ASYMPTOTIC FORMULÆ FOR THE DISTRIBUTION OF INTEGERS OF VARIOUS TYPES*

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1.

Statement of the problem.

1.1. We denote by q a number of the form

(1.11) $$2^{a_2}3^{a_3}5^{a_5}\dots p^{a_p},$$

where 2, 3, 5, ..., p are primes and

(1.111) $$a_2 \geqslant a_3 \geqslant a_5 \dots \geqslant a_p;$$

and by $Q(x)$ the number of such numbers which do not exceed x: and our problem is that of determining the order of $Q(x)$. We prove that

(1.12) $$Q(x) = \exp\left[\{1+o(1)\}\frac{2\pi}{\sqrt{3}}\sqrt{\left(\frac{\log x}{\log\log x}\right)}\right],$$

that is to say that to every positive ϵ corresponds an $x_0 = x_0(\epsilon)$, such that

(1.121) $$\left(\frac{2\pi}{\sqrt{3}}-\epsilon\right)\sqrt{\left(\frac{\log x}{\log\log x}\right)} < \log Q(x) < \left(\frac{2\pi}{\sqrt{3}}+\epsilon\right)\sqrt{\left(\frac{\log x}{\log\log x}\right)},$$

for $x > x_0$. These functions are of course of higher order than any power of $\log x$, but of lower order than any power of x.

The interest of the problem is threefold. In the first place the result itself, and the method by which it is obtained, are curious and interesting in themselves. Secondly, the method of proof is one which, as we show at the end of the paper, may be applied to a whole class of problems in the analytic theory of numbers: it enables us, for example, to find

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asymptotic formulæ for the number of partitions of n into positive integers, or into different positive integers, or into primes. Finally, the class of numbers q includes as a sub-class the "highly composite" numbers recently studied by Mr. Ramanujan in an elaborate memoir in these *Proceedings*.* The problem of determining, with any precision, the number $H(x)$ of highly composite numbers not exceeding x appears to be one of extreme difficulty. Mr. Ramanujan has proved, by elementary methods, that the order of $H(x)$ is at any rate greater than that of $\log x$†: but it is still uncertain whether or no the order of $H(x)$ is greater than that of any power of $\log x$. In order to apply transcendental methods to this problem, it would be necessary to study the properties of the function

$$\mathfrak{H}(s) = \Sigma \frac{1}{h^s},$$

where h is a highly composite number, and we have not been able to make any progress in this direction. It is therefore very desirable to study the distribution of wider classes of numbers which include the highly composite numbers and possess some at any rate of their characteristic properties. The simplest and most natural such class is that of the numbers q; and here progress is comparatively easy, since the function

$$\mathfrak{Q}(s) = \Sigma \frac{1}{q^s} \tag{1.13}$$

possesses a product expression analogous to Euler's product expression for $\zeta(s)$, viz.

$$Q(s) = \prod_1^{\infty} \left(\frac{1}{1-l_n^{-s}}\right), \tag{1.14}$$

where $l_n = 2.3.5 \ldots p_n$ is the product of the first n primes.

We have not been able to apply to this problem the methods, depending on the theory of functions of a complex variable, by which the Prime Number Theorem was proved. The function $Q(s)$ has the line $\sigma = 0$‡ as a line of essential singularities, and we are not able to obtain sufficiently accurate

* Ramanujan, "Highly Composite Numbers", *Proc. London Math. Soc.*, Ser. 2, Vol. 14, 1915, pp. 347–409.

† As great as that of $\dfrac{\log x \sqrt{(\log \log x)}}{(\log \log \log x)^{\frac{3}{2}}}$: see p. 385 of his memoir.

‡ We write as usual $s = \sigma + it$.

information concerning the nature of these singularities. But it is easy enough to determine the behaviour of $Q(s)$ as a function of the *real* variable s; and it proves sufficient for our purpose to determine an asymptotic formula for $Q(s)$ when $s \to 0$, and then to apply a "Tauberian" theorem similar to those proved by Messrs. Hardy and Littlewood in a series of papers published in these *Proceedings* and elsewhere.*

This "Tauberian" theorem is in itself of considerable interest as being (so far as we are aware) the first such theorem which deals with functions or sequences tending to infinity more rapidly than any power of the variable.

2.

Elementary results.

2.1. Let us consider, before proceeding further, what information concerning the order of $Q(x)$ can be obtained by purely elementary methods.

Let

(2.11) $$l_n = 2 \,.\, 3 \,.\, 5 \ldots p_n = e^{\vartheta(p_n)},$$

where $\vartheta(x)$ is Tschebyschef's function

$$\vartheta(x) = \sum_{p \leqslant x} \log p.$$

The class of numbers q is plainly identical with the class of numbers of the form

(2.12) $$l_1^{b_1} l_2^{b_2} \ldots l_n^{b_n},$$

where $$b_1 \geqslant 0, \quad b_2 \geqslant 0, \quad \ldots, \quad b_n \geqslant 0.$$

Now every b can be expressed in one and only one way in the form

(2.13) $$b_i = c_{i,m} 2^m + c_{i,m-1} 2^{m-1} + \ldots + c_{i,0},$$

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* See, in particular, Hardy and Littlewood, "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive", *Proc. London Math. Soc.*, Ser. 2, Vol. 13, pp. 174-191; and "Some theorems concerning Dirichlet's series", *Messenger of Mathematics*, Vol. 43, pp. 134–147.

where every c is equal to zero or to unity. We have therefore

$$(2.14) \qquad q = \prod_{i=1}^{n} \left(l_i^{\sum_{j=0}^{m} c_{i,j} 2^j} \right) = \prod_{j=0}^{m} \prod_{i=1}^{n} l_i^{c_{i,j} 2^j} = \prod_{j=0}^{m} r_j^{2^j},$$

say, where

$$(2.141) \qquad r_j = l_1^{c_{1,j}} l_2^{c_{2,j}} \dots l_n^{c_{n,j}}.$$

Let r denote, generally, a number of the form

$$(2.15) \qquad r = l_1^{c_1} l_2^{c_2} \dots l_n^{c_n},$$

where every c is zero or unity : and $R(x)$ the number of such numbers which do not exceed x. If $q \leqslant x$, we have

$$r_0 \leqslant x, \quad r_1^2 \leqslant x, \quad r_2^4 \leqslant x, \quad \dots .$$

The number of possible values of r_0, in the formula (2.14), cannot therefore exceed $R(x)$; the number of possible values of r_1 cannot exceed $R(x^{\frac{1}{2}})$; and so on. The total number of values of q can therefore not exceed

$$(2.16) \qquad S(x) = R(x)\, R(x^{\frac{1}{2}})\, R(x^{\frac{1}{4}}) \dots R(x^{2^{-\varpi}}),$$

where ϖ is the largest number such that

$$(2.161) \qquad x^{2^{-\varpi}} \geqslant 2, \qquad x \geqslant 2^{2^{\varpi}}.$$

Thus

$$(2.17) \qquad Q(x) \leqslant S(x).$$

2.2. We denote by f and g the largest numbers, such that

$$(2.211) \qquad l_f \leqslant x,$$

$$(2.212) \qquad l_1 l_2 \dots l_g \leqslant x.$$

It is known* (and may be proved by elementary methods) that constants A and B exist, such that

$$(2.221) \qquad \vartheta(x) \geqslant Ax \quad (x \geqslant 2),$$

and

$$(2.222) \qquad p_n \geqslant Bn \log n \quad (n \geqslant 1).$$

* See Landau, *Handbuch*, pp. 79, 83, 214.

We have therefore
$$e^{Ap_f} \leqslant x,$$
$$f \log f = O(\log x),$$

(2.23)
$$\log f = O(\log\log x);$$

and
$$\sum_1^g \vartheta(p_\nu) \leqslant \log x, \qquad \sum_1^g p_\nu = O(\log x),$$
$$\sum_1^g \nu \log \nu = O(\log x), \qquad g^2 \log g = O(\log x),$$

(2.24)
$$g = O\sqrt{\left(\frac{\log x}{\log\log x}\right)}.$$

But it is easy to obtain an upper bound for $R(x)$ in terms of f and g. The number of numbers $l_1, l_2, \ldots,$ not exceeding x, is not greater than f; the number of products, not exceeding x, of pairs of such numbers, is *a fortiori* not greater than $\frac{1}{2}f(f-1)$; and so on. Thus

$$R(x) \leqslant f + \frac{f(f-1)}{2!} + \frac{f(f-1)(f-2)}{3!} + \ldots,$$

where the summation need be extended to g terms only, since

$$l_1 l_2 \ldots l_g l_{g+1} > x.$$

A fortiori, we have

$$R(x) \leqslant 1 + f + \frac{f^2}{2!} + \ldots + \frac{f^g}{g!} < (1+f)^g = e^{g \log(1+f)}.$$

Thus

(2.25)
$$R(x) = e^{O(g \log f)} = e^{O\{\sqrt{(\log x \log\log x)}\}},$$

by (2.23) and (2.24). Finally, since

$$\log\sqrt{x} \log\log\sqrt{x} < \tfrac{1}{2}\log x \log\log x,$$

it follows from (2.16) and (2.17) that

(2.26)
$$Q(x) = \exp\left[O\left\{\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{\varpi}}\right)\sqrt{(\log x \log\log x)}\right\}\right]$$
$$= e^{O\{\sqrt{(\log x \log\log x)}\}}.$$

2.3. A *lower* bound for $Q(x)$ may be found as follows. If g is defined as in 2.2, we have

$$l_1 l_2 \ldots l_g \leqslant x < l_1 l_2 \ldots l_g l_{g+1}.$$

It follows from the analysis of 2 . 2 that

$$l_{g+1} = e^{\vartheta(p_{g+1})} = e^{O(g \log g)},$$

and
$$l_1 l_2 \dots l_g = \exp\left\{\sum_1^g \vartheta(p_\nu)\right\} = e^{O(g^2 \log g)}.$$

Thus
$$x < e^{O(g^2 \log g)};$$

which is only possible if g is greater than a constant positive multiple of

$$\sqrt{\left(\frac{\log x}{\log\log x}\right)}.$$

Now the numbers $l_1, l_2, \dots, l_g$ can be combined in 2^g different ways, and each such combination gives a number q not greater than x. Thus

(2 . 31)
$$Q(x) \geqslant 2^g > \exp\left\{K\sqrt{\left(\frac{\log x}{\log\log x}\right)}\right\},$$

where K is a positive constant. From (2 . 26) and (2 . 31) it follows that there are positive constants K and L such that

(2 . 32)
$$K\sqrt{\left(\frac{\log x}{\log\log x}\right)} < \log Q(x) < L\sqrt{(\log x \log\log x)}.$$

The inequalities (2 . 32) give a fairly accurate idea as to the order of magnitude of $Q(x)$. But they are much less precise than the inequalities (1 . 121). To obtain these requires the use of less elementary methods.

3.

The behaviour of $\mathfrak{Q}(s)$ when $s \to 0$ by positive values.

3 . 1. From the fact, already used in 2 . 1, that the class of numbers q is identical with the class of numbers of the form (2 . 12), it follows at once that

(1 . 14)
$$\mathfrak{Q}(s) = \Sigma \frac{1}{q^s} = \prod_1^\infty \left(\frac{1}{1-l_n^{-s}}\right).$$

Both series and product are absolutely convergent for $\sigma > 0$, and

(3 . 11)
$$\log \mathfrak{Q}(s) = \phi(s) + \tfrac{1}{2}\phi(2s) + \tfrac{1}{3}\phi(3s) + \dots,$$

where

(3 . 111)
$$\phi(s) = \sum_1^\infty l_n^{-s}.$$

We have also

$$(3.12)\quad \phi(s) = \frac{1-2^{-s}}{2^s-1} + 2^{-s}\frac{1-3^{-s}}{3^s-1} + 2^{-s}3^{-s}\frac{1-5^{-s}}{5^s-1} + \dots$$

$$= \frac{1}{2^s-1} + 2^{-s}\left(\frac{1}{3^s-1} - \frac{1}{2^s-1}\right) + 2^{-s}3^{-s}\left(\frac{1}{5^s-1} - \frac{1}{3^s-1}\right) + \dots$$

$$= \frac{1}{2^s-1} + \sum_1^\infty e^{-s\vartheta(p_n)} \int_{p_n}^{p_{n+1}} \frac{d}{dx}\left(\frac{1}{x^s-1}\right) dx$$

$$= \frac{1}{2^s-1} - s\int_2^\infty \frac{x^{s-1}}{(x^s-1)^2} e^{-s\vartheta(x)}\, dx.$$

3.2. LEMMA.—*If* $x > 1$, $s > 0$, *then*

$$(3.21)\qquad \frac{1}{(s\log x)^2} - \frac{1}{12} < \frac{x^s}{(x^s-1)^2} < \frac{1}{(s\log x)^2}.$$

Write $x^s = e^u$: then we have to prove that

$$(3.22)\qquad \frac{1}{u^2} - \frac{1}{12} < \frac{e^u}{(e^u-1)^2} < \frac{1}{u^2}$$

for all positive values of u; or (writing w for $\frac{1}{2}u$) that

$$(3.23)\qquad \frac{1}{w^2} - \frac{1}{3} < \frac{1}{\sinh^2 w} < \frac{1}{w^2}$$

for all positive values of w. But it is easy to prove that the function

$$f(w) = \frac{1}{w^2} - \frac{1}{\sinh^2 w}$$

is a steadily decreasing function of w, and that its limit when $w \to 0$ is $\frac{1}{3}$; and this establishes the truth of the lemma.

3.3. We have therefore

$$(3.31)\qquad \phi(s) = \frac{1}{2^s-1} - \phi_1(s) = \frac{1}{s\log 2} - \phi_1(s) + O(1),$$

where

$$(3.311)\quad \frac{1}{s}\int_2^\infty \left\{\frac{1}{(\log x)^2} - \frac{s^2}{12}\right\} e^{-s\vartheta(x)} \frac{dx}{x} < \phi_1(x) < \frac{1}{s}\int_2^\infty \frac{e^{-s\vartheta(x)}}{(\log x)^2}\frac{dx}{x}.$$

From the second of these inequalities, and (2 . 221), it follows that

$$\phi_1(s) < \frac{1}{s}\int_2^\infty \frac{e^{-Asx}}{(\log x)^2}\frac{dx}{x} = \frac{e^{-2As}}{s\log 2} - A\int_2^\infty \frac{e^{-Asx}}{\log x}dx$$

$$= \frac{1}{s\log 2} - A\int_2^\infty \frac{e^{-Asx}}{\log x}dx + O(1);$$

and so that

(3 . 32) $$\phi(s) > A\int_2^\infty \frac{e^{-Asx}}{\log x}dx + O(1).$$

On the other hand there is a positive constant B, such that

$$\vartheta(x) < Bx \quad (x \geqslant 2).^*$$

Thus

$$\phi_1(s) > \frac{1}{s}\int_2^\infty \left\{\frac{1}{(\log x)^2} - \frac{s^2}{12}\right\} e^{-Bsx}\frac{dx}{x} = \frac{1}{s}\int_2^\infty \frac{e^{-Bsx}}{(\log x)^2}\frac{dx}{x} + O(1)$$

$$= \frac{1}{s\log 2} - B\int_2^\infty \frac{e^{-Bsx}}{\log x}dx + O(1);$$

and so

(3 . 33) $$\phi(s) < B\int_2^\infty \frac{e^{-Bsx}}{\log x}dx + O(1).$$

3 . 4. LEMMA.—*If H is any positive number, then*

$$J(s) = H\int_2^\infty \frac{e^{-Hsx}}{\log x}dx \sim \frac{1}{s\log(1/s)},$$

when $s \to 0$.

Given any positive number ϵ, we can choose ξ and X, so that

$$\int_0^\xi He^{-Hx}dx < \epsilon, \quad \int_X^\infty He^{-Hx}dx < \epsilon.$$

Now $$s\log\left(\frac{1}{s}\right)J(s) = \int_{2s}^\infty \frac{He^{-Hu}\log(1/s)}{\log u + \log(1/s)}du = \int_{2s}^{\sqrt{s}} + \int_{\sqrt{s}}^{\xi} + \int_\xi^X + \int_X^\infty$$

$$= j_1(s) + j_2(s) + j_3(s) + j_4(s),$$

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* Landau, *Handbuch*, *l.c.*

say. And we have

$$0 < j_1(s) < \frac{\log(1/s)}{\log 2} \int_{2s}^{\sqrt{s}} He^{-Hu}\, du = O\{\sqrt{s}\log(1/s)\} = o(1),$$

$$0 < j_2(s) < 2\int_0^{\xi} He^{-Hu}\, du < 2\epsilon,$$

$$j_3(s) = \int_{\xi}^{X} He^{-Hu}\, du + o(1),$$

$$0 < j_4(s) < \int_X^{\infty} He^{-Hu}\, du < \epsilon;$$

and so $\left|1 - s\log\left(\frac{1}{s}\right) J(s)\right| = \left|\int_0^{\infty} He^{-Hu}\, du - j_1(s) - j_2(s) - j_3(s) - j_4(s)\right|$

$$< 5\epsilon + o(1) < 6\epsilon,$$

for all sufficiently small values of s.

3.5. From (3.32), (3.33), and the lemma just proved, it follows that

$$\phi(s) = \Sigma\, l_n^{-s} \sim \frac{1}{s\log(1/s)}. \tag{3.51}$$

From this formula we can deduce an asymptotic formula for $\log \mathfrak{Q}(s)$. We choose N so that

$$\sum_{N<n} \frac{1}{n^2} < \epsilon, \tag{3.52}$$

and we write

$$\log \mathfrak{Q}(s) = \Sigma \frac{1}{n}\, \phi(ns) = \sum_{1 \leqslant n \leqslant N} + \sum_{N<n<1/\sqrt{s}} + \sum_{1/\sqrt{s} \leqslant n \leqslant 1/s} + \sum_{1/s<n} \tag{3.53}$$

$$= \Phi_1(s) + \Phi_2(s) + \Phi_3(s) + \Phi_4(s),$$

say.

In the first place

$$\Phi_1(s) = \frac{1+o(1)}{s\log(1/s)} \sum_1^N \frac{1}{n^2}. \tag{3.541}$$

In the second place

$$\phi(ns) = O\left\{\frac{1}{ns\log(1/ns)}\right\},$$

and $$\log(1/ns) > \tfrac{1}{2}\log(1/s),$$

if $N < n < 1/\sqrt{s}$. It follows that a constant K exists such that

(3.542) $$\Phi_2(s) < \frac{K}{s \log(1/s)} \sum_{N<n} \frac{1}{n^2} < \frac{K\epsilon}{s \log(1/s)}.$$

Thirdly, $\sqrt{s} \leqslant ns \leqslant 1$ in $\Phi_3(s)$, and a constant L exists such that

$$\phi(ns) < \frac{L}{\sqrt{s} \log(1/s)}.$$

Thus

(3.543) $$\Phi_3(s) < \frac{L}{\sqrt{s} \log(1/s)} \sum_{1}^{1/s} \frac{1}{n} < \frac{2L}{\sqrt{s}},$$

for all sufficiently small values of s.

Finally, in $\Phi_4(s)$ we have $ns > 1$, and a constant M exists such that

$$\phi(ns) < M 2^{-ns}.$$

Thus

(3.544) $$\Phi_4(s) < M \sum_{1/s<n} \frac{2^{-ns}}{n} < sM \sum_{1/s<n} 2^{-ns} < \frac{sM}{1-2^{-s}} = O(1).$$

From (3.53), (3.541)–(3.544), and (3.52) it follows that

(3.55) $$\log \mathfrak{A}(s) = \frac{1}{s \log(1/s)} \left[\{1+o(1)\} \left(\frac{\pi^2}{6} + \rho \right) + \rho' \right] + O\left\{ \frac{1}{\sqrt{s} \log(1/s)} \right\} + O(1),$$

where $$|\rho| < \epsilon, \quad |\rho'| < K\epsilon.$$

Thus

(3.56) $$\log \mathfrak{A}(s) \sim \frac{\pi^2}{6s \log(1/s)},$$

or

(3.57) $$\mathfrak{A}(s) = \exp\left[\{1+o(1)\} \frac{\pi^2}{6s \log(1/s)} \right].$$

4.

A Tauberian theorem.

4.1. The passage from (3.57) to (1.12) depends upon a theorem of the "Tauberian" type.

THEOREM A.—*Suppose that*

(1) $\lambda_1 \geqslant 0, \quad \lambda_n > \lambda_{n-1}, \quad \lambda_n \to \infty$;

(2) $\lambda_n/\lambda_{n-1} \to 1$;

(3) $a_n \geqslant 0$;

(4) $A > 0, \quad \alpha > 0$;

(5) $\Sigma a_n e^{-\lambda_n s}$ *is convergent for* $s > 0$;

(6) $f(s) = \Sigma a_n e^{-\lambda_n s} = \exp\left[\{1+o(1)\} A s^{-\alpha} \left\{\log\left(\frac{1}{s}\right)\right\}^{-\beta}\right]$,

when $s \to 0$. *Then*

$$A_n = a_1 + a_2 + \ldots + a_n = \exp\left[\{1+o(1)\} B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)}\right],$$

where
$$B = A^{1/(1+\alpha)} \alpha^{-\alpha/(1+\alpha)} (1+\alpha)^{1+[\beta/(1+\alpha)]},$$

when $n \to \infty$.

We are given that

$$(4.11) \quad (1-\delta) A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} < \log f(s) < (1+\delta) A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta},$$

for every positive δ and all sufficiently small values of s; and we have to show that

$$(4.12) \quad (1-\epsilon) B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)} < \log A_n$$
$$< (1+\epsilon) B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)},$$

for every positive ϵ and all sufficiently large values of n.

In the argument which follows we shall be dealing with three variables, δ, s, and n (or m), the two latter variables being connected by an equation or by inequalities, and with an auxiliary parameter ζ. We shall use the letter η, without a suffix, to denote generally a function of δ, s, and n (or m)*, which is not the same in different formulæ, but in all cases tends to zero when δ and s tend to zero and n (or m) to infinity; so that, given any positive ϵ, we have

$$0 < |\eta| < \epsilon,$$

for
$$0 < \delta < \delta_0, \quad 0 < s < s_0, \quad n > n_0.$$

* η may, of course, in some cases be a function of some of these variables only.

We shall use the symbol η_ζ to denote a function *of* ζ *only* which tends to zero with ζ, so that

$$0 < |\eta_\zeta| < \epsilon,$$

if ζ is small enough. It is to be understood that the choice of a ζ to satisfy certain conditions is in all cases prior to that of δ, s, and n (or m). Finally, we use the letters H, K, ... to denote positive numbers independent of these variables *and of* ζ.

The *second* of the inequalities (4 . 12) is very easily proved. For

$$(4\,.\,131)\qquad A_n e^{-\lambda_n s} < a_1 e^{-\lambda_1 s} + a_2 e^{-\lambda_2 s} + \ldots + a_n e^{-\lambda_n s}$$

$$< f(s) < \exp\left\{(1+\delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}\right\},$$

$$(4\,.\,132)\qquad A_n < \exp \chi,$$

where

$$(4\,.\,1321)\qquad \chi = (1+\delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta} + \lambda_n s.$$

We can choose a value of s, corresponding to every large value of n, such that

$$(4\,.\,14)\qquad (1-\delta) A\alpha s^{-1-\alpha} \left(\log \frac{1}{s}\right)^{-\beta} < \lambda_n < (1+\delta) A\alpha s^{-1-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}.$$

From these inequalities we deduce, by an elementary process of approximation,

$$(4\,.\,151)\qquad (1-\eta)(A\alpha)^{-1/(1+\alpha)} \lambda_n^{1/(1+\alpha)} \left(\log \frac{1}{s}\right)^{\beta/(1+\alpha)}$$

$$< \frac{1}{s} < (1+\eta)(A\alpha)^{-1/(1+\alpha)} \lambda_n^{1/(1+\alpha)} \left(\log \frac{1}{s}\right)^{\beta/(1+\alpha)},$$

$$(4\,.\,152)\qquad \frac{1-\eta}{1+\alpha} \log \lambda_n < \log \frac{1}{s} < \frac{1+\eta}{1+\alpha} \log \lambda_n,$$

$$(4\,.\,153)\qquad (1-\eta) \frac{\alpha B}{1+\alpha} \lambda_n^{-1/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)}$$

$$< s < (1+\eta) \frac{\alpha B}{1+\alpha} \lambda_n^{-1/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)},$$

$$(4\,.\,154)\qquad \chi < (1+\eta) B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha}$$

We have therefore

(4 . 16) $$\log A_n < (1+\epsilon) B \lambda_n^{\alpha/(1+\alpha)} (\log \lambda_n)^{-\beta/(1+\alpha)},$$

for every positive ϵ and all sufficiently large values of n.*

4 . 2. We have

(4 . 21) $$f(s) = \Sigma a_n e^{-\lambda_n s} = \Sigma A_n (e^{-\lambda_n s} - e^{-\lambda_{n+1} s})$$
$$= s \sum_1^\infty A_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-sx} dx = s \int_0^\infty \boldsymbol{A}(x) e^{-sx} dx,$$

where $\boldsymbol{A}(x)$ is the discontinuous function defined by

$$\boldsymbol{A}(x) = A_n \quad (\lambda_n \leqslant x < \lambda_{n+1}),\dagger$$

so that, by (4.16),

(4.22) $$\log \boldsymbol{A}(x) < (1+\epsilon) B x^{\alpha/(1+\alpha)} (\log x)^{-\beta/(1+\alpha)}$$

for every positive ϵ and all sufficiently large values of x.

We have therefore

(4 . 23) $$\exp\left\{(1-\delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}\right\} < s \int_0^\infty \boldsymbol{A}(x) e^{-sx} dx$$
$$< \exp\left\{(1+\delta) A s^{-\alpha} \left(\log \frac{1}{s}\right)^{-\beta}\right\}$$

for every positive δ and all sufficiently small values of s.

We define λ_x, a steadily increasing and continuous function of the continuous variable x, by the equation

$$\lambda_x = \lambda_n + (x-n)(\lambda_{n+1} - \lambda_n) \quad (n \leqslant x \leqslant n+1).$$

We can then choose m so that

(4 . 24) $$\frac{1}{s} = \frac{1+\alpha}{\alpha B} \lambda_m^{1/(1+\alpha)} (\log \lambda_m)^{\beta/(1+\alpha)}.$$

We shall now show that the limits of the integral in (4 . 23) may be re-

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* We use the second inequality (4 . 12) in the proof of the first. It would be sufficient for our purpose to begin by proving a result cruder than (4 . 16), with any constant K on the right-hand side instead of $(1+\epsilon) B$. But it is equally easy to obtain the more precise inequality. Compare the argument in the second of the two papers by Hardy and Littlewood quoted on p. 114 (pp. 143 *et seq.*).

† Compare Hardy and Riesz, "The General Theory of Dirichlet's Series", *Cambridge Tracts in Mathematics*, No. 18, 1915, p. 24.

placed by $(1-\zeta)\lambda_m$ and $(1+\zeta)\lambda_m$, where ζ is an arbitrary positive number less than unity.

We write

(4 . 25)
$$J(s) = s\int_0^\infty A(x)e^{-sx}\,dx = s\left\{\int_0^{\lambda_m/H} + \int_{\lambda_m/H}^{(1-\zeta)\lambda_m} + \int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m} + \int_{(1+\zeta)\lambda_m}^{H\lambda_m} + \int_{H\lambda_m}^{\infty}\right\}$$
$$= J_1+J_2+J_3+J_4+J_5,$$

where H is a constant, in any case greater than 1, and large enough to satisfy certain further conditions which will appear in a moment; and we proceed to show that J_1, J_2, J_4, and J_5 are negligible in comparison with the exponentials which occur in (4 . 23), and so in comparison with J_3.

4 . 3. The integrals J_1 and J_5 are easily disposed of. In the first place we have

(4 . 31)
$$J_1 = s\int_0^{\lambda_m/H} A(x)e^{-sx}\,dx < A\left(\frac{\lambda_m}{H}\right)$$
$$< \exp\left\{(1+\delta)B\left(\frac{\lambda_m}{H}\right)^{\alpha/(1+\alpha)}\left(\log\frac{\lambda_m}{H}\right)^{-\beta/(1+\alpha)}\right\},$$

by (4 . 22).* It will be found, by a straightforward calculation, that this expression is less than

(4 . 32)
$$\exp\left\{(1+\eta)A(1+\alpha)H^{-\alpha/(1+\alpha)}s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\},$$

and is therefore certainly negligible if H is sufficiently large.

Thus J_1 is negligible. To prove that J_5 is negligible we prove first that
$$sx > 4Bx^{\alpha/(1+\alpha)}(\log x)^{-\beta/(1+\alpha)},$$
if $s > H\lambda_m$ and H is large enough†. It follows that
$$J_5 = s\int_{H\lambda_m}^{\infty} A(x)e^{-sx}\,dx < s\int_{H\lambda_m}^{\infty}\exp\{(1+\delta)Bx^{\alpha/(1+\alpha)}(\log x)^{-\beta/(1+\alpha)}-sx\}\,dx$$
$$< s\int_0^\infty e^{-\frac{1}{2}sx}\,dx = \tfrac{1}{2},$$

and is therefore negligible.

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* With δ in the place of ϵ.

† We suppress the details of the calculation, which is quite straightforward.

4.4. The integrals J_2 and J_4 may be discussed in practically the same way, and we may confine ourselves to the latter.

We have

$$(4.41)\qquad J_4(s) = s\int_{(1+\zeta)\lambda_m}^{H\lambda_m} A(x)\,e^{-sx}\,dx < s\int_{(1+\zeta)\lambda_m}^{H\lambda_m} e^{\psi}\,dx,$$

where

$$(4.411)\qquad \psi = (1+\delta)\,B x^{\alpha/(1+\alpha)}(\log x)^{-\beta/(1+\alpha)} - sx.$$

The maximum of the function ψ occurs for $x = x_0$, where

$$(4.42)\qquad \frac{1}{s} = (1+\eta)\,\frac{1+\alpha}{\alpha B}\,x_0^{1/(1+\alpha)}(\log x_0)^{\beta/(1+\alpha)}.$$

From this equation, and (4.24), it plainly results that

$$(4.43)\qquad (1-\eta)\lambda_m < x_0 < (1+\eta)\lambda_m,$$

and that x_0 falls (when δ and s are small enough) between $(1-\zeta)\lambda_m$ and $(1+\zeta)\lambda_m$.

Let us write
$$x = x_0+\xi$$
in J_4. Then

$$\psi(x) = \psi(x_0)+\tfrac{1}{2}(1+\delta)B\xi^2\,\frac{d^2}{dx_1^2}\left\{x_1^{\alpha/(1+\alpha)}(\log x_1)^{-\beta/(1+\alpha)}\right\},$$

where $x_0 < x_1 < x$ and *a fortiori*

$$(1-\zeta)\lambda_m < x_1 < H\lambda_m.$$

It follows that

$$(4.44)\qquad \frac{d^2}{dx_1^2}\left\{x_1^{\alpha/(1+\alpha)}(\log x_1)^{-\beta/(1+\alpha)}\right\} < -K\lambda_m^{\alpha/(1+\alpha)-2}(\log\lambda_m)^{-\beta/(1+\alpha)}.$$

On the other hand, an easy calculation shows that

$$(4.45)\qquad (1-\eta)A\,s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} < \psi(x_0) < (1+\eta)A\,s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}.$$

Thus

$$(4.46)\qquad J_4 < \exp\left\{(1+\eta)A\,s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}$$
$$\times\int_{(\zeta-\eta)\lambda_m}^{\infty}\exp\left\{-L\xi^2\lambda_m^{\alpha/(1+\alpha)-2}(\log\lambda_m)^{-\beta/(1+\alpha)}\right\}d\xi$$
$$< \exp\left\{(1+\eta)A\,s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta} - M\zeta^2\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\right\}$$
$$< \exp\left\{(1+\eta-N\zeta^2)A\,s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}.$$

Since ζ is independent of δ and s, this inequality shows that J_4 is negligible; and a similar argument may be applied to J_2.

4.5. We may therefore replace the inequalities (4.23) by

$$(4.51)\quad \exp\left\{(1-\delta)A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} < s\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m} A(x)e^{-sx}\,dx < \exp\left\{(1+\delta)A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\}.$$

Since $A(x)$ is a steadily increasing function of x, it follows that

$$(4.521)\quad \exp\left\{(1-\delta)A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} < sA\{(1+\zeta)\lambda_m\}\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m} e^{-sx}\,dx,$$

$$(4.522)\quad \exp\left\{(1+\delta)A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}\right\} > sA\{(1-\zeta)\lambda_m\}\int_{(1-\zeta)\lambda_m}^{(1+\zeta)\lambda_m} e^{-sx}\,dx;$$

or

$$(4.531)\quad (e^{\zeta s\lambda_m}-e^{-\zeta s\lambda_m})A\{(1-\zeta)\lambda_m\} < \exp\left\{(1+\delta)A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}+\lambda_m s\right\},$$

$$(4.532)\quad (e^{\zeta s\lambda_m}-e^{-\zeta s\lambda_m})A\{(1+\zeta)\lambda_m\} > \exp\left\{(1-\delta)A s^{-\alpha}\left(\log\frac{1}{s}\right)^{-\beta}+\lambda_m s\right\}.$$

If we substitute for s, in terms of λ_m, in the right-hand sides of (4.531) and (4.532), we obtain expressions of the form

$$\exp\{(1+\eta)B\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\}.$$

On the other hand $\qquad e^{\zeta s\lambda_m}-e^{-\zeta s\lambda_m}$

is of the form $\qquad \exp\{\eta_\zeta\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\}.$

We have thus

$$(4.541)\quad A\{(1-\zeta)\lambda_m\} < \exp\{(1+\eta_\zeta+\eta)B\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\},$$

$$(4.542)\quad A\{(1+\zeta)\lambda_m\} > \exp\{(1-\eta_\zeta-\eta)B\lambda_m^{\alpha/(1+\alpha)}(\log\lambda_m)^{-\beta/(1+\alpha)}\}.$$

Now let ν be any number such that

$$(4.55)\qquad (1-\zeta)\lambda_m \leqslant \lambda_\nu \leqslant (1+\zeta)\lambda_m.$$

Since $\lambda_n/\lambda_{n-1}\to 1$, it is clear that *all* numbers n from a certain point onwards will fall among the numbers ν. It follows from (4.541) and

(4 . 542) that

(4 . 56) $$\exp\{(1-\eta_\zeta-\eta)(1-\eta_\zeta)B\lambda_\nu^{\alpha/(1+\alpha)}(\log\lambda_\nu)^{-\beta/(1+\alpha)}\} < A(\lambda_\nu) < \exp\{(1+\eta_\zeta+\eta)(1+\eta_\zeta)B\lambda_\nu^{\alpha/(1+\alpha)}(\log\lambda_\nu)^{-\beta/(1+\alpha)}\};$$

and therefore that, given ϵ, we can choose first ζ and then n_0 so that

(4 . 57) $$\exp\{(1-\epsilon)B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)} < A(\lambda_n) < \exp\{(1+\epsilon)B\lambda_n^{\alpha/(1+\alpha)}(\log\lambda_n)^{-\beta/(1+\alpha)}\},$$

for $n \geqslant n_0$. This completes the proof of the theorem.

4 . 6. There is of course a corresponding "Abelian" theorem, which we content ourselves with enunciating. This theorem is naturally not limited by the restriction that the coefficients a_n are positive.

THEOREM B.—*Suppose that*

(1) $\lambda_1 \geqslant 0,\ \lambda_n > \lambda_{n-1},\ \lambda_n \to \infty$;

(2) $\lambda_n/\lambda_{n-1} \to 1$;

(3) $A > 0,\ 0 < \alpha < 1$;

(4) $A_n = a_1+a_2+\ldots+a_n = \exp[\{1+o(1)\}A\lambda_n^\alpha(\log\lambda_n)^{-\beta}]$,

when $n \to \infty$. *Then the series* $\Sigma a_n e^{-\lambda_n s}$ *is convergent for* $s > 0$, *and*

$$f(s) = \Sigma a_n e^{-\lambda_n s} = \exp\left[\{1+o(1)\}Bs^{-\alpha/(1-\alpha)}\left(\log\frac{1}{s}\right)^{-\beta/(1-\alpha)}\right],$$

where $$B = A^{1/(1-\alpha)}\alpha^{\alpha/(1-\alpha)}(1-\alpha)^{1+[\beta/(1-\alpha)]}],$$

when $s \to 0$.

The proof of this theorem, which is naturally easier than that of the correlative Tauberian theorem, should present no difficulty to anyone who has followed the analysis which precedes.

4 . 7. The simplest and most interesting cases of Theorems A and B are those in which

$$\lambda_n = n, \quad \beta = 0.$$

It is then convenient to write x for e^{-s}. We thus obtain

THEOREM C.—*If* $A > 0,\ 0 < \alpha < 1,$ *and*

$$\log A_n = \log(a_1 + a_2 + \ldots + a_n) \sim An^{\alpha},$$

then the series $\Sigma a_n x^n$ *is convergent for* $|x| < 1,$ *and*

$$\log f(x) = \log(\Sigma a_n x^n) \sim B(1-x)^{-\alpha/(1-\alpha)},$$

where

$$B = (1-\alpha)\,\alpha^{\alpha/(1-\alpha)} A^{1/(1-\alpha)},$$

when $x \to 1$ *by real values.*

If the coefficients are positive the converse inference is also correct. That is to say, if

$$A > 0, \qquad \alpha > 0,$$

and

$$\log f(x) \sim A(1-x)^{-\alpha},$$

then

$$\log A_n \sim Bn^{\alpha/(1+\alpha)},$$

where

$$B = (1+\alpha)\,\alpha^{-\alpha/(1+\alpha)} A^{1/(1+\alpha)}.$$

5.

Application to our problem, and to other problems in the Theory of Numbers.

5 . 1. We proved in 3 that

$$\log \mathfrak{Q}(s) \sim \frac{\pi^2}{6s\log(1/s)}. \tag{3 . 56}$$

In Theorem A take

$$\lambda_n = \log n, \qquad A = \frac{\pi^2}{6}, \qquad \alpha = 1, \qquad \beta = 1.$$

Then all the conditions of the theorem are satisfied. And A_n is $Q(n)$, the number of numbers q not exceeding n. We have therefore

$$\log Q(n) \sim B\sqrt{\left(\frac{\log n}{\log\log n}\right)}, \tag{5 . 11}$$

where

$$B = 2^{\frac{3}{2}}\sqrt{\left(\frac{\pi^2}{6}\right)} = \frac{2\pi}{\sqrt{3}}. \tag{5 . 12}$$

5 . 2. The method which we have followed in solving this problem is one capable of many other interesting applications.

Suppose, for example, that $R_r(n)$ is the number of ways in which n can be represented as the sum of any number of r-th powers of positive integers.* We shall prove that

$$(5.21)\qquad \log R_r(n) \sim (r+1)\left\{\frac{1}{r}\Gamma\left(\frac{1}{r}+1\right)\zeta\left(\frac{1}{r}+1\right)\right\}^{r/(r+1)} n^{1/(r+1)}.$$

In particular, if $P(n) = R_1(n)$ is the number of partitions of n, then

$$(5.22)\qquad \log P(n) \sim \pi\sqrt{\left(\frac{2n}{3}\right)}.$$

We need only sketch the proof, which is in principle similar to the main proof of this paper. We have

$$\sum_1^\infty R_r(n)e^{-ns} = \prod_1^\infty\left(\frac{1}{1-e^{-s\nu^r}}\right),$$

and so

$$(5.23)\qquad f(s) = \sum_1^\infty \{R_r(n)-R_r(n-1)\}\, e^{-ns} = \prod_2^\infty\left(\frac{1}{1-e^{-s\nu^r}}\right).\dagger$$

It is obvious that $R_r(n)$ increases with n and that all the coefficients in $f(s)$ are positive. Again,

$$(5.24)\qquad \log f(s) = \sum_2^\infty \log\left(\frac{1}{1-e^{-s\nu^r}}\right) = \sum_2^\infty (e^{-s\nu^r}+\tfrac{1}{2}e^{-2s\nu^r}+\ldots)$$

$$= \sum_{k=1}^\infty \frac{1}{k}\,\phi(ks),$$

where

$$(5.241)\qquad \phi(s) = \sum_2^\infty e^{-s\nu^r}.$$

But

$$(5.25)\qquad \phi(s) \sim \Gamma\left(\frac{1}{r}+1\right)s^{-1/r},$$

when $s\to 0$; and we can deduce, by an argument similar to that of 3.5, that

$$(5.26)\qquad \log f(s) \sim \Gamma\left(\frac{1}{r}+1\right)\zeta\left(\frac{1}{r}+1\right)s^{-1/r}.$$

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* Thus $28 = 3^3+1^3 = 3.2^3+4.1^3 = 2.2^3+12.1^3 = 2^3+20.1^3 = 28.1^3$:

and $R_3(28) = 5$.

The *order* of the powers is supposed to be indifferent, so that (*e.g.*) 3^3+1^3 and 1^3+3^3 are not reckoned as separate representations.

† $R_r(0)$ is to be interpreted as zero.

We now obtain (5.21) by an application of Theorem A, taking

$$\lambda_n = n, \quad a = \frac{1}{r}, \quad \beta = 0, \quad A = \Gamma\left(\frac{1}{r}+1\right) \zeta\left(\frac{1}{r}+1\right).$$

In a similar manner we can show that, if $S(n)$ is the number of partitions of n into *different* positive integers, so that

$$\Sigma S(n) e^{-ns} = (1+e^{-s})(1+e^{-2s})(1+e^{-3s})+\ldots$$
$$= \frac{1}{(1-e^{-s})(1-e^{-3s})(1-e^{-5s})\ldots},$$

then

(5.27) $$\log S(n) \sim \pi \sqrt{\left(\frac{n}{3}\right)};$$

that if $T_r(n)$ is the number of representations of n as the sum of r-th powers of *primes*, then

(5.28) $$\log T_r(n) \sim (r+1) \left\{\Gamma\left(\frac{1}{r}+2\right) \zeta\left(\frac{1}{r}+1\right)\right\}^{r/(r+1)} n^{1/(r+1)} (\log n)^{-r/(r+1)};$$

and, in particular, that if $T(n) = T_1(n)$ is the number of partitions of n into primes, then

(5.281) $$\log T(n) \sim \frac{2\pi}{\sqrt{3}} \sqrt{\left(\frac{n}{\log n}\right)}.$$

Finally, we can show that if r and s are positive integers, $a > 0$, and $0 \leqslant b \leqslant 1$, and

(5.291) $$\Sigma \phi(n) x^n = \frac{\{(1+ax)(1+ax^2)(1+ax^3)\ldots\}^r}{\{(1-bx)(1-bx^2)(1-bx^3)\ldots\}^s},$$

then

(5.292) $$\log \phi(n) \sim 2\sqrt{(cn)},$$

where

(5.2921) $$c = r\int_0^a \frac{\log(1+t)}{t}\, dt - s\int_0^b \frac{\log(1-t)}{t}\, dt.$$

In particular, if $a = 1$, $b = 1$, and $r = s$, we have

(5.293) $$\Sigma \phi(n) x^n = (1-2x+2x^4-2x^9+\ldots)^{-r},$$

(5.294) $$\log \phi(n) \sim \pi\sqrt{(rn)}.$$

[*Added March 28th*, 1917.—Since this paper was written M. G. Valiron ("Sur la croissance du module maximum des séries entières", *Bulletin de la Société mathématique de France*, Vol. 44, 1916, pp. 45–64) has published a number of very interesting theorems concerning power-series which are more or less directly related to ours. M. Valiron considers power-series only, and his point of view is different from ours, in some respects more restricted and in others more general.

He proves in particular that *the necessary and sufficient conditions that*

$$\log M(r) \sim \frac{A}{(1-r)^{\alpha}},$$

where $M(r)$ *is the maximum modulus of* $f(x) = \Sigma a_n x^n$ *for* $|x| = r$, *are that*

$$\log |a_n| < (1+\epsilon)(1+\alpha) A^{1/(1+\alpha)} \left(\frac{n}{\alpha}\right)^{\alpha/(1+\alpha)}$$

for $n > n_0(\epsilon)$, *and*

$$\log |a_n| > (1-\epsilon_p)(1+\alpha) A^{1/(1+\alpha)} \left(\frac{n}{\alpha}\right)^{\alpha/(1+\alpha)}$$

for $n = n_p$ $(p = 1, 2, 3, \ldots)$, *where* $n_{p+1}/n_p \to 1$ *and* $\epsilon_p \to 0$ *as* $p \to \infty$.

M. Valiron refers to previous, but less general or less precise, results given by Borel (*Leçons sur les séries à termes positifs*, 1902, Ch. 5) and by Wiman ("Über dem Zusammenhang zwischen dem Maximal-betrage einer analytischen Funktion und dem grössten Gliede der zugehörigen Taylor'schen Reihe", *Acta Mathematica*, Vol. 37, 1914, pp. 305–326). We may add a reference to Le Roy, "Valeurs asymptotiques de certaines séries procédant suivant les puissances entières et positives d'une variable réelle", *Bulletin des sciences mathématiques*, Ser. 2, Vol. 24, 1900, pp. 245–268.

We have more recently obtained results concerning $P(n)$, the number of partitions of n, far more precise than (5 . 22). A preliminary account of these researches has appeared, under the title "Une formule asymptotique pour le nombre des partitions de n", in the *Comptes Rendus* of January 2nd, 1917; and a fuller account has been presented to the Society. See *Records of Proceedings at Meetings*, March 1st, 1917.]