THE PRODUCTION OF NOISE AND VIBRATION BY CERTAIN SQUIRREL-CAGE INDUCTION MOTORS.*

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SUMMARY.

(1) A suggestion is made that the high-pitched notes emitted by some induction motors are due to a side-pull arising from an unsymmetrical field which may be produced when certain numbers of rotor slots are used.
(2) A simple case is first considered and the dissymmetry shown.
(3) and (4) The forces produced are indicated and their effects considered.
(5) The field is analysed and found to include pairs of components, such that in each pair the numbers of poles differ by two. The interferences between such pairs of fields produce the effects observed.
(6) An elementary investigation is made which gives an expression for the frequency of the note produced.
(7) Illustrative examples are considered.
(8) The principle is extended.
(9) A more general investigation is made.
(10) A rule is developed for determining what numbers of rotor slots should be avoided.
(11) Some experimental results are given.

(1.)

Induction motors with squirrel-cage rotors usually have excellent characteristics as regards efficiency, power factor and robustness, while a high starting torque is not expected of them, in general. The rotor is extremely simple, but in spite of this simplicity it presents one or two minor, though intricate, problems in connection with the choice of the number of slots. In order to secure a high power factor at full load and a large pull-out torque it is desirable to employ a large number of slots per pole, say 10 or more, but such a rotor may exhibit a tendency to crawl at or about certain sub-multiples of synchronous speed, viz. in the case of a three-phase motor, 1/7th, 1/13th, etc., of full speed. This tendency to crawl can be sufficiently prevented (*a*) by adopting suitable fractional-pitch windings in the stator, and (*b*) by skewing the rotor slots through one stator slot-pitch. There are cases, however, where these remedies cannot be applied and where crawling is guarded against by using a suitably chosen number of slots less than the number in the stator. Such rotors have the additional advantage that they are cheaper to construct than those having larger numbers of slots.

Considerable care is necessary in choosing the number of slots in the latter case, as such rotors are liable to be very noisy in starting and on load, they may exhibit a tendency to crawl at speeds higher than those mentioned above, and they may give rise to serious vibration. Experienced designers are familiar with certain numbers of slots which are safe, but, so far as the author knows, the matter has never been very fully investigated nor have any reliable rules been published. Most manufacturers have trouble from this source from time to time, but direct experimental investigation is not possible on account of the large expense involved in testing a sufficiently great number of combinations of numbers of poles, and of rotor and stator slots. A motor which has the defects mentioned may produce a fairly high-pitched musical note at speeds within a certain range, and the intensity of the sound may reach a maximum near the middle of the range. In some cases there are several noisy speed-ranges and full-load speed may come within one of them, although a motor which is very noisy at one point during the starting period may operate quite quietly at full speed. When the maximum effect is felt at some speed during the starting period, the vibration may be sufficiently severe to introduce a considerable retarding torque due to friction at the bearings and also, perhaps, to the currents induced in the rotor by its peculiar motion in the field, and the remaining torque may be zero or insufficient to accelerate the machine further.

These effects are probably due to unbalanced magnetic fields, arising as indicated below, which produce rotating forces similar to those caused by mechanical unbalance; the speed of rotation of these forces is much higher than that of the rotor itself, so that the critical speed of the rotor is reached and exceeded even in moderate-speed machines.

(2.)

It is convenient to commence by studying a definite, simple case and we will consider a rotor with five slots acted on by a four-pole, sine-shaped rotating field. We shall neglect the slot openings and assume that the rotor is surrounded by a uniform air-gap.

Fig. 1 shows the relative positions of rotating field and rotor conductors at a certain instant. The teeth are numbered and the arrow indicates the direction of motion of the field with respect to the rotor, the latter being supposed at rest. Positive ordinates indicate flux entering the rotor. It is clear that all the rotor conductors will experience the same virtual E.M.F. and they will all have the same inductance and

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resistance, hence the virtual current and the angle of lag will be the same for all. The phase difference between the E.M.F.'s or currents in successive conductors is $4\pi/5 = 144°$. The vector diagram in Fig. 2 refers to the same instant as Fig. 1, and the full lines represent the E.M.F.'s in the rotor conductors. The vector marked (5, 1) refers to the conductor in the slot between tooth No. 5 and tooth No. 1. In this investigation we shall neglect the resistance of the conductors, so that the current will lag 90° behind the E.M.F. in all cases. On this assumption the current vectors are shown in Fig. 2 by broken lines.

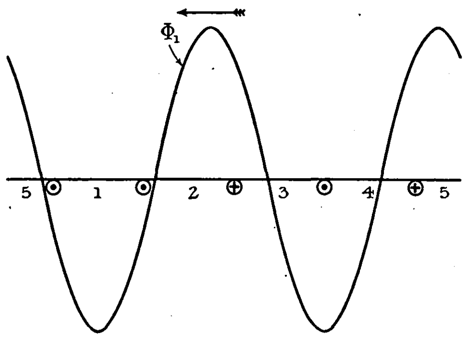

FIG. 1.

Now, the conductors are short-circuited at each end by rings, which rings are not connected together by any other path, therefore the sum of the five currents must be zero at every instant. This means that the vectors representing them must form a closed polygon when taken in the same order as the slots. This polygon is

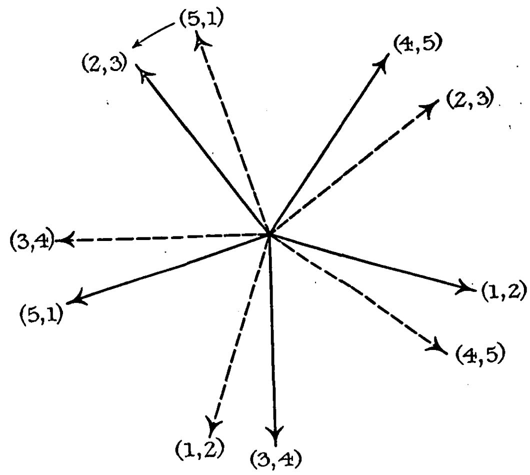

FIG. 2.

drawn in Fig. 3 and is so numbered that the vector (5, 1) represents the current in the conductor between tooth No. 5 and tooth No. 1, and so on. These rotor currents produce a magnetic field which combines with the original four-pole field and produces a very irregular resultant. We shall assume that the effects of saturation and of the slot openings can be neglected, and in the first place we shall examine the form of the field which the rotor currents would produce if they acted alone.

It is a well-known proposition that the M.M.F. acting on the air-gap opposite tooth 5 differs from that acting opposite tooth 1 by the M.M.F. due to the current in conductor (5, 1); hence, if 0 be the centre of the circumscribing circle in Fig. 3, the vectors (0, 1), (0, 2), (0, 3), etc., will represent the maximum ampere-turns acting on the air-gap opposite teeth Nos. 1, 2, 3, etc., respec-

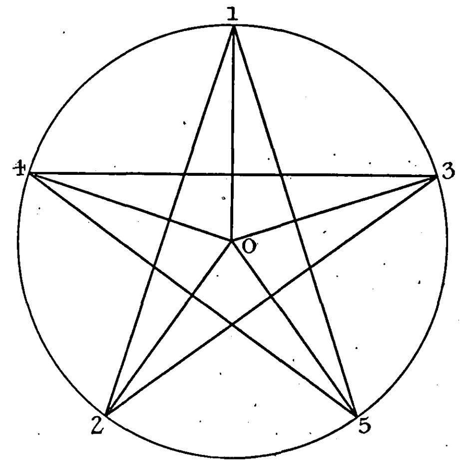

FIG. 3.

tively, to the same scale as (5, 1), (1, 2), etc., represent the maximum current per conductor. This follows because the vector (5, 1) is the difference between the vectors (0, 5) and (0, 1), and because the algebraic sum of the instantaneous values of the fluxes in all the teeth must be zero at every instant. With the aid of Fig. 3 we can draw the stepped curve of Fig. 4 (abcdefghkl) which shows the distribution of the field, Φ_2, due to the rotor currents at the instant

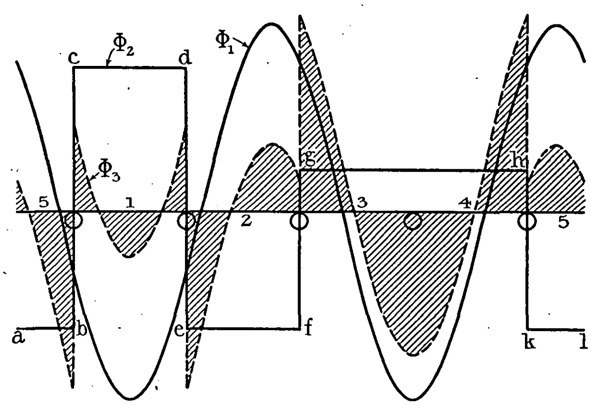

FIG. 4.

when the flux in tooth 1 is at its maximum value. It will be seen that a kind of four-pole field is produced which satisfies the condition that the total flux entering the rotor is equal to the total flux leaving it.

The curve Φ_1 shows the position of the main flux wave at this instant. The curve Φ_3 which bounds the shaded areas is obtained by adding together at each point along the rotor surface the ordinates of curves Φ_1 and Φ_2. The values of the rotor currents and fluxes have been calculated on the assumption that the end leakage may be neglected. The formulæ employed were those given by the author in a paper on "The Air-gap Field of the Induction Motor," * and they indicate that the maximum height of the stepped curve is 0·757 times the height of the Φ_1 curve. A useful check on this figure is provided by the fact that, since the rotor conductors are assumed to have no resistance, the net flux traversing any tooth must be zero at every instant, a condition which the curve Φ_3 fulfils.

(3.)

Now the force with which any element of the rotor surface is attracted towards the stator is proportional to the square of the flux density at that point. The circle in Fig. 5 represents the rotor of the previous figures, the conductors being shown and the teeth numbered. Outside the circle radial lines have been drawn, the lengths of which are proportional to the squares of the resultant densities (Φ_3) shown in Fig. 4.

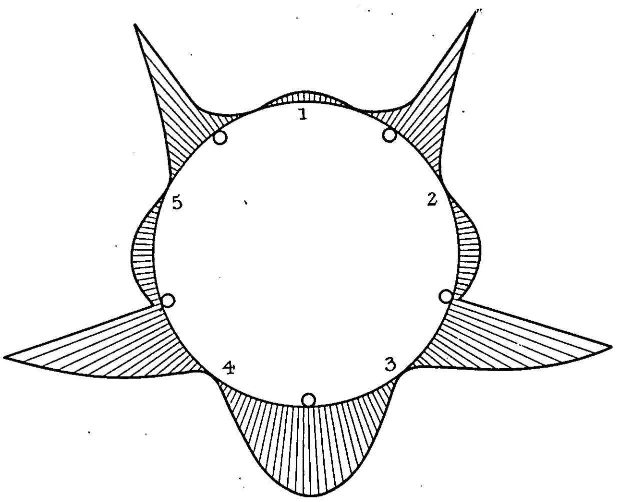

FIG. 5.

It is clear from this diagram that the rotor experiences a resultant force directed radially outwards through the centre of slot (3, 4).

Fig. 6 has been drawn to show the flux distribution 1/10th cycle later when Φ_1 has moved through 1/20th revolution and the vectors of Figs. 2 and 3 have moved through 1/10th revolution. The resultant field is similar to that of Fig. 4, except that it is moved one slot to the right and reversed in sign. If a new figure corresponding to Fig. 5 were drawn for this case it would be found that the resultant force had the same value as before, but was now directed outwards between tooth No. 4 and tooth No. 5, hence in 1/10th cycle its direction has turned through 1/5th revolution. Further diagrams drawn in the same way would show that the force vector makes two revolutions per cycle and therefore rotates four times as fast as Φ_1, relatively to the rotor.

(4.)

This unbalanced force tends to bend the shaft of the rotor and, under steady conditions, the plane of the neutral axis will rotate at the same speed in space as the unbalanced force; this speed is quite different from the speed of rotation of the rotor, if any. The shaft being bent, the air-gap is reduced on one side and an increased unbalanced magnetic attraction is produced, which is added to the original disturbing force, and further, since the centre of gravity of the rotor is whirling with the neutral axis in a circle, a centripetal mass-acceleration is required to maintain the motion. Steady motion is obtained when the shaft is bent to such an extent that the elastic forces set up by the bending are just sufficient to balance the magnetic attraction and to provide the requisite mass-acceleration. It should be noted that such steady conditions cannot exist while the rotational speed of the rotor is changing, because the frequency of the forces brought into play is changing continuously.

The forces acting on the stator and rotor due to magnetic unbalance are equal and opposite, and therefore are balanced if the motor is regarded as a whole, but it is otherwise with the mass-acceleration, which leads to reactions at the bearings that are transmitted to the foundations, producing the same effects as regards

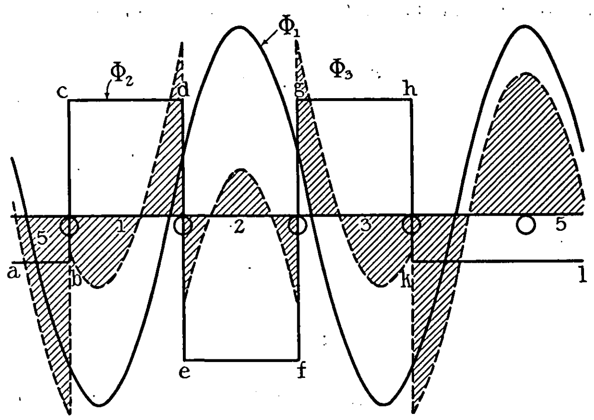

FIG. 6.

noise and vibration as if the rotor were mechanically out of balance and were rotating at the speed at which it now whirls. We shall see later that conditions may exist in a motor which cause the frequency of these forces to be that of a high-pitched musical note, and therefore comparatively small forces (i.e. small deflections) will cause a loud noise. It will also be obvious from the dynamics of the problem that with a given initial disturbing force there will be a certain speed of rotation of this force at which the deflection of the shaft becomes unstable and stator and rotor may come into contact with one another. From the examples given below we shall see that, as the rotor accelerates, the speed of the disturbing force varies over a very wide range, and the critical whirling speed to which we have just referred (which is practically identical with the mechanical critical speed) may be included within this range; the vibration may then be so severe that further acceleration is prevented. Thus we get a crawling speed of a new kind, which is necessarily accompanied by loud noise and vibration.

The action of the main magnetic field under these

* *Electrician*, 1916, vol. 77, p. 663.

circumstances is somewhat obscure, since the presence of the closed squirrel-cage winding prevents rapid changes in the flux distribution, which might otherwise be caused by the whirling. It may be expected, however, to flatten out the resonance curve to a considerable extent and to damp the whirling. It will be clear that if two motors which are otherwise identical have shafts with different degrees of stiffness, the one with the more rigid shaft is less likely to give trouble than the other. Any looseness in the bearings, between the rotor and the shaft or between the stator and its housing, would accentuate the effects here discussed. It may be possible for a given combination of stator and rotor slots to be troublesome in one case and to have little effect in another.

(5.)

The method of investigation which has been given in Section 2 is too laborious for general use, and a further analysis is necessary in order to establish rules to indicate how these effects may be avoided.

A useful light is thrown on the phenomenon if we apply Fourier's analysis to the fields shown in Figs. 4 and 6. In the paper * referred to above, the author

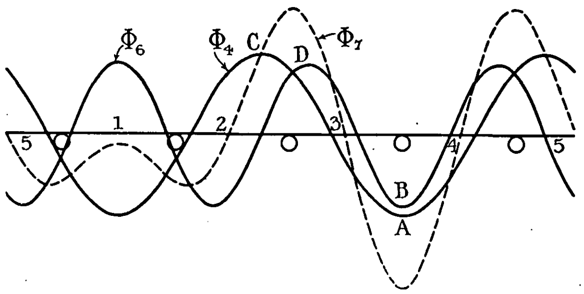

FIG. 7.

has shown that if an inducing field having p poles acts on a squirrel-cage rotor winding with G conductors the fields produced by the rotor currents can be resolved into a series of fields, each of which has cp poles, and may rotate in the same direction as the original field, or in the opposite direction with respect to the rotor. The possible values for c form an infinite series and the general term is given by the expression

$$c = \frac{2Gd}{p} + 1 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where d is any positive or negative integer. Those fields for which c is negative rotate in the opposite direction to the original field, the others in the same direction. The speeds of the fields are such that they all produce E.M.F.'s of the same frequency, in phase with one another, in any rotor conductor.

Formula (1) can be written

$$cp = 2Gd + p \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The amplitude of any one of these fields is equal to

$$\frac{2G \sin (cp\pi/2G)}{cp\pi} \times y$$

* Loc. cit.

where y is the maximum density in the air-gap opposite any tooth, as determined from such a vector diagram as that given in Fig. 3.

If we consider the case of Section 2 in which $G = 5$ and $p = 4$, and substitute successively in Equation (2) the following values for d, viz. 0, -1, $+1$, -2, $+2$, etc., we get the following as the numbers of poles in the series of multiple fields that are present,

$$cp = 4, \ -6, \ 14, \ -16, \ 24, \text{ etc.}$$

The important point about this series is that it consists of a number of pairs of oppositely rotating fields in which the numbers of poles differ by two, thus (4, $-$ 6), (14, $-$ 16), etc.

In Fig. 7 the curve Φ_4 represents the difference between Φ_1 and the four-pole component of Φ_2, a flux wave which moves, like Φ_1, from right to left; Φ_6 represents the six-pole component of Φ_2 and moves from left to right. Φ_7 represents the resultant of these two fields at a certain

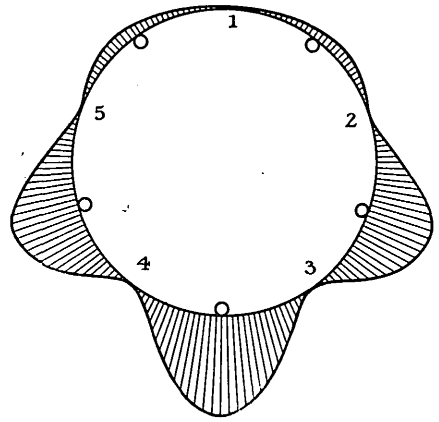

FIG. 8.

instant, and the corresponding force diagram is shown in Fig. 8. Figs. 7 and 8 show the characteristic feature of the resultant field produced by combining a p-pole field with a $(p \pm 2)$-pole field, viz. that there is a strong zone the centre of which is immediately opposite to that of a weak zone, which leads to an unbalanced magnetic pull. This force can be represented by a radial vector rotating about the axis of the rotor.

If a p-pole field be combined with a $(p + x)$-pole field, where $x = 4$, 6, 8, 10, etc., there will be $x/2$ strong and $x/2$ weak zones distributed regularly round the periphery of the rotor, and the forces will constitute a balanced system. It is only when $x = 2$ that unbalanced forces arise.

(6.)

The speed at which the force vector rotates can be determined in the following manner. Suppose we have a p_1-pole field rotating at n_1 revs. per sec., and a p_2-pole field rotating at n_2 revs. per sec. If $p_2 = p_1 + 2$, then interference effects will be produced as shown above. When a crest of the p_1-pole field coincides with a crest of the p_2-pole field, as shown in Fig. 7 at A and B, the force vector is directed outwards through the corresponding point on the rotor, viz. slot (3, 4). We will take a movement from right to left as positive. At a certain instant, earlier or later than that of Fig. 7 according to the direction of relative motion of the two fields, the crests C and D will coincide and the force vector will have moved to this new point of coincidence. The speed of the p_2-pole field with respect to the p_1-pole field is $(n_2 - n_1)$ revs. per sec. Relatively to the p_1-pole field the force vector will have moved from A to C, i.e. through $1/p_1$th of the circumference, while the p_2 field moves through $(1/p_1 - 1/p_2)$th of the circumference, hence the velocity of the force vector relative to the p_1-pole field is greater than the relative velocity of the p_2-pole field in the ratio

$$1/p_1 : (1/p_1 - 1/p_2)$$

therefore the speed of the force vector relative to the p_1-pole field is

$$\begin{aligned}(n_2 - n_1) \times \frac{1/p_1}{1/p_1 - 1/p_2} &= \frac{(n_2 - n_1)p_2}{p_2 - p_1} \\ &= \tfrac{1}{2}[(n_2 - n_1)p_2] \quad . \quad . \quad . \quad . \quad (4)\end{aligned}$$

Now the speed of the p_1-pole field is n_1, so that the speed of the force vector in space is

$$\begin{aligned}n_1 + \tfrac{1}{2}[(n_2 - n_1)p_2] &= \tfrac{1}{2}[n_2p_2 - n_1(p_2 - 2)] \\ &= \tfrac{1}{2}(n_2p_2 - n_1p_1) \quad . \quad . \quad . \quad . \quad (5)\end{aligned}$$

If either of these fields is produced by the direct magnetizing effect of the stator currents, its speed is known from the relation $np = 2f$, where f is the primary frequency; but if one or both arise from rotor currents the speeds must be calculated as follows. Suppose a p_4 field is produced by rotor currents which are induced by a p_3 stator field, then

$$(\pm)p_4 = 2Gd + p_3 \quad . \quad . \quad . \quad . \quad (6)$$

If the speed of the p_3 field is n_3 and that of the rotor n_0, the frequency of the rotor currents under consideration is

$$(n_3 - n_0)p_3/2 \quad . \quad . \quad . \quad . \quad (6a)$$

n_3 must here be given its proper sign, whilst p_3 is taken as a positive quantity.

(The difference between a positive and a negative frequency is expressed in the difference in phase sequence of the currents in the rotor bars. For example, in order to produce the same frequency in this sense, the four-pole and six-pole fields of Fig. 7 must rotate in opposite directions with respect to a five bar rotor, whereas if both rotated in the same direction the frequencies of the E.M.F.'s produced would differ in sign.)

The speed of the p_4 field relative to the rotor is such that the frequency of the E.M.F.'s which it induces in the rotor conductors is the same as the frequency due to the p_3 field, i.e.

$$(n_4 - n_0) \times (\pm)\, p_4 = (n_3 - n_0)p_3 \quad . \quad . \quad (7)$$

hence
$$n_4 = n_0 + (n_3 - n_0)p_3/(\pm)p_4 \quad . \quad . \quad . \quad (8)$$

or, to avoid ambiguity of sign, we may write

$$\begin{aligned}n_4 &= n_0 + \frac{(n_3 - n_0)p_3}{2Gd + p_3} \\ &= \frac{2Gdn_0 + n_3p_3}{2Gd + p_3} \quad . \quad . \quad . \quad . \quad (9)\end{aligned}$$

n_4 may be either positive or negative, and when required for substitution in Equation (5) as n_1 or n_2 it must always be associated with its proper sign; on the other hand, when p_4 is used in this equation as p_1 or p_2, it must be taken as a positive quantity; the examples given in Section 7 will help to make this point clear.

Since the force acting on each element of the rotor surface depends on the total induction density there, this method of singling out two components of the field and considering them apart from the rest requires some justification; it explains the phenomena clearly and correctly, but it does not constitute a sufficiently rigorous treatment of the problem. A more general treatment is given in Section 9, where it is shown that an unbalanced force can only arise when there are component fields with numbers of poles differing by two, and that the magnitude of this force is proportional to the product of the amplitude of these two components.

(7.)

Some numerical examples will serve to illustrate the matter.

Example (1). *A four-pole,* 50-*period three-phase motor has a squirrel-cage rotor with* 19 *slots; the fifth multiple field of this machine has* 20 *poles and runs backwards at a speed which is one-fifth of that of the main field, i.e. at* − 5 *revs. per sec.*—From Equation (2) we see that the rotor currents induced by this field produce a series of fields the numbers of poles in which are

20, − 18, 58, − 56, and so on,

a series of pairs of numbers differing by two in each case. We will consider the first pair in which

$$p_3 = 20 = p_2, \text{ and } p_4 = 18 = p_1, \ (d = -1)$$

Case (*a*).—If the rotor is stationary we have $n_0 = 0$

$$n_3 = n_2 = -5, \text{ and } n_4 = n_1 = \frac{0 - 5 \times 20}{-18}$$

[from Equation (9)]

$$\begin{aligned}n_2p_2 &= -5 \times 20 = -100 \\ n_1p_1 &= +5 \times 20 = +100\end{aligned}$$

hence the speed of the force vector, by Equation (5), is

$$\frac{-100 - 100}{2} = -100 \text{ revs. per sec.}$$

Case (*b*).—At synchronism, $n_0 = 25$ revs. per sec.

$$\begin{aligned}n_3 &= n_2 = -5 \\ n_4 &= n_1 = \frac{-2 \times 19 \times 25 - 5 \times 20}{-18} = \frac{1\,050}{18}\end{aligned}$$

hence the speed of the force vector is

$$\frac{-5 \times 20 - 1\,050}{2} = -575 \text{ revs. per sec.}$$

This speed corresponds to a fairly high musical note. Between standstill and full speed the force vector will probably have passed through the critical speed of the rotor.

Example (2). *A four-pole,* 50-*period three-phase motor* *has a squirrel-cage rotor with 29 slots. The seventh multiple field of this machine has 28 poles and its speed is + 3·57 revs. per sec.*—The numbers of poles in the components of the field set up by the rotor currents induced by the seventh stator field are, by Equation (2)

28, − 30, 86, − 88, etc.

Case (*a*).—When the rotor is stationary, $n_0 = 0$. Considering the first pair of fields, we have

$$p_3 = p_1 = 28;\ p_4 = p_2 = 30,$$
$$n_3 = n_1 = 3 \cdot 57,$$
$$n_4 = n_2 = \frac{0 + 3\cdot 57 \times 28}{-30}, \text{ (since } d = -1)$$

The speed of the force vector is

$$\frac{-3\cdot 57 \times 28 - 3\cdot 57 \times 28}{2} = -100 \text{ revs. per sec.}$$

Case (*b*).—At synchronism, $n_0 = 25$ revs. per sec.

$$n_3 = n_1 = 3\cdot 57$$
$$n_4 = n_2 = \frac{-58 \times 25 + 3\cdot 57 \times 28}{-30}$$
$$= \frac{1\,350}{30}$$

The speed of the force vector is

$$\frac{1\,350 - 3\cdot 57 \times 28}{2} = 625 \text{ revs. per sec.}$$

Between standstill and synchronism there is obviously some speed of the rotor at which the force vector is stationary. This speed can be determined from the equations given above, and is 3·45 revs. per sec. The magnitude of the force at this speed, however, is extremely small.

(8.)

In the two cases considered in Section 7 the rotor currents themselves give rise to pairs of fields which interfere with one another and produce the magnetic " beats " which cause the rotating side pull, but there are cases of another kind in which a field produced by the stator currents interferes with an entirely independent one produced by the rotor currents. For instance, if a four-pole, three-phase motor has a squirrel-cage rotor with 25 slots, the currents induced in the rotor bars by the main field will produce a complex field with components having

4, − 46, + 54, etc., poles.

The − 46 field interferes with the 11th multiple field of the stator which has 44 poles. If the stator should have 24 slots the 11th field would be very important, especially if the slots were open rather wide, in which case its amplitude at full load might exceed that of the main field. On the other hand, the currents induced in the rotor winding by the 11th stator field produce a complex field the components of which have

44, − 6, 94, − 56, etc., poles.

The (− 6)-pole field is the most important of this series; its amplitude is greater than that of the 44-pole field, and it produces interference effects with the main field of the motor. The force vectors corresponding to these two sets of " beats " rotate at the same speed, but in general they differ in magnitude and in phase and produce a resultant. If the frequency of supply were 50 periods per sec. and the motor were running with 20 per cent slip, the speed of the force vectors would be 560 revs. per sec.

A 25-slot rotor in a four-pole motor produces a large number of other sets of interfering fields and so has a great capacity for producing noise.

(9.)

General Expression for the Radial Forces acting on a Rotor.

Let the circle of Fig. 9 represent an end view of the periphery of a rotor, round which a complex field exists. In a perfectly general case the induction density at

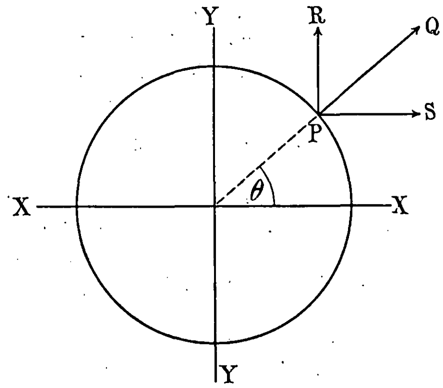

Fig. 9.

any point P on the surface of the rotor is given by the expression

$$A \sin(\theta + a) + B \sin(2\theta + b) + C \sin(3\theta + c) + \ldots$$

The radial force PQ acting on an element of the surface measuring $\delta\theta$ tangentially and of unit length axially is given by the expression

$$K\,\delta\theta[A \sin(\theta + a) + B \sin(2\theta + b) + C \sin(3\theta + c) + \ldots]^2$$

where K is a numerical coefficient.

In order to examine to what extent the forces on all the elements of the surface are balanced we will resolve PQ into its components

$$PR = PQ \sin\theta = K[A \sin(\theta + a) + B \sin(2\theta + b) + \ldots]^2 \sin\theta\,\delta\theta \quad . \quad (10)$$

and

$$PS = PQ \cos\theta = K[A \sin(\theta + a) + B \sin(2\theta + b) + \ldots]^2 \cos\theta\,\delta\theta \quad . \quad (11)$$

Case (*a*). *Permeance of air-gap uniform.*—K is now a constant and by integrating expression (13) between $\theta = 0$ and $\theta = 2\pi$ we get a value for the unbalanced force in the direction of the y-axis, while the integration of (14) gives the unbalanced force in the direction of the x-axis.

These integrations furnish a first series of terms such as

$$K\int_0^{2\pi} N^2 \sin^2 (\nu\theta + n) \sin\theta \, d\theta$$

and $$K\int_0^{2\pi} N^2 \sin^2 (\nu\theta + n) \cos\theta \, d\theta, \text{ respectively}$$

the values of which are all zero, and a second series of terms such as

$$K\int_0^{2\pi} 2MN \sin (\mu\theta + m) \,.\, \sin (\nu\theta + n) \sin\theta \, d\theta$$

and

$$K\int_0^{2\pi} 2MN \sin (\mu\theta + m) \,.\, \sin (\nu\theta + n) \cos\theta \, d\theta, \text{ respectively,}$$

where ν is greater than μ.

The first of these becomes

$$NMK\int_0^{2\pi} \cos [(\mu - \nu)\theta + (n - m)] \sin\theta \, d\theta$$
$$- NMK\int_0^{2\pi} \cos [(\mu + \nu)\theta + (n + m)] \sin\theta \, d\theta$$

which develops into

$$\tfrac{1}{2}NMK\int_0^{2\pi} \sin [\theta + (\mu - \nu)\,\theta + m - n] d\theta$$
$$- \tfrac{1}{2}NMK\int_0^{2\pi} \sin [\theta - (\mu - \nu)\theta - (m - n)] d\theta$$
$$- \tfrac{1}{2}NMK\int \sin [\theta + (\mu + \nu)\theta + n + m] d\theta$$
$$+ \tfrac{1}{2}NMK\int_0^{2\pi} \sin [\theta + (\mu + \nu)\theta - m - n] d\theta$$

Since μ and ν are both positive integers and ν is greater than μ, the last three terms are all zero. The first term is also zero except in the one case where $\nu - \mu = 1$; its value is then

$$PR = \pi NMK \sin (m - n)$$

Similarly the cosine component, PS, is

$$\pi NMK \cos (m - n)$$

when $\nu - \mu = 1$, and is zero in all other cases. The resultant unbalanced force, PQ, is πMNK and is inclined to the x-axis at an angle $(m - n)$.

Thus we see that if the actual field can be resolved into components two of which have numbers of poles, 2μ and 2ν, which differ by 2, these two fields will cause a resultant radial force to act on the rotor. In a perfectly general case it is conceivable that there might be a number of such pairs of fields, the effects of which might cancel one another, but in the induction motor such pairs only occur in the special circumstances investigated above.

In the induction motor the components of the field are rotating at various speeds, and therefore n and m are functions of the time. If we suppose the μ field to be rotating with angular velocity ω_1, and the ν field with angular velocity ω_2, then

$$m = -\mu\omega_1 t + m'$$
and $$n = -\nu\omega_2 t + n'$$
hence $$m - n = m' - n' + (\nu\omega_2 - \mu\omega_1)t \quad . \quad . \quad (12)$$

The instantaneous components of the unbalanced force are

$$PR = \pi NMK \sin [(\nu\omega_2 - \mu\omega_1)t + (m' - n')] \quad . \quad . \quad (13)$$
and $$PS = \pi NMK \cos [\nu\omega_2 - \mu\omega_1)t + (m' - n') \quad (14)$$

which indicate a constant force of magnitude proportional to the product of the amplitudes of the two fields, applied to the rotor in a direction which rotates with an angular velocity of $(\nu\omega_2 - \mu\omega_1)$ radians per sec.

Each pair of fields with numbers of poles differing by two will produce such a rotating force, but in the practical case the resulting forces differ in magnitude and in relative position, and generally they cannot cancel one another.

The speed of the rotating force in revs. per sec. is

$$\nu\frac{\omega_2}{2\pi} - \mu\frac{\omega_1}{2\pi}$$

and, since $\nu = \tfrac{1}{2}p_2$ and $\mu = \tfrac{1}{2}p_1$, this expression is identical with Equation (5).

Case (b). Variable permeance.—When the permeance of the air-gap varies periodically owing to the presence of slot apertures, etc., K is not constant but varies in a regular manner from a maximum to a minimum and back again to a maximum. If we suppose the effect to be due to the slot openings of the rotor only, the permeance will have G maximum and G minimum values. Let us consider G points on the circumference at each of which the permeance will have the same value. For each point we shall have a pair of expressions such as (13) and (14) above. Taking first expression (13) and assuming θ to be measured from one of the G points, we get a series of expressions for the sine components of the forces acting at the several points, such as

$$K[A \sin a + B \sin b + \ldots N \sin n \ldots]^2 \sin 0 \,.\, \delta\theta$$
$$K[A \sin (2\pi/G + a) + B \sin (4\pi/G + b) + \ldots$$
$$+ N \sin (2\nu\pi/G + n) + \ldots]^2 \times \sin (2\pi/G) \,.\, \delta\theta$$
etc.

The $(R + 1)$th expression is

$$K[A \sin (2\pi R/G + a) + B \sin (4\pi R/G + b) + \ldots$$
$$+ N \sin (2\nu\pi R/G + n) + \ldots]^2 \sin (2\pi R/G) \,.\, \delta\theta$$

Each of these G expressions contains similar terms to those set out for the $(R + 1)$th. Expanding the $(R + 1)$th expression we obtain two series of terms such as

$$KN^2 \sin^2 (2\nu\pi R/G + m) \,.\, \sin (2\pi R/G) \,.\, \delta\theta \quad . \quad . \quad (15)$$

and

$$2KMN \sin (2\mu\pi R/G + m) \,.\, \sin (2\nu\pi R/G + n)$$
$$. \sin (2\pi R/G) \,.\, \delta\theta \quad . \quad . \quad (16)$$

When we add together the expanded expressions for all the G components we find that the terms can be arranged in two classes of series. Those of the first class contain G terms in A^2, or in B^2, . . . or in N^2, . . . similar to (15), and in each series R has successive values of 0, 1, 2, . . . $(G - 1)$. Those of the second class contain terms in AB, or BC, or . . . MN . . . similar to (16) in which R has the successive values just mentioned.

Let us consider a typical series of the first class. The general term given in (15) can be further expanded to the following form

$$\tfrac{1}{2}KN^2\,\delta\theta\left[\sin 2\pi R/G + \tfrac{1}{2}\sin\left\{(2\nu - 1)2\pi R/G + 2n\right\} - \tfrac{1}{2}\sin\left\{(2\nu + 1)2\pi R/G + 2n\right\}\right]$$

from which it is obvious that the sum of the series in N^2 is zero unless $G = (2\nu \pm 1)$, and that the sum of all the series in this class is zero unless there is at least one component field for which the number of poles is $(G \pm 1)$. Another form of this statement would be: If a component of the field has p poles and $G = (p \pm 1)$ a side-pull will result from this combination, which is obviously true without further explanation.

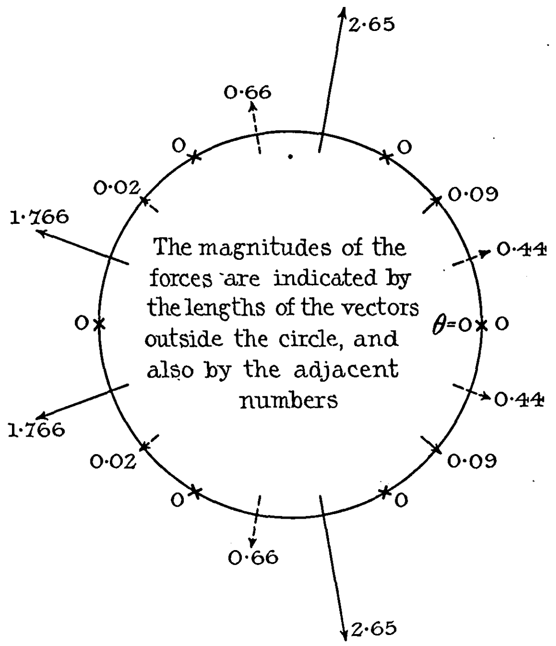

FIG. 10.

Returning now to the series of the second class, the general term (16) can be expanded to the following form

$$\tfrac{1}{2}KMN\,\delta\theta\left[\sin\left\{(\nu - \mu + 1)2\pi R/G + n - m\right\} - \sin\left\{(\nu - \mu - 1)2\pi R/G + n - m\right\} - \sin\left\{(\nu + \mu + 1)2\pi R/G + n + m\right\} + \sin\left\{(\nu + \mu - 1)2\pi R/G + n + m\right\}\right]$$

from which it is clear that the sum of the series in MN is zero unless $G = (\nu \pm \mu \pm 1)$, and the sum of all the series of this class will be zero unless G differs by unity from the sum or difference of the numbers of pairs of poles in any two of the component fields.

The significance of this condition is not very obvious and an example will help to make it clear. Suppose we have a field of four poles ($\mu = 2$) with a fifth multiple of 20 poles ($\nu = 10$), and suppose that $G = 9(=\nu-\mu+1)$. For the sake of simplicity we will assume that both fields have the same amplitude, so that the flux density at the $(R + 1)$th point will be proportional to

$$\sin 2\pi R/G + \sin 10\pi R/G$$

The full-line vectors in Fig. 10 represent the squares of the nine values of this quantity and clearly indicate the presence of a resultant side-pull. The dotted vectors in the same figure show the values of the same quantities at points midway between the G points already considered; these have been drawn on the assumption that the permeance at the second set of points is one-half that at the first set. The two sets of forces evidently produce a resultant which would not have been present if the permeance had been uniform. If other intermediate sets of points are considered it will be seen that a resultant side-pull must be produced unless the permeance at all points separated by $1/2G$ of the circumference of the rotor is the same

Returning to the expressions for the cosine terms corresponding to (15) and (16), we find that they can be expanded to

$$\tfrac{1}{2}KN^2\,\delta\theta\left[\cos 2\pi R/G - \tfrac{1}{2}\cos\left\{(2\nu + 1)2\pi R/G + 2n\right\} - \tfrac{1}{2}\cos\left\{(2\nu - 1)2\pi R/G + 2n\right\}\right]$$

and

$$\tfrac{1}{2}KMN\,\delta\theta\left[\cos\left\{(\nu - \mu + 1)2\pi R/G + n - m\right\} + \cos\left\{(\nu - \mu - 1)2\pi R/G + n - m\right\} - \cos\left\{(\nu + \mu + 1)2\pi R/G + n + m\right\} - \cos\left\{(\nu + \mu - 1)2\pi R/G + n + m\right\}\right]$$

Taking first the case of $G = 2\nu + 1$, the components of the resultant force are

$$-\tfrac{1}{4}GKN^2\,\delta\theta\,.\,\sin 2n, \quad\text{and}\quad -\tfrac{1}{4}GKN^2\,\delta\theta\,.\,\cos 2n,$$
respectively.

If the field is moving with respect to the rotor with velocity $+\omega$ we can write

$$n = -\nu\omega t + n'$$

whence we see that the resultant force vector rotates with velocity $-2\nu\omega$ with respect to the rotor. On the other hand, when $G = 2\nu - 1$ the velocity of the resultant force vector is $+2\nu\omega$ with respect to the rotor.

When $G = (\nu - \mu + 1)$ the resultant sine and cosine components are

$$\tfrac{1}{2}GKMN\,\delta\theta\,.\,\sin(n - m) \text{ and } \tfrac{1}{2}GKMN\,\delta\theta\,.\,\cos(n - m).$$

Writing

$$m = -\mu\omega_1 t + m' \quad\text{and}\quad n = -\nu\omega_2 t + n'$$

where ω_1 and ω_2 are measured with respect to the rotor, we see that the speed of the resultant force vector with respect to the rotor is $(\mu\omega_1 - \nu\omega_2)$.

In the same way we find that

when $G = \nu - \mu - 1$ the speed is $-(\mu\omega_1 - \nu\omega_2)$
when $G = \nu + \mu + 1$ the speed is $+(\mu\omega_1 + \nu\omega_2)$

and when $G = \nu + \mu - 1$ the speed is $-(\mu\omega_1 + \nu\omega_2)$.

In all these cases it will be found that the speed of the force vector in space is $Gn_0 - 2f$, where n_0 is the speed of the rotor in revs. per sec., and f is the frequency of supply.

The effect of staggering the rotor slots through one slot-pitch is to make constant the average permeance along any line on the circumference drawn parallel to the axis, and therefore if we consider only the resultant of the radial forces across such a line these resultants will always be in equilibrium, so far as the effects considered under Case (*b*) are concerned, but the radial force is not uniformly distributed along such a line, and if G has one of the values referred to above the forces may be greatest in the middle and least at the ends of one axial line, and least in the middle and greatest at the ends of a diametrically opposite line. Thus there may still be a rotating bending moment acting on the rotor which may conceivably produce appreciable effects on a long rotor.

(10.)

We must now see what rules can be deduced to enable us to choose numbers of slots for squirrel-cage rotors which will avoid these troubles.

(1) *To avoid the effects of interference between fields with numbers of poles differing by two.*—In a three-phase motor the numbers of poles in the component stator fields are p, $5p$, $7p$, $11p$, etc., or, generally $(6d \pm 1)p$, and in a single-phase or two-phase motor they are p, $3p$, $5p$, $7p$, $9p$, etc., or, generally, $(4d \pm 1)p$.

Therefore for a three-phase motor, by Equation (2), we must have

$$(6d_1 \pm 1)p \pm 2 \neq 2Gd + (6d_2 \pm 1)p$$

d_1 and d_2 being positive integers. This reduces to $Gd \neq pd_3 \pm 1$, where d_3 is a positive or negative integer, and since $d = -1$ is the only case of practical importance we finally obtain $G \neq pd_3 \pm 1$, i.e. the number of slots in the rotor must not differ by unity from any multiple of the number of poles for which the stator is wound. For single- and two-phase motors we get the same result. In the case of a four-pole machine, for instance, the rotor must not have an odd number of slots.

(2) In Case (*b*) of Section 9 we have seen that we should not have $G = 2\nu \pm 1$, nor $G = \nu \pm \mu \pm 1$.

For three-phase motors, therefore, we must have

$$G \neq p(6d_1 \pm 1) \pm 1$$

and $$G \neq \tfrac{1}{2}p[(6d_1 \pm 1) \pm (6d_2 \pm 1)] \pm 1$$

Both of these conditions are included in the general condition in Case (1) and the same is true for single- and two-phase motors; thus *if the number of rotor slots is chosen so that it does not differ by unity from any multiple of the number of poles, noise arising from all the causes considered in this paper will be avoided.*

The results of an investigation have been published * recently in which a four-pole, 24-slot stator was tested with several rotors having different numbers of slots. Without exception the rotors with odd numbers of slots were noisy, whilst those with even numbers were quiet.

* W. Stiel: *Zeitschrift des Vereines deutscher Ingenieure*, 1921, vol. 65, p. 147.

(11.)

In order to obtain some experimental confirmation of the foregoing theory the following measurements have been carried out by Mr. W. G. Spencer and Mr. E. L. Sainsbury at Woolwich Polytechnic. The machine employed in the experiments was a 4-h.p. six-pole, 50-period, three-phase motor by a well-known British maker. The squirrel-cage rotor had 41 slots, skewed through one slot-pitch, and the stator had a full-pitch winding in 54 slots. During the starting period this machine ran quite silently at low speeds and also at full speed, but when supplied at normal frequency a definite musical note was emitted between speeds of 600 and 750 r.p.m., and it may be assumed that the speed of the rotating side-pull coincides with the critical speed of the rotor at about the middle of this range. The machine was very rigidly constructed and consequently the note was so faint as to be negligible from a commercial point of view, though it was clear enough for experimental purposes.

For this machine, since $p = 6$, we have $G = 41 = 7p - 1$, a number which should be avoided, according to Section 9.

If we consider the action of the 7th multiple field on this rotor we have $p_3 = 42$, and the possible values of p_4 are

$$p_4 = (2 \times 41 \times d) + 42$$

substituting successively $0, -1, +1, -2, +2$, etc., for d, we obtain the following series of values for p_4 .. $42, -40, +124, -122$, etc. The 42-pole field produced by the rotor currents combines with the original $7p$-pole field (reducing its value), and this resultant, together with the field for which $p_4 = -40$, produces a rotating side-pull which is the cause of the note observed. The two fields of 122 and 124 poles also produce a rotating side-pull.

Taking f as the frequency of supply and n_0 as the speed of the rotor we have

$$p_3 = p_2 = 42\,; \quad p_4 = p_1 = 40$$

$$n_3 = \frac{2 \times f}{6} \times \frac{1}{7} = \frac{2f}{42}$$

i.e. one-seventh of the speed of the main field, and, since $d = -1$,

$$n_4 = \frac{-82n_0 + n_3 \times 42}{-40} \quad \text{[by Eqn. (9)]}$$

$$= \frac{-82n_0 + 2f}{-40}$$

$$= n_1$$

Since $n_2 \neq n_3$, the speed of the force vector [by Eqn. (5)] is

$$\tfrac{1}{2}(42n_3 - 40n_4) = 2f - 41n_0$$

We are not concerned with the direction of this rotation, and in order to deal with a positive quantity we may write for the frequency of the note emitted

$$41n_0 - 2f$$

For the other two fields, with 122 and 124 poles respectively, the speeds must be calculated as for n_4 above.

The speed of the 122-pole field is (d being -2)

$$\frac{-164n_0 + 2f}{-122} = n_1$$

while that of the 124-pole field is (d being $+1$)

$$\frac{82n_0 + 2f}{124} = n_2$$

The speed of the force vector in this case is

$$\tfrac{1}{2}(124n_2 - 122n_1)$$
$$= -41n_0 + 2f, \text{ as before}$$

The pitch of the loudest note emitted by the machine was rather lower than 384 per sec., but for convenience this pitch was taken as standard during the experiments, and a tuning-fork was used as a standard of reference. Power was supplied to the motor at several different frequencies between 0 and 60 periods per sec., and at each frequency the speed was observed at which the note emitted coincided with that of the tuning-fork. The points marked in Fig. 11 show the actual readings obtained, and the line drawn amongst them represents $41n_0 - 2f$. It will be seen that substantial agreement has been obtained. There are two points on the extreme left of the figure for which it does not seem possible to account on the basis of the theory here given.

The faintness of the note emitted by this machine is due to the fact that the slots were skewed, and the effect would have been eliminated entirely if the obliquity had been such that the slot openings extended across two slot-pitches instead of one, though this is scarcely practicable and, in the present case, is quite unnecessary.

Considerable difficulties had to be overcome in order to obtain the results shown in Fig. 11, and the author

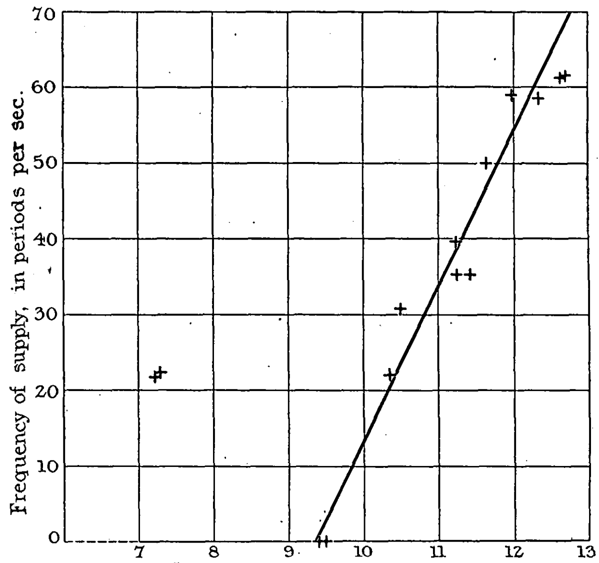

FIG. 11.

wishes to express his indebtedness to Messrs. Spencer and Sainsbury for the care and trouble which they took in making the experiments. It was hoped to show that the critical speed of the rotor was in the neighbourhood of 384 revs. per sec. In this, however, we have not yet been successful, and further experiments are in progress.