a point. And it is easy to see what that point is. For, the *second* external common tangent of the circles B' and γ , is AB , one of the sides of the given triangle; and the *second* external common tangent of the circles B' and α is BC , one of the sides of the triangle.

But we know the point of concurrence of AB and BC , to be at B , one of the vertices of the triangle. Consequently, we attain this important result, that the second common internal tangent of the circles α and γ passes through the angular point B of the triangle. In a similar way, it may be shown that the second common internal tangent of the circles α and β passes through the angle C ; and that the second common internal tangent of the circles β and γ passes through the angle A .

These three tangents are therefore three lines proceeding from A , B , C , the three angles of the given triangle, and meeting *in a single point* (which a few lines previously we named O). Now STEINER affirms that this point O is the centre of the inscribed circle of the given triangle ABC ; and this is the theorem which PLÜCKER and other geometers have been unable to prove except by the use of analytical methods of investigation. But here we call to our assistance the Lemma 11, which we have demonstrated above; and we proceed as follows. Since we have shown that the line BO touches the circle α , and also the circle γ . And since we have also shown that the line DY drawn from the point of contact

D to the circle γ , is equal to the line EZ drawn from the point of contact E to touch the circle α . Therefore by the corollary to Lemma 11, the line BO necessarily bisects the angle B of the original triangle. Similarly it is shown that CO bisects the angle C, and that AO bisects the angle A. Therefore, O is the centre of the inscribed circle of the triangle ABC:—which is STEINER'S Theorem. The solution of MALFATTI'S Problem is therefore as follows:—

Bisect the angles of the triangle ABC, by the lines AO, BO, CO. In two of the smaller triangles thus made AOB, BOC, inscribe the circles γ and α . From D, the point of contact of α with the side BC, draw a line DY, touching the circle γ . Then DY will touch one of the required circles also; which circle also touches AB, BC, two sides of the triangle, and is therefore wholly determined.