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XIII. *On the Theory of the Figure of the Planets contained in the Third Book of the Mécanique Céleste. By J. IVORY, Esq. M.A. F.R.S.*

[Concluded from p. 37.]

II. **T**HERE can be no other apology for the observations which I have made on the analysis of Laplace, except that they are true and rigorously proved. And as this is the best apology that can be made, so I am not aware that any other is necessary. To avoid as much as possible all objection and cavil, I have employed in the proof the author's own mode of investigation. Speculations of this kind are at present entirely out of fashion, or rather they are discouraged and undervalued as much as possible. They seem even to be excluded from what is popularly called the inductive philosophy, forgetting that they form a part of the noblest and most successful induction that, we may venture to predict, will ever do honour to the human intellect. What has occupied the attention of Maclaurin and Simpson, of D'Alembert, Lagrange and Laplace, is now utterly condemned as useless, with a degree of levity that will not easily be believed. But the exclusive spirit which reigns so powerfully at present, whether it proceeds from particular interests or from narrow views of science, will at length spend its force; and the discussion I have undertaken may then contribute to place an important branch of the philosophy of Newton on a solid foundation. It follows from what has been shown, that the method of Laplace, when freed from series without convergency, and reduced to what is strictly demonstrative, is confined to a class of spheroids first proposed by D'Alembert. We cannot, therefore, allow that the method is perfectly general, unless it were proved that the class of spheroids mentioned comprehends every case in which the conditions of equilibrium can possibly be fulfilled. But it is greatly to be wished that so important a part of the system of the world as the theory of the figure of the planets, were deduced from

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sure principles, by a process of reasoning not depending upon any dubious or intricate point of analysis.

In treating of the figure of the planets there are three different cases that principally engage attention. We may consider the equilibrium of a fluid mass that is homogeneous; or of one composed of strata varying in density according to any law; or we may suppose a solid nucleus wholly or partially covered with a fluid. Although the principles on which I proceed are equally applicable in every case, yet for the sake of brevity and simplicity I here confine myself to the first case only; namely, the equilibrium of a homogeneous mass of fluid. Such is the intimate connexion which binds together the different parts of the same theory, that if we can fairly overcome the difficulties which obstruct our progress in one case, every other case will readily be brought within our power.

Now the principles by which Laplace has determined the equilibrium of a homogeneous fluid mass are these two: first, the direction of gravity must be every where perpendicular to the outer surface; secondly, the radius r of the spheroid must come under this formula, viz. $r = a(1 + \alpha y)$, y being a function of the angles which determine the position of the radius, and α a small coefficient of which the square and higher powers are to be neglected. In the theory of Laplace, the equilibrium of a fluid of uniform density is a necessary consequence of the two conditions mentioned. Of these the first is entirely mathematical; the only purpose it can serve is to allow the rejecting of certain quantities which would otherwise embarrass calculation; but it can in no respect contribute to make out the proof of the equilibrium, which must be deduced from hydrostatical principles alone. Whether there be an equilibrium or not must depend entirely on the first condition. If that is sufficient, Laplace's solution will be exact; otherwise we must conclude that it is defective, and we can consider it only as a method of calculation which accidentally leads to a result that we know to be true from other considerations.

We have now then to inquire what are the conditions necessary to the equilibrium of a homogeneous fluid. The whole received doctrine on this head is contained in the single proposition following. Conceive a homogeneous fluid contained within a continuous surface, and let x, y, z denote the rectangular co-ordinates of a point in the surface drawn to three planes intersecting in the centre of gravity of the mass; then, ϕ denoting a function of x, y, z , if the equation of the outer surface be

$$\phi = C;$$

the fluid will be *in equilibrio*, if every molecule, whether situated

tuated in the surface or any where in the interior of the fluid, is urged by the forces $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, $\frac{d\phi}{dz}$ in the direction of the co-ordinates and tending to shorten them, it being always understood that the co-ordinates of the molecule are to be substituted in the expressions of the forces.

In order to demonstrate this proposition take the differential of the equation of the surface ; then

$$\frac{d\phi}{dx} dx + \frac{d\phi}{dy} dy + \frac{d\phi}{dz} dz = 0 :$$

now $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, $\frac{d\phi}{dz}$ are the forces which act upon a molecule in the surface at the point of which x, y, z are the co-ordinates ; and if we put

$$p = \sqrt{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2},$$

then p , which is the resultant of the partial forces, will represent the gravity at the exterior surface ; and it is easy to deduce from the differential equation that the direction of p will be perpendicular to that surface. Suppose that the constant quantity C in the equation of the fluid's surface decreases by a small variation, then

$$\phi = C - \delta C,$$

which will be the equation of a surface in the interior of the mass indefinitely near the outer surface : and since the forces which urge every molecule of the fluid are expressed by the same functions of the co-ordinates of the molecule, it follows that the resultant of the forces acting at any point of this new surface will be perpendicular to it, for the same reason that the like resultant is perpendicular to the outer surface. And as we may conceive that C decreases by indefinitely small gradations till it is entirely exhausted, it is evident that the whole fluid mass may be supposed to be divided into thin strata separated from one another by surfaces perpendicular to the forces which urge the molecules contained in them. Clairaut has called such surfaces *Couches de niveau*, or level surfaces, from the property which they possess in common of being perpendicular to the direction of gravity. Again, let k denote the perpendicular distance between the outer surface of the fluid and the level surface immediately below it ; and let x, y, z be the co-ordinates of the extremity of k in the first surface, and $x - \delta x, y - \delta y, z - \delta z$ the co-ordinates of the other extremity in the other surface ; then if we substitute the respective co-ordinates in the equations

$$\begin{aligned} \phi &= C \\ \phi &= C - \delta C, \end{aligned}$$

and

and take the difference, we shall get

$$\frac{d\phi}{dx} \delta x + \frac{d\phi}{dy} \delta y + \frac{d\phi}{dz} \delta z = \delta C.$$

As before, put

$$p = \sqrt{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2},$$

and it will be easy to prove that the cosines of the angles which the lines δx , δy , δz make with k are respectively equal

$$\text{to } \frac{1}{p} \times \frac{d\phi}{dx}, \frac{1}{p} \times \frac{d\phi}{dy}, \frac{1}{p} \times \frac{d\phi}{dz};$$

hence

$$\delta x = \frac{k}{p} \times \frac{d\phi}{dx}, \delta y = \frac{k}{p} \times \frac{d\phi}{dy}, \delta z = \frac{k}{p} \times \frac{d\phi}{dz};$$

wherefore, by substituting these values in the foregoing expression, we shall obtain

$$\frac{k}{p} \times \left\{ \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2 \right\} = k \times \frac{p^2}{p} = kp = \delta C.$$

Suppose, further, that the level surface is divided into equal elementary parts, one of which is ds ; then $k \times ds$ will represent the small mass of fluid contained between the two surfaces upon the base ds ; and, p being the resultant of the forces that urge every molecule of the small mass, $k \times p \times ds = ds \times \delta C$ will represent its pressure upon the level surface: wherefore, since δC and ds are the same at every point of the surface, it follows that the pressure will be equal upon all equal spaces of the level surface. In the very same manner it may be shown that the second stratum presses equally upon the level surface below it; and the same thing is manifestly true of all the successive strata in order. But as the number of strata increases, the fluid within them continually decreases in quantity; ultimately therefore it will be reduced to a particle which exerts no force, and which, being pressed equally on all sides, cannot fail to be *in equilibrio*. Remounting now from this central drop to the exterior surface, it is evident that the whole mass of fluid, and every part of it bounded by a level surface, will be separately *in equilibrio*.

In the foregoing demonstration we have followed the ideas of Clairaut in his excellent work on the figure of the earth*. It appears from the reasoning that all the conditions of equilibrium are contained in the equation of the external surface, viz.

$$\phi = C,$$

nothing more being necessary than that ϕ be a function of

* *Figure de la Terre*, prem. partie, § xxi.

three independent co-ordinates. Instead of the equation itself we may substitute its fluxion, viz.

$$\frac{d\phi}{dx} dx + \frac{d\phi}{dy} dy + \frac{d\phi}{dz} dz = 0;$$

which again amounts to affirming that the resultant of the forces $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, $\frac{d\phi}{dz}$ urging a molecule in the external surface must be perpendicular to that surface.

From what has now been shown, it appears that Laplace in his investigation has strictly adhered to the received theory of fluids. But the algebraic calculus, in its generalizations, is apt to overlook distinctions, the neglect of which sometimes leads to error and inconclusive reasoning. It never can be too often repeated, that analysis is merely an engine of investigation, although a very powerful one. All its force and all its beauty are derived, as in the ancient geometry, from the certainty of the principles on which it proceeds, and from the clearness with which it traces their consequences. The matter we are considering will furnish an example of a theory which is correct in general, but which becomes defective when applied in circumstances where a modification is necessary. In the foregoing investigation p is the gravity at the level surface immediately below the external surface; or rather, it is the gravity which, according to the principles of the differential calculus, is supposed to remain without change from the one surface to the other. The force p therefore depends entirely on the level surface and the matter within it; $p \times k \times ds$ is the pressure which this force causes by its action on an elementary part of the superincumbent stratum. In the investigation of Clairaut the pressure mentioned is supposed to be the only force which the stratum exerts on the fluid below it; and if we allow that it is equable, and likewise that the whole mass is *in equilibrio*, it will follow that the part bounded by the level surface will be *in equilibrio* separately, if the stratum above it were taken away or annihilated. But, in the case of a planet, there is another force that must be taken into account, besides the pressure of the stratum caused by the gravitation at the level surface: it is the attraction of the stratum itself upon all the particles within it. The existence of this force is a consequence of the law universally prevailing in nature, that every particle of matter attracts every other particle. Thus, in the case of a planet, the stratum is made to press upon the fluid below it by two different forces,—by the gravitation at the level surface, and by the attraction which its own matter exerts upon the particles within it. Both these pressures

pressures must be equable over the whole level surface, otherwise Clairaut's reasoning will not apply; and as they are produced by independent causes, they cannot possibly be included in one and the same equation. In the case of a planet it therefore becomes necessary to add to Clairaut's theory a new condition, which can only be derived from the figure of the stratum. If we suppose that the stratum is possessed of such a figure as to attract every particle in the inside with equal force in opposite directions, it is manifest that the attraction of the stratum will not disturb the equilibrium of the fluid below it, and consequently that the pressure upon the level surface must be equable. Every level stratum being subjected to this new condition at the same time that the usual equation of the level surface is retained, we may extend Clairaut's demonstration to the case of a planet; and as the two conditions are sufficient, so we must infer that they are necessary, for the equilibrium. Thus are we led to the true conditions requisite to the equilibrium of a homogeneous mass of fluid consisting of particles that mutually attract one another, which are these two; viz. 1st, The resultant of the forces acting at every point of the external surface must be perpendicular to that surface; 2dly, A stratum of the fluid contained between two level surfaces must attract every particle in the inside with equal forces in opposite directions.

It follows from what has been proved, that the solution of Laplace is defective, because one of the conditions that must be attended to in the case of a planet is omitted. We have no direct evidence that the figure brought out is one of equilibrium. Whether the result be exact or not, is accidental, and can be known in no other way than by a comparison with other solutions derived from unexceptionable principles.

We can now assign the reason why Maclaurin, and all those who supposed an elliptical spheroid, have succeeded in this investigation. That condition of equilibrium which has always been omitted in the hydrostatical theory, is contained in the assumed figure. In a homogeneous ellipsoid the level surfaces are similar to the outer surface; and the author of the *Principia* has proved that a hollow shell of homogeneous matter, contained between two similar elliptical surfaces, attracts a particle in the inside with equal force in opposite directions. Thus the most difficult and abstruse part of the investigation being contained in the very hypothesis assumed, the rest of the solution is readily completed by the more obvious principles of hydrostatics. It deserves to be mentioned that of the two requisite conditions, the second, or the one which has always been omitted, determines the kind of figure without

without which the equilibrium cannot subsist; the other ascertains the proportion of the forces necessary to make that figure possible,

We can now likewise discover, *à priori*, why Laplace's method, in the circumstances supposed, brings out a result which is exact as a first approximation. It arises from this,—that, when the fluid is a perfect sphere, the two conditions of equilibrium coincide in one; for Newton has proved that a spherical shell of homogeneous matter attracts a particle in the inside equally in opposite directions. What is exactly true in the case of a sphere, must be nearly so when the fluid approaches indefinitely to that figure; and in these circumstances, the second condition being implied to a certain extent in the first, we obtain an approximation by means of the latter alone. This, however, is true only of the first step in approximating to the figure of a planet; in proceeding further it becomes necessary to take in all the conditions of equilibrium.

I shall only add one more remark on the analysis of Laplace. The radius of the spheroid, drawn from the centre of gravity, is represented by a series of this form, viz.

$$r = a + aa \{Y^{(2)} + Y^{(3)} + Y^{(4)} + \&c.\},$$

the terms on the right-hand side being the development of the function y . This expression has nothing to do with the conditions of equilibrium; it belongs to every spheroid nearly spherical. Now the second of the conditions of the problem, or that one which determines the species of the figure necessary to the equilibrium, proves immediately that the radius can consist only of two terms, viz.

$$r = a + aa Y^{(2)}.$$

In the method of Laplace the superfluous terms are got rid of by a particular mode of reasoning, which does not cohere well with the rest of the investigation, and certainly hurts the unity of solution. The process, indeed, is not long in the case of a uniform density; but when the fluid is composed of strata of variable density, the reasoning is both long and complicated, and the result is at length brought out by dint of calculation*. There is therefore a great advantage in solving the problem, as it ought to be solved, by the true principles of the case, both because it is more satisfactory to the mind, and because it is more simple. The observation here made is of the greater importance as it extends to the theory of the tides, and to the other questions treated of in the *Mécanique Céleste*, where the figure of the planets is concerned, the same form of the radius being assumed in all these cases.

* Livre iii. § 29 & 30.

In a paper printed in the Philosophical Transactions for 1824 I have investigated the conditions of the equilibrium of a homogeneous mass of fluid, and have applied them to determine the figure which it will assume when it is urged by the attraction of its particles and a centrifugal force caused by a rotation about an axis passing through the centre of gravity. This is the first general solution of the problem that has been deduced from the principles of hydrostatics, without having recourse to approximations, and without introducing arbitrary suppositions. It must be allowed, however, that the investigation is in some respects not so simple as it might be made. But I propose to return to this subject, and in a particular work to treat the theory of the figure of the planets from its fundamental principles.

Feb. 3, 1826.

JAMES IVORY.

N. B. In the *Conn. des Tems.* for 1828, M. Puissant, in a note at p. 220, notices a mistake in naming an angle I fell into in the solution of a geodetical problem inserted in the Philosophical Magazine for July 1824, and which is corrected in the same Journal for April 1825. The least attention to the solution would have shown M. Puissant that the import of the angle, which I have inadvertently called the *true latitude*, is fixed by the assumed values of the co-ordinates. Hence it cannot possibly be what I have said it is; it is the *reduced latitude*, and can be nothing else. There certainly is a misnomer; but the accuracy of the solution is not affected by it, because the meaning of the angle is determined independently of the name given to it. M. Puissant uses these words, "*Il est aisé de voir que le calcul n'est pas fondé sur des considérations analytiques assez rigoureuses, puisque le théorème qui en découle n'est pas parfaitement exact.*" Now here M. Puissant is passing sentence without having examined the case. My solution is perfectly exact: it is deduced from the most rigorous principles of analysis; although, in enunciating the theorem in question, I have inadvertently given the name of the true latitude to an angle, which in the analysis stands for the reduced latitude, and can possibly stand for nothing else. In the pages of the same Journal I have made many observations relating to subjects treated of in the *Conn. des Tems.*; and although it would excite no surprise to find all these discussions passed by in silence in that work, yet it is rather remarkable that an inadvertence alone, substantially of no moment, and which has been corrected so long ago, is held up to public notice and carelessly misrepresented.

J. I.