the doublets also varying periodically. Each doublet would occupy a volume c/ω cube, and so would give interference effects. It is also seen that if both polarized rays are eliminated by crossed nicols no energy will get through, and it would be impossible to have "longitudinal radiation" by itself.

A source of radiation of frequency ω would consist of a number of rotating doublets. Energy would be dissipated in two ways:—(1) A doublet losing all its energy, as its periphery is moving with velocity c, would send out a pulse to infinity—if all the doublets were rotating in phase, the disruption of a number at a fairly steady rate would give the ordinary periodic radiation and interference effects; (2) a doublet owing to a "collision" would leave the cluster and travel as a whole in a straight line from the source until through another collision it gave up all its energy—its velocity would be less than c, but would probably give effects that would appear instantaneous. Accordingly, it seems that a source could be very weak and give interference effects, and yet be able to start the photoelectric effect in an inappreciable time.

LXXXV. *On Oscillation Hysteresis in a Triode Generator with Two Degrees of Freedom.* *By* Balth. van der Pol Jun., *D.Sc.**

When two oscillatory circuits are linked together by means of a magnetic, electrostatic, or resistance coupling, it is well known that the circuit combination possesses two degrees of freedom. If one of these oscillatory circuits is, moreover, a part of a triode generator, it is natural to ask whether the two modes of vibration can exist simultaneously or, if this is not the case, whether the one or the other mode of vibration will obtain for any particular conditions.

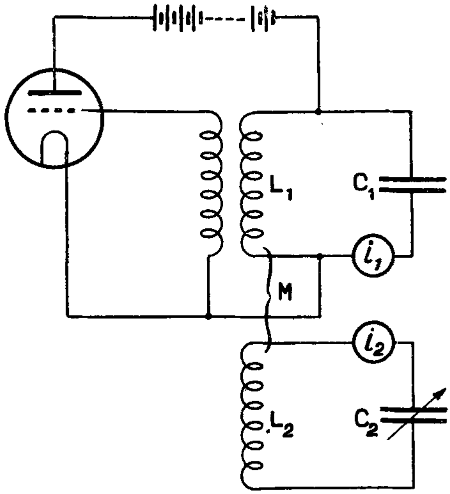
Now it is found experimentally that, when the system oscillates in one of the two modes of vibration and the natural frequency of the secondary circuit is varied gradually, the system suddenly jumps at a certain point from the first mode of vibration to the other. If afterwards the natural frequency of the secondary is varied in the reverse direction it is found that the system jumps from the second into the first mode of vibration, but at a point which is *not* identical with the first one mentioned above, and thus

* Communicated by Professor H. A. Lorentz, For.M.R.S.

a kind of oscillation hysteresis is obtained, which, apart from its importance in technical applications, is of interest from a physical point of view.

The normal experimental arrangement is shown in fig. 1, where an oscillatory circuit L_2C_2 is shown coupled through

Fig. 1.



the mutual induction M to the circuit L_1C_1 belonging to a normal triode generator. We may consider the indications of the thermal ammeters i_1 and i_2 when

(i.) C_2 is brought from a small value, through the point of resonance ($L_2C_2 = L_1C_1$)L, to a large value ; and

(ii.) when the value of C_2 is thereupon gradually decreased through resonance to the first small value.

The relations between i_1 and $\omega_2{}^2$ and i_2 and $\omega_2{}^2$ thus obtained are shown in figs. 2 and 3, where the arrows indicate the paths followed. Further, in these two diagrams the total system is found to vibrate for conditions represented by EFB in one of the two modes of vibration, *i. e.* with the higher one of the two coupling frequencies, while for conditions represented by DCA the system vibrates with the lower coupling frequency. Hence at the points B and A discontinuities occur in the modes of vibration, resulting in a discontinuity both in frequency and amplitudes of the currents. But, further, it is also seen from these graphs that the system has the tendency to go on oscillating as long as possible in the mode of vibration in which it is already oscillating, though the other mode of vibration is possible for the same parameters.

Fig. 2.

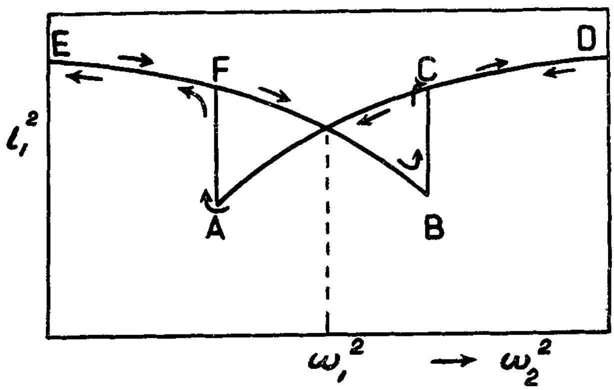
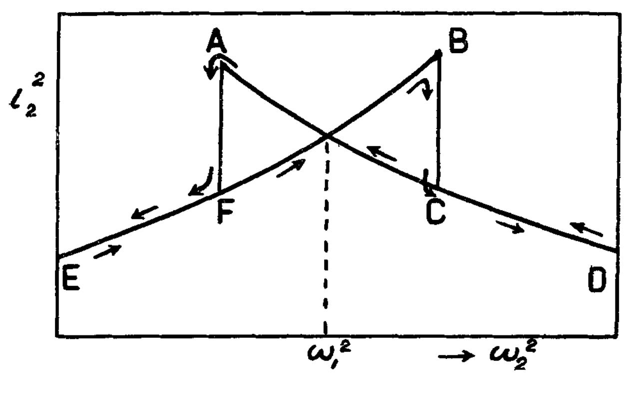


Fig. 3.



These phenomena were noticed by the author in February 1920, but it was felt that no satisfactory explanation could be given unless progress was first made in the development of a non-linear theory of sustained oscillations. For it is obvious that, when the problem is treated with linear differential equations, the principle of superposition is valid, and in this case oscillations in the one mode are uninfluenced by oscillations in the other. It is therefore somewhat surprising that up to the present, though several theoretical contributions to the problem have already appeared *, the phenomenon has, as far as we are aware, only been dealt with in a linear theory. The solutions of the differential equations in this case are of the form $e^{\pm \alpha t} \sin \omega t$, and it depends on the sign of α whether an oscillation will build up or decay. But whether both oscillations will be present simultaneously or whether the one mode of vibration will suddenly be replaced by the other when a parameter of the circuits is gradually varied, and whether a hysteresis loop will be obtained, these questions can *not* be answered by a linear theory. In order therefore to retain in the analysis the necessary interaction of simultaneous vibrations which determines the stability of the oscillations, non-linear terms which occur through the curvature of the triode characteristics may not be ignored.

Before attempting, however, to set out a non-linear theory of the phenomena under consideration, a few remarks may first be made concerning the terminology.

The notion of one or two degrees of freedom is used here as an extension of the usual meaning attached to these terms in the ordinary linear treatment of oscillation problems. We are well aware that, *e. g.* to speak of a system as having one degree of freedom when more than one stable oscillation is possible for a given set of parameters † is not altogether satisfactory, but it is hoped that from the description of the phenomena the meaning will be sufficiently clear.

Further, we shall discriminate between a "possible" vibration and a vibration that can actually be realized. With "possible" is here meant a solution representing a stationary oscillation with a constant amplitude. It may, however, be that this oscillation cannot be realized, it being unstable.

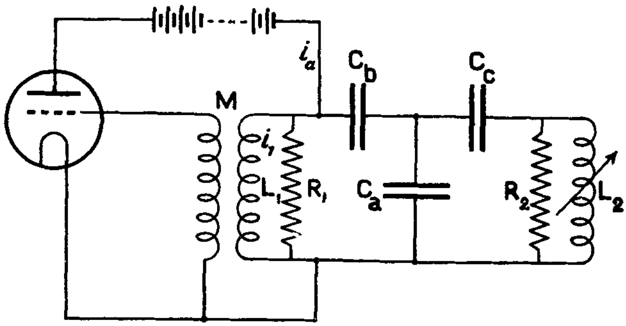
Finally, it is obvious that for a system such as shown in fig. 1, when the secondary circuit is very loosely coupled to the primary, the reaction of the secondary on the primary will be small. It is found experimentally that under these circumstances an ordinary resonance curve can be obtained as the secondary circuit. This case will, however, not be considered here and e shall confine our considerations to cases where the coupling is strong.

* J. S. Townsend, Radio Review, i. p. 369 (May 1920). K. Heegner, *Archiv für Elektrotechnik*, ix. p. 127 (1920). F. Harms, *Jahrbuch für drahtl. Telegraphie*, xv. p. 442 (1920). H. Vogel und M. Wien, *Ann. d. Phys.* lxii. p. 649 (1920). H. G Möller, *Jahrbuch für drahtl. Telegraphie*, xvi. p. 402 (1920). H. Pauli, *ibid.* xvii. p. 322 (1921). W. Rogowski, *Archiv für Elektrotechnik*, x. pp. 1, 15 (1921) See also Möller, *Die Elektronenröhre* (Vieweg, 1920).

† See, *e. g.* Appleton and Van der Pol, Phil. Mag. Jan. 1922.

In order to simplify the analysis we shall not treat the actual circuit shown in fig. 1, which, from an experimental point of view, is the simpler, but will replace the magnetic coupling by an equivalent capacity coupling. We shall also replace the series resistance of the oscillatory circuits by equivalent shunt resistances, and therefore deal with the circuit of fig. 4. In this way we retain all the essentials of the problem while, through simple phase relations, a considerable simplification of the analysis is obtained.

Fig. 4.



In fig. 4 the total equivalent capacity $\left\{\begin{matrix} C_1 \\ C_2 \end{matrix}\right.$ of the $\left\{\begin{matrix} \text{primary} \\ \text{secondary} \end{matrix}\right.$ circuit is given by

$$\frac{1}{C_1}=\frac{1}{C_a}+\frac{1}{C_b},$$

$$\frac{1}{C_2}=\frac{1}{C_a}+\frac{1}{C_c},$$

and the square of the coupling coefficient, k^2, becomes

$$k^2=\frac{C_1C_2}{C_a^2}.$$

The free angular frequency $\left\{\begin{matrix} \omega_1 \\ \omega_2 \end{matrix}\right.$ of the $\left\{\begin{matrix} \text{primary} \\ \text{secondary} \end{matrix}\right.$ circuit is further

$$\omega_1^2=\frac{1}{C_1L_1},$$

$$\omega_2^2=\frac{1}{C_2L_2}.$$

Similarly the damping coefficient $\begin{cases}\alpha_1'' \\ \alpha_2''\end{cases}$ of the $\begin{cases}\text{primary} \\ \text{secondary}\end{cases}$ circuit is given by

$$\alpha_1'' = \frac{1}{C_1 R_1},$$

$$\alpha_2'' = \frac{1}{C_2 R_2}.$$

We call i_a the variable part of the anode current and v_a the variable part of the anode potential. The application of Kirchhoff's laws to the circuits then leads (with neglect of the grid current) to :

$$\frac{d^4 v_a}{dt^4} + (\alpha_1'' + \alpha_2'')\frac{d^3 v_a}{dt^3} + \{\omega_1^2 + \omega_2^2 + (1-k^2)\alpha_1''\alpha_2''\}\frac{d^2 v_a}{dt^2}$$

$$+ (1-k^2)(\omega_1^2\alpha_2'' + \omega_2^2\alpha_1'')\frac{dv_a}{dt} + (1-k^2)\omega_1^2\omega_2^2 v_a$$

$$= -\frac{1}{C_1}\left[\frac{d^3 i_a}{dt^3} + \alpha_2''(1-k^2)\frac{d^2 i_a}{dt^3} + \omega_2^2(1-k^2)\frac{di_a}{dt}\right]. \quad . \quad (1)$$

We further notice that

$$v_a = L_1 \frac{di_1}{dt},$$

$$v_g = -M\frac{di_1}{dt},$$

where v_g is the grid potential, so that a constant ratio exists between the variable anode potential and grid potential, namely,

$$\frac{v_g}{v_a} = -\frac{M}{L_1}.$$

Hence, though in general the anode current is a function of both the anode and grid potentials, by means of this constant ratio we are able to express the anode current as a function of the variable anode potential alone. A method of determining experimentally this relation $i_a = \psi(v_a)$ has been previously described *. For conditions for which free oscillations are possible this characteristic has in general a negative slope † for $v_a = 0$. It is therefore appropriate to develop the function $i_a = \psi(v_a)$ as

$$i_a = -\alpha' v + \beta' v^2 + \gamma' v^3, \quad . \quad . \quad . \quad . \quad (1\,a)$$

where the index of v_a has been dropped for simplicity as will be done in the further treatment.

* Appleton and Van der Pol, Phil. Mag. xlii. p. 201 (1921).

† When developed in this way the theory applies equally well to "dynatron" circuits.

It may here be noticed that stable oscillations are only possible when both α' and γ' as defined by (1 *a*) are positive. No further terms are needed in this series to enable us to account for the hysteresis phenomenon under consideration, though, naturally, in order to obtain a more exact numerical result in all details, further terms may be necessary.

We further write

$$\frac{\alpha'}{C_1}-\alpha_1''=\alpha_1,$$

$$\alpha_2''=\alpha_2,$$

$$\frac{\beta'}{C_1}=\beta,$$

$$\frac{\gamma'}{C_1}=\gamma,$$

and assume, in agreement with the usual circuit dimensions, that the initial logarithmic increment of the total primary (triode included) and the logarithmic decrement of the secondary are small compared with unity, *i. e.* that

$$0<\frac{\alpha_1}{\omega_1}\leqslant 1, \quad 0<\frac{\alpha_2}{\omega_2}\leqslant 1.$$

On making these substitutions in (1) we arrive at

$$\frac{d^4v}{dt^4}+(\alpha_2-\alpha_1)\frac{d^3v}{dt^3}+\{\omega_1^2+\omega_2^2-(1-k^2)\alpha_1\alpha_2\}\frac{d^2v}{dt^2}$$
$$+(1-k^2)(\omega_1^2\alpha_2-\omega_2^2\alpha_1)\frac{dv}{dt}+(1-k^2)\omega_1^2\omega_2^2v$$
$$=-\left\{\frac{d^3}{dt^3}+\alpha_2(1-k^2)\frac{d^2}{dt^2}+\omega_2^2(1-k^2)\frac{d}{dt}\right\}(\beta v^2+\gamma v^3). \quad . \quad . \quad . \quad (2)$$

On neglecting small terms in (2) this equation can further be simplified, and we obtain as the fundamental differential equation of our problem

$$\frac{d^4v}{dt^4}+(\omega_1^2+\omega_2^2)\frac{d^2v}{dt^2}+(1-k^2)\omega_1^2\omega_2^2v$$
$$+\left\{\frac{d^3}{dt^3}+\omega_2^2(1-k^2)\frac{d}{dt}\right\}(\beta v^2+\gamma v^3)+(\alpha_2-\alpha_1)\frac{d^3v}{dt^3}$$
$$+(1-k^2)(\omega_1^2\alpha_2-\omega_2^2\alpha_1)\frac{dv}{dt}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The general solution of (3) seems not to be possible, but since we consider $\frac{\alpha_1}{\omega_1}$ and $\frac{\alpha_2}{\omega_2}$ as small compared with unity, an approximate solution can be obtained.

From the general nature of the problem two modes of vibration may be expected to be possible, and to express this mathematically we are thus led to a trial solution,

$$v = a \sin \omega_{\text{I}} t + b \sin (\omega_{\text{II}} t + \lambda),$$

where a and b are certain unknown functions of the time and ω_{I} and ω_{II} are unknown frequencies. λ is an arbitrary phase constant.

As
$$\frac{\alpha_1}{\omega_1} \leqslant 1, \qquad \frac{\alpha_2}{\omega_2} \leqslant 1,$$

and ω_1 and ω_2 are of the same order of magnitude, we may expect the possible building up or decay of the amplitudes to occur slowly compared with the oscillations themselves, that is,

$$\frac{da}{dt} \leqslant \omega_{\text{I}} a,$$

$$\frac{db}{dt} \leqslant \omega_{\text{II}} b,$$

Hence the second and higher differential coefficients of a and b with respect to time will be neglected. We thus have

$$\left.\begin{aligned}
v &= a \sin \omega_{\text{I}} t + b \sin (\omega_{\text{II}} t + \lambda), \\
\dot{v} &= \omega_{\text{I}} a \cos \omega_{\text{I}} t + \dot{a} \sin \omega_{\text{I}} t + \omega_{\text{II}} b \cos (\omega_{\text{II}} t + \lambda) \\
&\qquad + \dot{b} \sin (\omega_{\text{II}} t + \lambda), \\
\ddot{v} &= -\omega_{\text{I}}^2 a \sin \omega_{\text{I}} t + 2\omega_{\text{I}} \dot{a} \cos \omega_{\text{I}} t - \omega_{\text{II}}^2 b \sin (\omega_{\text{II}} t + \lambda) \\
&\qquad + 2\omega_{\text{II}} \dot{b} \cos (\omega_{\text{II}} t + \lambda). \\
\dddot{v} &= -\omega_{\text{I}}^3 a \cos \omega_{\text{I}} t - 3\omega_{\text{I}}^2 \dot{a} \sin \omega_{\text{I}} t - \omega_{\text{II}}^3 b \cos (\omega_{\text{II}} t + \lambda) \\
&\qquad - 3\omega_{\text{II}}^2 \dot{b} \sin (\omega_{\text{II}} t + \lambda), \\
\ddddot{v} &= \omega_{\text{I}}^4 a \sin \omega_{\text{I}} t - 4\omega_{\text{I}}^3 \dot{a} \cos \omega_{\text{I}} t + \omega_{\text{II}}^4 b \sin (\omega_{\text{II}} t + \lambda) \\
&\qquad - 4\omega_{\text{II}}^3 \dot{b} \cos (\omega_{\text{II}} t + \lambda).
\end{aligned}\right\} \quad (4)$$

We shall further have to consider the terms involving v^2 and v^3. These non-linear terms obviously suggest the presence of higher harmonics and combination tones, but as the increment and decrement are small, the main effect of the non-linear terms is in their influence on the amplitudes.

For under these circumstances the series representing the amplitudes of the harmonics may be expected to converge rapidly so that the influence of the harmonics on the amplitude of the fundamental may, as a first approximation, be neglected. We are thus justified in neglecting as a first approximation the presence of these higher harmonics and combination tones and shall retain, therefore, in the terms with v^2 and v^3 only those parts involving the frequencies ω_I and ω_{II}. We thus see that the term βv^2 has no influence on the result. In considering, however,

$$v^3 = \{a \sin \omega_I t + b \sin (\omega_{II} t + \lambda)\}^3,$$

terms of several frequencies occur, such as ω_I, ω_{II}, $3\omega_I$, $3\omega_{II}$, $\omega_I + 2\omega_{II}$, $\omega_I - 2\omega_{II}$, $2\omega_I + \omega_{II}$, ... etc., but only the terms involving the frequencies ω_I and ω_{II} will be retained.

Hence we have

$$v^3 = \tfrac{3}{4}a(a^2 + 2b^2) \sin \omega_I t + \tfrac{3}{4}b(b^2 + 2a^2) \sin (\omega_{II} t + \lambda). \quad (5)$$

It may here be noticed that b occurs in the coefficient of $\sin \omega_I t$ and a in the coefficient of $\sin (\omega_{II} t + \lambda)$. This fundamental fact in the non-linear treatment of our problem shows the mutual influence of simultaneous vibrations, and it will further be found that the presence of one oscillation makes it more difficult for the other to develop. When more than three terms are used in the series expansion for i_a, as is advisable when working on the lower bottom part or higher top part of the $i_a - v_g$ characteristic, then the presence of one oscillation is, however, occasionally favourable to the development of another oscillation. Such special cases will, however, not be considered here.

We now proceed to substitute from (4) and (5) in (3) and thus get an equation of the form

$$A \sin \omega_I t + B \sin (\omega_{II} t + \lambda) + C \cos \omega_I t + D \cos (\omega_{II} t + \lambda) = 0, \quad (5a)$$

where A, B, C, and D are functions of the variables ω_I, ω_{II}, a, b, but they also contain $\dot{a}$, $\dot{b}$, $\frac{da^3}{dt}$, $\frac{db^3}{dt}$.

These expressions A, B, C, and D contain terms of three orders of magnitude, viz.:

first order: $\omega^4 a$, $\omega^4 b$;

second order: $\alpha\omega^3 a$, $\gamma\omega^3 a^3$, $\omega^3 \dot{a}$ .... etc.;

third order: $\alpha\omega^2 \dot{a}$, $\gamma\omega^2 \frac{da^3}{dt}$ .... etc.:

but we only retain the first two orders of magnitude.

In order to satisfy (5 a) identically we equate separately to zero the four coefficients A, B, C, D. Thus four equations are obtained for the four variables a, b, ω_I, and ω_{II}. They are found to be

$$\omega_I^4 - \omega_I^2(\omega_1^2+\omega_2^2) + (1-k^2)\omega_1^2\omega_2^2 = 0, \quad . \quad (6\,a)$$

$$\omega_{II}^4 - \omega_{II}^2(\omega_1^2+\omega_2^2) + (1-k^2)\omega_1^2\omega_2^2 = 0, \quad . \quad (6\,b)$$

$$2(\omega_1^2+\omega_2^2-2\omega_I^2)\frac{da}{dt} + \{(1-k^2)(\omega_1^2\alpha_2-\omega_2^2\alpha_1) + \omega_I^2(\alpha_1-\alpha_2)\}\,a + \tfrac{3}{4}\gamma a(a^2+2b^2)\,\{\omega_2^2(1-k^2)-\omega_I^2\} = 0, \quad . \quad . \quad . \quad (7\,a)$$

$$2(\omega_1^2+\omega_2^2-2\omega_{II}^2)\frac{db}{dt} + \{(1-k^2)(\omega_1^2\alpha_2-\omega_2^2\alpha_1) + \omega_{II}^2(\alpha_1-\alpha_2)\}\,b + \tfrac{3}{4}\gamma b(b^2+2a^2)\,\{\omega_2^2(1-k^2)-\omega_{II}^2\} = 0. \quad . \quad . \quad . \quad (7\,b)$$

Equations (6 a) and (6 b) give us the coupling frequencies ω_I and ω_{II}, while (7 a) and (7 b) enable us to find the possible stationary amplitudes a and b and to determine their stability.

Since (6 a) and (6 b) are of the same form it is necessary to define ω_I and ω_{II} quite definitely. We shall take

$$\left.\begin{aligned}\omega_I^2 &= \tfrac{1}{2}(\omega_1^2+\omega_2^2) + \tfrac{1}{2}\sqrt{(\omega_1^2+\omega_2^2)^2 - 4(1-k^2)\omega_1^2\omega_2^2},\\ \omega_{II}^2 &= \tfrac{1}{2}(\omega_1^2+\omega_2^2) - \tfrac{1}{2}\sqrt{(\omega_1^2+\omega_2^2)^2 - 4(1-k^2)\omega_1^2\omega_2^2},\end{aligned}\right\} \quad (8)$$

where the roots are to be taken positive, so that

$$\omega_I^2 > \omega_{II}^2,\ \omega_1^2,\ \omega_2^2,$$

and

$$\omega_{II}^2 < \omega_I^2,\ \omega_1^2,\ \omega_2^2.$$

The equations (7 a) and (7 b) can further be written

$$\left.\begin{aligned}\frac{da^2}{dt} &= E_I a^2(a_0^2 - a^2 - 2b^2),\\ \frac{db^2}{dt} &= E_{II} b^2(b_0^2 - b^2 - 2a^2),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

where, with the aid of (6 a), (6 b), and (8), we have

$$\left.\begin{aligned}E_I &= \tfrac{3}{4}\gamma\frac{\omega_I^2}{\omega_1^2}\cdot\frac{\omega_I^2-\omega_2^2}{\omega_I^2-\omega_{II}^2},\\ E_{II} &= \tfrac{3}{4}\gamma\frac{\omega_{II}^2}{\omega_1^2}\cdot\frac{\omega_2^2-\omega_{II}^2}{\omega_I^2-\omega_{II}^2},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

and thus
$$E_{I} > 0,$$
$$E_{II} > 0.$$

Further, the term a_0^2 introduced in (9), represents the square of the stationary amplitude which would be obtained when an oscillation in the first mode of vibration, *i. e.* with a frequency ω_I, alone was present. Similarly b_0 is the stationary amplitude which would be attained if the system vibrated only in the second mode of vibration. These amplitudes a_0^2 and b_0^2 are obtained directly from (7 *a*) and (7 *b*) by putting $\frac{da}{dt}=\frac{db}{dt}=0$ and are found to be

$$\left.\begin{aligned} a_0^2 &= \frac{\alpha_1}{\frac{3}{4}\gamma} - \frac{\omega_I^2-\omega_1^2}{\omega_1^2-\omega_2^2}\cdot\frac{\omega_1^2}{\omega_2^2}\cdot\frac{\alpha_2}{\frac{3}{4}\gamma}, \\ b_0^2 &= \frac{\alpha_1}{\frac{3}{4}\gamma} - \frac{\omega_1^2-\omega_{II}^2}{\omega_2^2-\omega_{II}^2}\cdot\frac{\omega_1^2}{\omega_2^2}\cdot\frac{\alpha_2}{\frac{3}{4}\gamma}. \end{aligned}\right\} \quad . \quad . \quad . \quad (9\,a)$$

However, a_0 and b_0 are not the only "possible" stationary amplitudes as may be seen from (9) by putting

$$\frac{da^2}{dt}=0, \quad \frac{db^2}{dt}=0.$$

We thus have in general for the "possible" stationary amplitudes a_s and b_s the two equations

$$a_s^2(a_0^2-a_s^2-2b_s^2)=0,$$
$$b_s^2(b_0^2-b_s^2-2a_s^2)=0,$$

the four sets of solutions of which are

$$\left.\begin{array}{lll} \text{(i.)} & a_s^2=0, & b_s^2=0, \\ \text{(ii.)} & a_s^2=\frac{1}{3}(2b_0^2-a_0^2), & b_s^2=\frac{1}{3}(2a_0^2-b_0^2), \\ \text{(iii.)} & a_s^2=a_0^2, & b_s^2=0, \\ \text{(iv.)} & a_s^2=0, & b_s^2=b_0^2. \end{array}\right\} \quad (11)$$

But we shall further have to investigate separately which one of these four stationary solutions (11) will be attained in any given circumstances. Therefore (9) would have to be solved, which is a difficult, if not impossible, matter. However, in order to investigate the *stability* of each of the four solutions (11) we may consider the effect of a small forced change of amplitude from the stationary value due to some disturbing cause, and investigate the tendency of the amplitude either to return to or to depart further from its initial stationary value, thus applying the usual method in questions of dynamic stability. In this way we shall find a certain conservatism of the system in that the particular mode of vibration persists (when one parameter is varied gradually) even when conditions have been reached which are not favourable to it and which are such that, were this mode of vibration not actually present, yet the other one would exist. In other words, *metastable* oscillation conditions may arise.

Let these small changes of amplitude be represented by the type δ. We thus substitute in (9)

$$a^2 = a_s^2 + \delta a_s^2,$$
$$b^2 = b_s^2 + \delta b_s^2,$$

and only retain first powers of the small quantities δa_s^2 and δb_s^2. Hence we have

$$\left.\begin{aligned} \frac{d(\delta a_s^2)}{dt} &= \mathrm{E_I}(a_0^2 - 2a_s^2 - 2b_s^2)\delta a_s^2 - 2\mathrm{E_I} a_s^2 \delta b_s^2, \\ \frac{d(\delta b_s^2)}{dt} &= \mathrm{E_{II}}(b_0^2 - 2b_s^2 - 2a_s^2)\delta b_s^2 - 2\mathrm{E_{II}} b_s^2 \delta a_s^2. \end{aligned}\right\} \qquad (12)$$

These linear equations (12) are solved by putting

$$\delta a_s^2 = \mathrm{A}' e^{kt},$$
$$\delta b_s^2 = \mathrm{B}' e^{kt},$$

and we obtain as characteristic equation for k

$$k^2 + k\{\mathrm{E_I}(2a_s^2 + 2b_s^2 - a_0^2) + \mathrm{E_{II}}(2a_s^2 + 2b_s^2 - b_0^2)\}$$
$$+ \mathrm{E_I E_{II}}\{(2a_s^2 + 2b_s^2 - a_0^2)(2a_s^2 + 2b_s^2 - b_0^2) - 4a_s^2 b_s^2\} = 0. \quad . \quad . \quad . \quad (13)$$

In order that a set of stationary values a_s and b_s should be stable neither of the two roots k of (13) may be positive as this would show the tendency of the system to depart from the stationary solution in question.

A set of stationary amplitudes a_s and b_s is therefore only stable when

$$\left.\begin{aligned} &\mathrm{E_I}(2a_s^2 + 2b_s^2 - a_0^2) + \mathrm{E_{II}}(2a_s^2 + 2b_s^2 - b_0^2) > 0 \\ \text{and}\quad &(2a_s^2 + 2b_s^2 - a_0^2)(2a_s^2 + 2b_s^2 - b_0^2) - 4a_s^2 b_s^2 > 0. \end{aligned}\right\} \qquad (14)$$

We shall now proceed to investigate the conditions of stability of our four solutions i., ii , iii., iv. of (11) separately.

i. $a_s^2=0, \quad b_s^2=0.$

After substitution of these values in (14) we find as the condition for which both amplitudes remain zero :

$$-a_0^2 E_I - b_0^2 E_{II} > 0$$

and $$a_0^2 b_0^2 > 0,$$

or, as E_I and E_{II} are both positive,

$$a_0^2 < 0$$

and $$b_0^2 < 0.$$

These inequalities are expressions for the fact that *only* when the circuit conditions (resistances, retroaction, etc.) are such that no oscillations are "possible" at all, can the system be kept in the non-oscillatory state, from which it may be concluded that, when oscillations *are* "possible" at all, some form of oscillation (either ii., iii., or iv.) will build up automatically.

ii. $$a_s^2=\tfrac{1}{3}(2b_0^2-a_0^2),$$
$$b_s^2=\tfrac{1}{3}(2a_0^2-b_0^2).$$

This represents the case in which *both* coupling frequencies would be present simultaneously. But the conditions of stability here are from (14) easily found to be

$$\left.\begin{aligned} E_I a_s^2 + E_{II} b_s^2 > 0 \\ -a_s^2 b_s^2 > 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (15\,a, b)$$

Now for a_s and b_s to be possible at all we must obviously have

$$a_s^2 > 0$$

$$b_s^2 > 0,$$

which relations are incompatible with (15 *b*). We thus see that the simultaneous occurrence of finite stationary oscillations of both the coupling frequencies represents an unstable condition and can therefore not be realized in practice.

This is in complete agreement with the experimental results.

We saw (i.) that when a vibration is possible at all the system will automatically start vibrating in some form. It cannot, however, produce stationary oscillations in both frequencies at the same time (ii.), so that only one of the two coupling frequencies will build up. Which one this will be depends on the circumstances and can be found from a consideration of

iii. $a_s^2 = a_0^2, \quad b_s^2 = 0,$

iv. $a_s^2 = 0, \quad b_s^2 = b_0^2.$

These cases may be conveniently treated together. Before considering, however, in detail the stability of the system when oscillating in *one* mode of vibration only, we shall first determine the conditions for which such an oscillation is "possible" at all, apart from its stability. Moreover, as the peculiar discontinuities, described in the introduction, occur when the natural frequency of the secondary circuit is altered, we shall leave all parameters unaltered except the detuning of the secondary and consider how these possible amplitudes a_0 and b_0 vary as a function of this detuning. For the circuit under consideration this variation of ω_2 is brought about by varying L_2 (fig. 4), which is equivalent to a variation of the secondary capacity in a case where the electrostatic coupling here considered is replaced by its electromagnetic equivalent.

Now (9 a) can be written

$$\left.\begin{aligned} \tfrac{3}{4}\gamma \frac{a_0^2}{\alpha_2} &= \frac{\alpha_1}{\alpha_2} - f_{\rm I}(\omega_2^2), \\ \tfrac{3}{4}\gamma \frac{b_0^2}{\alpha_2} &= \frac{\alpha_1}{\alpha_2} - f_{\rm II}(\omega_2^2), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

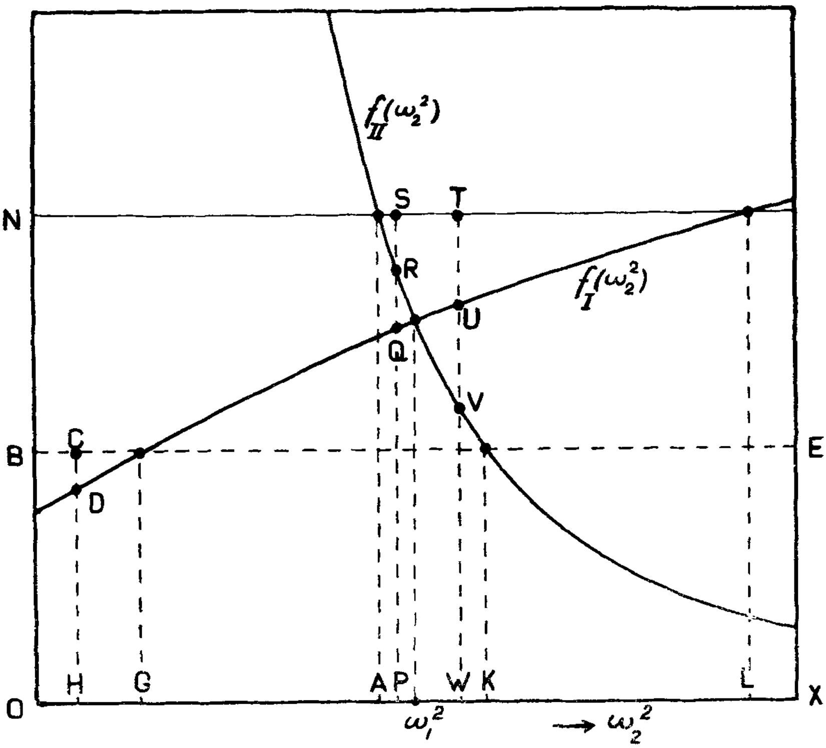
where

$$\left.\begin{aligned} f_{\rm I}(\omega_2^2) &= \frac{\omega_{\rm I}^2 - \omega_1^2}{\omega_{\rm I}^2 - \omega_2^2} \cdot \frac{\omega_1^2}{\omega_2^2} \\ \text{and} \quad f_{\rm II}(\omega_2^2) &= \frac{\omega_1^2 - \omega_{\rm II}^2}{\omega_2^2 - \omega_{\rm II}^2} \cdot \frac{\omega_1^2}{\omega_2^2} . \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (17)$$

These functions $f_{\rm I}$ and $f_{\rm II}$ are the coefficients with which the damping coefficient of the secondary must be multiplied in order to transpose the secondary damping to the primary circuits. They are represented in fig. 5 for a constant $k^2=0{\cdot}5$. Since in (16) α_1, α_2 and γ are independent of ω_2 we can trace in the latter diagram the dependence of a_0^2 and b_0^2 on $\frac{\alpha_1}{\alpha_2}$ and ω_2^2.

For example, if $\frac{\alpha_1}{\alpha_2}$ is represented by OB the amplitude a_0^2 is by (16) proportional to the vertical distance between the

Fig. 5.



line BE and the curve $f_I(\omega_2^2)$. In a similar way b_0^2 is proportional to the distance between BE and $f_{II}(\omega_2^2)$. Thus when $\omega_2^2=OH < OG$, a_0^2 is proportional to CD, but when $\omega_2^2 > OG$, a_0^2 would be negative and oscillations of the frequency ω_I are impossible. In the same way oscillations of the frequency ω_{II} are only possible when $\omega_2^2 > OK$.

If we next consider a larger value of $\frac{\alpha_1}{\alpha_2}$, *e.g.* $\frac{\alpha_1}{\alpha_2}=ON$, markedly different possibilities arise. For example, the range of values of ω_2^2 for which a_0^2 is possible is now represented by OL and is seen to extend *beyond* the resonance position $\omega_2^2=\omega_1^2$. In the same way the amplitude b_0^2 is now possible for all values of ω_2^2 greater than OA. Hence a region AL for ω_2^2 exists in which both modes of vibration are separately "possible." We thus must have recourse to a consideration of the conditions of *stability* in order to decide which mode of vibration will actually be present for any given conditions.

Now in order that

$$a_s^2=a_0^2, \quad b_s^2=0 \quad \text{be stable.}$$

we must have according to (14)

$$\left.\begin{aligned} E_I a_0^2+E_{II}(2a_0^2-b_0^2)&>0 \\ \text{and} \qquad a_0^2(2a_0^2-b_0^2)&>0, \end{aligned}\right\} \quad . \quad . \quad . \quad (18\, a, b)$$

where the second condition is the more stringent one.

Hence for a_0^2 only to be stable, we must have

$$a_0^2>\tfrac{1}{2}b_0^2,$$

and similarly for b_0^2 only to be stable we must have

$$b_0^2>\tfrac{1}{2}a_0^2.$$

These conditions are represented in fig. 5 for $\frac{\alpha_1}{\alpha_2}=\text{ON}$. The vertical SP is so chosen that SR=RQ and the vertical TW such that TU=UV. Hence, though we found previously AL as the region where a_0^2 as well as b_0^2 were separately "*possible*", we may now further conclude that the common region where a_0^2 as well as b_0^2 are separately *stable* is given by the smaller distance PW only.

Which one of the two possible and stable oscillations $a_0 \sin \omega_I t$ or $b_0 \sin(\omega_{II} t+\lambda)$ will be attained in the region PW? The answer to this question, which must also include the explanation of the hysteresis effect, is given by (9), and will be seen to depend on the initial conditions. For let us see what happens when ω_2^2 is gradually brought from a small value such as represented by OH, through resonance to a big value represented by OX. (We again assume $\frac{\alpha_1}{\alpha_2}$ to be given by ON.)

First, when $\omega_2^2=\text{OH}$, only the first mode of vibration is possible and stable and we therefore have

$$a_s^2=a_0^2, \qquad b_s^2=0.$$

Whether b^2 has the tendency to build up when once $a_s^2=a_0^2$ and $b^2=0$, is seen from (9 *b*), which can be written

$$\frac{d \log b^2}{dt}=\mathrm{E}_{\mathrm{II}}(b_0^2-2a_0^2),$$

which shows that $\log b^2$ or b^2 itself will only increase when ω_2^2 has been given such a value that

$$b_0^2-2a_0^2>0.$$

This is the case when

$$\omega_2^2=\mathrm{OW} \quad (\text{fig. } 5).$$

We can therefore bring ω_2^2 from a value $\omega_2^2<\omega_1^2$ through resonance $(\omega_2^2=\omega_1^2)$ to a value $\omega_2^2=\mathrm{OW}>\omega_1^2$ while all the time the system continues to vibrate in the first mode only. But as soon as ω_2^2 has reached the value OW where the square of the amplitude which would obtain if the system vibrated in the second mode only equals twice the square of the amplitude of the vibrations in the first mode actually present ($b_0^2=2a_0^2$ or TV$=$2TU), then the oscillation suddenly jumps from the first mode to the second. A further increase of ω_2^2 to any bigger value will leave the second mode only present. In the same way bringing ω_2^2 back from a big value through resonance to a smaller value results in the second mode being present up to the point P where $2b_0^2=a_0^2$. It is therefore clear that the mode of vibration once obtained persists up to the point where it is no longer *stable* (*not*, as occasionally stated in a linear treatment, where it is no longer *possible*) though a region may have been traversed where the other mode would separately be stable, and thus the hysteresis effect found experimentally can be explained by theory.

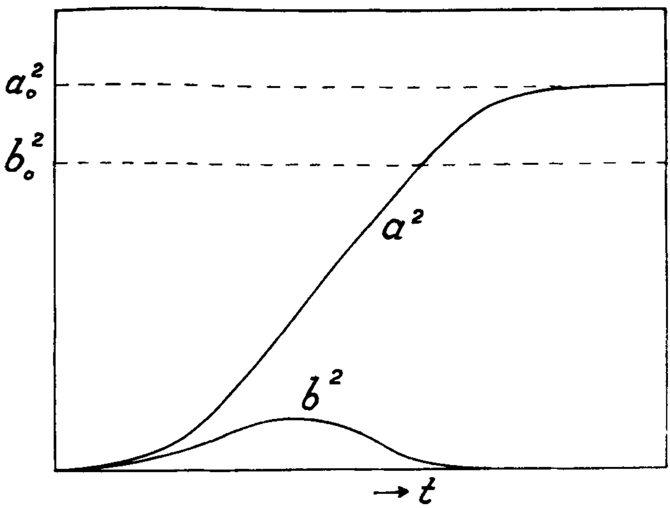
Another experimental fact can further be found theoretically. When one branch of the primary circuit, such *e. g.* as the grid circuit, is first open and thereupon the circuit is closed, then it is found that in general when $\omega_2^2<\omega_1^2$ only vibrations of frequency ω_{I} build up, while, when $\omega_2^2>\omega_1^2$, the system starts vibrating in the second mode.

The initial conditions here are therefore

$$\text{for} \quad t=0, \quad a^2=b^2=0.$$

Now (9) may be integrated graphically for these initial conditions, and fig. 6 is the result for a special case where $\omega_2^2<\omega_1^2$ and therefore $a_0^2>b_0^2$. This figure shows clearly how originally both vibrations build up simultaneously but also that the initial rate of increase of b^2 is smaller than that of a^2. Further, as follows directly from (9), b^2 reaches its

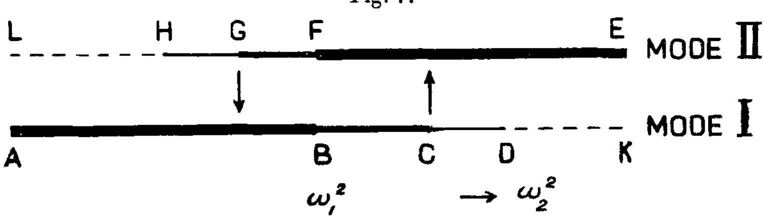
Fig. 6.



maximum when $b^2 = b_0^2 - 2a^2$, and thereupon it goes back asymptotically to zero while a^2 increases up to a_0^2, its stationary value.

Summing up, our results can be described in short with the aid of the schematic fig. 7. Mode I. is *stable* for

Fig. 7.



$\omega_2^2 < \omega_1^2$ (AB), Mode II. for $\omega_2^2 > \omega_1^2$ (EF). Mode I. is *metastable* for $\omega_2^2 > \omega_1^2$ but only up to the point C (BC); Mode II. is metastable for $\omega_2^2 > \omega_1^2$ up to the point G (FG). The part CD is *unstable* for Mode I., and the part GH for Mode II. DK represents the part where Mode I. is *impossible* at all, LH the impossible part for Mode II. On closing the primary circuit only stable oscillations are obtained, while the metastable states can only be realized when the system is first in a stable state and thereupon slowly (compared with the damping coefficients) (or adiabatically) brought to the metastable state.

Some doubt, however, still exists whether it is exactly the point of resonance which separates the regions where on closing the primary circuit either a_0^2 or b_0^2 is finally attained, as some dissymmetry still exists in the formulæ ($E_{II} \neq E_I$).

But a value for ω_2^2 very close to ω_1^2 may easily be obtained experimentally for which it is a mere matter of chance whether a_0 or b_0 will finally be obtained. A simple way of demonstrating this fact is by putting a big leaky condenser in series with the grid of the triode circuit. It is well known that with this arrangement the oscillations are periodically quenched in an automatic way, so that regular trains of vibrations are obtained. The group frequency may *e. g.* be made of the order of one second. When next the secondary circuit is coupled to the primary we can, with a heterodyne arrangement, produce an audible combination tone corresponding to either the one or the other of the two frequencies ω_I and ω_{II}. In general, each time only one of these two combination tones is obtained, but with ω_2 close to or equal to ω_1, the combination tone heard every second jumps erratically between the two tones corresponding to ω_I and ω_{II} respectively, and it is a mere matter of chance which one of the two occurs.

Finally, (9 *a*) yields for $\omega_1 = \omega_2$

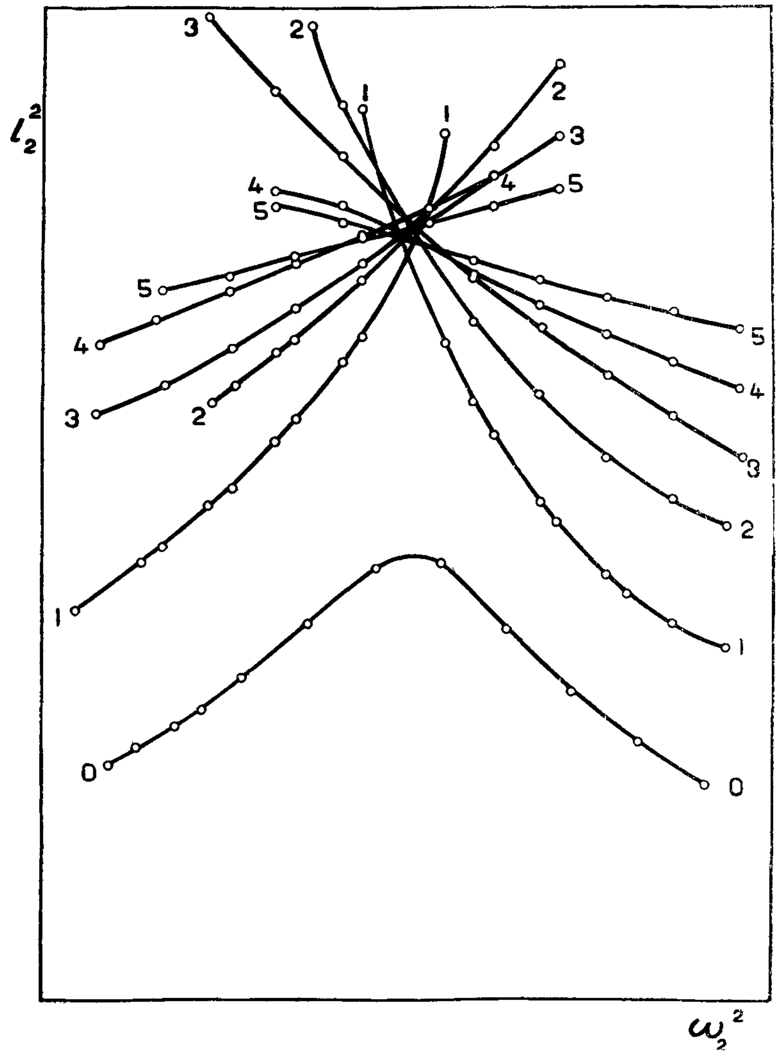
$$a_0^2 = \frac{1}{\frac{3}{4}\gamma}(\alpha_1 - \alpha_2),$$

$$b_0^2 = \frac{1}{\frac{3}{4}\gamma}(\alpha_1 - \alpha_2),$$

and thus shows that, as far as our approximations go, the two amplitude curves intersect at the point of resonance. But, moreover, these amplitudes at the resonance point are independent of the coupling coefficient. Fig. 8, which gives a set of observations of the mean square secondary current (in a circuit like that of fig 1) as a function of ω_2^2 for different coupling coefficients (increasing with the numbers 0, 1, 2, 3, 4, 5), is a confirmation of this theoretical result. For very loose coupling (Curve O) an ordinary resonance curve is obtained, but for closer coupling (1, 2, 3, 4, 5) the figure clearly shows that the intersection of the two branches

(1—1), (2—2), etc., occurs practically at resonance and for all curves at the same height.

Fig. 8.



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