RATE PROBLEMS.

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The following are typical of certain kinds of problems which ordinarily cause high-school pupils more trouble than almost any other kind:

1. The rate of an express train is $\frac{5}{8}$ that of a slow train, and it travels 36 miles in 32 minutes less time than the slow train does. Find the rate of each in miles and hours.

2. A and E can do a piece of work in 9 hours. After working together 7 hours, B finishes the work in 5 hours. In how many hours could each alone do the work?

3. A man bought a certain number of eggs. If he had bought 56 more for the same money, they would have cost a cent apiece less; if 24 less, a cent apiece more. How many eggs did he buy and at what price each?

4. A crew can row 10 miles in 50 minutes down stream and 12 miles in an hour and a half against the stream. Find the rate in miles an hour of the current, and of the crew in still water.

5. The fore wheel of a carriage makes 8 revolutions more than the hind wheel in going 180 feet; but, if the circumference of the fore wheel were $\frac{2}{3}$ as great and of the hind wheel $\frac{3}{4}$ as great, the fore wheel would make only 5 revolutions more than the hind wheel in going the same distance. Find circumference of each wheel.

6. A sum of money at 6 per cent interest, amounts to \$5,900 for a certain time and to \$7,100 for a time longer by 4 years. Find the principal and the time.

7. A party at a tavern found, on paying their bill, that had there been 4 more each would have paid 75 cents less; but if there had been 4 fewer, each would have paid \$1.50 more. How many were there, and how much did each pay?

8. A tank can be filled by one pipe in 9 hours and emptied by another in 21 hours. In what time will the tank be filled if both pipes be opened?

9. A laborer was engaged for 32 days on condition that for every day he worked he should receive \$1.75 and for every day he was idle he should forfeit \$1.00. At the end of the time he received \$28.50. How many days did he work and how many days was he idle?

10. The telegraph poles along a certain railway are at equal intervals. If there had been two more in each mile, the intervals between the poles would have been decreased by 20 feet. Find the number of poles in a mile.

11. If sound travels 1100 feet per second, how far away is a gun when the report of firing is heard $3\frac{1}{2}$ seconds after it occurred?

12. Two school track teams held a dual meet. One team scored 26 points, winning 7 first places and 5 second places; the other team scored 21 points, winning 6 first places and 3 second places. How many points did each place count?

13. A man hires a piano at \$5 a month of 30 days, and is to pay pro rata for any part of a month. What should he pay if he keeps the piano 51 days?

Problems like the foregoing are usually considered separately without attention being called to the fact that they all contain three

elements which are practically the same, although ordinarily called by different names in the different problems.

The equations giving the relation between the three elements are as follows, the numbers corresponding to those of the previously given problems:

1. Distance = rate $\times$ time.
2. Total amount of work = rate of work $\times$ time.
(Represented by unity.)
3. Total cost = price for one object $\times$ number of objects.
4. Distance = rate $\times$ time.
(Compound rate.)
5. Distance = Circumference $\times$ number of revolutions.
6. Total interest = rate $\times$ principal.
7. Total cost = cost per person $\times$ number persons.
8. Full tank = rate at which pipe emptied water $\times$ time.
(Unity.)
9. Total wages = wage per day $\times$ number of days.
10. Distance = length of one interval $\times$ number of intervals.
11. Distance = rate of travel $\times$ time.
12. Total number of points = number of points for a place $\times$ number of such places.
13. Total charge = rent for one day $\times$ number of days.

All of these, however, are only special cases and are all covered by the general rate equation,

$$R = rt,$$

in which,

R = Total result of process, operation, work, transaction, or whatever action of any kind is referred to in the problem.

r = rate at which the action, process, and so on, takes place (the rate must be uniform).

t = number of times the rate is employed.

In all of the problems mentioned and all others similar to them, t is greater than one.

However the same relation exists when $t = 1$, although problems of this nature do not usually involve much difficulty for the student. For example. The cost of a house and lot was \$6,500; the house cost \$3,500 more than the lot. What was the cost of each?

The writer has found by experience that high-school pupils are much interested in noting the similarity between these various kinds of problems, and in picking out problems of this nature (rate problems) from any list of miscellaneous problems. With a little practice under the direction and encouragement of the teacher, many pupils readily acquire considerable facility in adopting the general equation and general methods of solution to the particular conditions and requirements of any specific problem. Other-

wise all of these different problems are worked out separately, a solution of one of them being of comparatively little assistance in the attempt to solve one of another kind.

General Methods of Solution.

I. One unknown used.

1. Let unknown quantity represent one of the three component elements.
2. Use one of the given conditions of the problem in order to obtain an expression for a second one of the three elements.
3. Use the remaining data of the problem to form the desired equation.

Note. The information obtainable from the problem may be given either explicitly or implicitly. The equation $R = rt$ must frequently be used as an aid in forming expressions and equation.

Example, one unknown quantity. A man has 11 hours at his disposal. How far can he ride in a coach which travels $4\frac{1}{2}$ miles an hour so as to return in time, walking back at the rate of $3\frac{3}{4}$ miles an hour?

Three elements $\left\{ \begin{array}{l} \text{time} \\ \text{rate} \\ \text{distance} \end{array} \right.$

Let x = distance.

$$\left. \begin{array}{l} \frac{x}{4\frac{1}{2}} = \text{time going} \\ \frac{x}{3\frac{3}{4}} = \text{time returning} \end{array} \right\} \text{using the equation } R = rt.$$

$$\frac{x}{4\frac{1}{2}} + \frac{x}{3\frac{3}{4}} = 11.$$

II. Two unknowns used. (In this case each of the three general elements will occur twice.)

1. Let x and y represent the two magnitudes corresponding to one of the three elements.
2. Use data given by the problem to form expressions corresponding to a second one of the three elements.
3. Use the remaining data of the problem for forming the equations.

Note: See note under I.

Example, two unknown quantities: A and B run a race of 450 feet. The first heat, A gives B a start of 135 feet and is beaten by 4 seconds; the second heat, A gives B a start of 30 feet

and beats him by 3 seconds. How many feet can each run in a second?

Three elements $\begin{cases} \text{time} \\ \text{rate} \\ \text{distance} \end{cases}$

Let $x = \text{A's rate of running.}$
 $y = \text{B's rate of running.}$

$$\text{First heat } \left\{ \begin{array}{l} \frac{450}{x} = \text{A's time} \\ \frac{315}{y} = \text{B's time} \\ \frac{450}{x} - \frac{315}{y} = 4. \end{array} \right\} \begin{array}{l} \text{use equation } R = rt \\ \text{and given data.} \end{array}$$

$$\text{Second heat } \left\{ \begin{array}{l} \frac{450}{x} = \text{A's time} \\ \frac{420}{y} = \text{B's time} \\ \frac{420}{y} - \frac{450}{x} = 3. \end{array} \right\} \begin{array}{l} \text{equation } R = rt, \\ \text{and given data.} \end{array}$$

Problems of this kind can also frequently be solved by using only one unknown quantity. Thus problem 9 on page 241.

Let $x =$ number working days.

$$\begin{array}{l} \text{use } R = rt \left\{ \begin{array}{l} 32 - x = \text{number of idle days.} \\ 175x = \text{number of cents received for work.} \end{array} \right. \\ \text{and} \\ \text{given data } \left\{ \begin{array}{l} 100(32 - x) = \text{number cents forfeited.} \end{array} \right. \end{array}$$

$$175x - 100(32 - x) = 2850.$$

It is not sufficient just simply to call the attention of pupils to the analogy existing between the various types of rate problems. The teacher must first see that every pupil comprehends the meaning of "rate" and "rate problem." Then the "general rate equation" should be introduced and applied to various problems, in every case definitely noting that the three particular elements in any given problem are only special forms of the three general elements in the "general formula." The teacher should require every pupil to practice picking out rate problems from miscellaneous lists, in each case mentioning the three particular component elements involved and specifying which one of the three general elements each one of them corresponds to. The solution should then be developed on the basis of the "general formula" and "general methods of solution." If this procedure be *insisted* upon by the

teacher for a time, until the pupils have become thoroughly familiar with it and habituated to it, I venture to predict that both teacher and pupils will be agreeably surprised at the facility with which such problems will be solved ever afterwards.

TWO USEFUL THEOREMS IN GEOMETRY.

BY ALBERT BABBITT.

Theorem I. *The product of the two arms of a right-angled triangle is equal to the product of half of its perimeter by the difference between the sum of the two arms and the hypotenuse.*

Given the right-angled triangle ABC, whose arms are a and b , and the hypotenuse is c . Let $p \equiv \frac{1}{2}(a+b+c)$.

To prove that $ab = p(a+b-c)$.

$$\begin{aligned} \text{Proof} \quad p(a+b-c) &= \frac{(a+b+c)}{2}(a+b-c), \\ &= \frac{(a+b)^2 - c^2}{2}, \\ &= \frac{a^2 + b^2 + 2ab - c^2}{2}. \end{aligned}$$

Since ABC is a right-angled triangle,

$$a^2 + b^2 = c^2.$$

Hence, $p(a+b-c) = ab$, or $ab = p(a+b-c)$.

Theorem II. *The square root of the area of the middle section of a frustum of a quadrangular pyramid is equal to half of the sum of the square roots of the areas of the upper and lower bases of the given frustum.*

Let B and b denote respectively the area of the lower and upper bases of the given frustum ABCD-A'B'C'D', and let M denote the area of the middle section A''B''C''D''. Also denote the diagonals AD, A'D', and A''D'' respectively by l , k , and m .

To prove that $\sqrt{M} = \frac{1}{2}(\sqrt{B} + \sqrt{b})$.

Proof: A''D'', being the median of the trapezoid ADA'D', is equal to half of the sum of the bases, i. e.

$$m = \frac{1}{2}(k+l).$$

Hence, $k+l = 2m$.

Now, it is known that

$$\begin{aligned} \frac{B}{b} &= \frac{l^2}{k^2}, \\ \frac{b}{M} &= \frac{k^2}{m^2}, \end{aligned}$$

(taking positive value of square root only).

Whence,

$$\begin{aligned} \frac{\sqrt{B}}{\sqrt{b}} &= \frac{l}{k}, \\ \frac{\sqrt{b}}{\sqrt{M}} &= \frac{k}{m}. \end{aligned}$$

From the theory of proportion, it follows that

$$\frac{\sqrt{B} + \sqrt{b}}{\sqrt{b}} = \frac{l+k}{k}.$$

Multiplying this relation member by member by the relation