

## On Theories of Thermal Transpiration

This content has been downloaded from IOPscience. Please scroll down to see the full text.

1918 Proc. Phys. Soc. London 31 278

(<http://iopscience.iop.org/1478-7814/31/1/328>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 155.69.4.4

This content was downloaded on 01/10/2015 at 04:22

Please note that [terms and conditions apply](#).

**XX.** *On Theories of Thermal Transpiration.* By GILBERT D. WEST, M.Sc (Lond.).

RECEIVED MAY 8, 1919.

IN connection with an experimental research arising out of measurements of the pressure of light,\* the author has had occasion to consider the various theories that have been advanced to explain the phenomena of thermal transpiration. Of those considered, that formulated by Sutherland in 1896† has been most helpful. It is the object of the present Paper

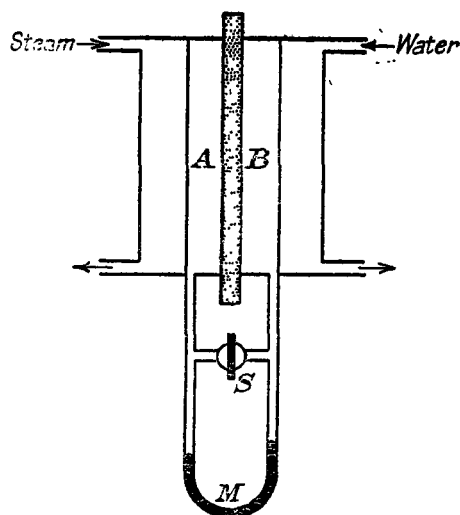


FIG. 1.

to indicate the relation of this theory to other theories, to show how it is capable of explaining work already done, and to put it into a form that will be of use in the further research it is hoped shortly to publish.

It is to Osborne Reynolds that we owe the discovery of thermal transpiration.‡ Reynolds used an apparatus somewhat similar to that shown in Fig. 1. The apparatus consists essentially of two chambers *A* and *B*, separated by a plate of porous material such as meerschaum. Means are provided for

\* "Proc." Phys. Soc., XXVIII, p. 259, 1916.

† "Phil. Mag.," XLII, p. 373, 1896.

‡ Phil. "Trans.," CLXX., p. 727, 1879.

maintaining the chambers at different temperatures.  $M$  is a mercury manometer, and  $S$  is a stopcock.

On opening the latter, the pressures in  $A$  and  $B$  are equalised. When, however, it is closed again, the pressure in  $A$  gradually rises, as the result of the passage of gas from  $B$  to  $A$  through the meerschäum. When the mean free path of the gas molecules is large compared with the size of the pores, it is found that the final pressures in the chambers  $A$  and  $B$  are proportional to the square root of their absolute temperatures,  $T_A$  and  $T_B$ . It should be noted that the nature of the gas is unimportant.

The usual explanation of this result is somewhat as follows. Suppose the porous plate replaced by a non-conducting lamina with a single perforation, small compared to the mean free path of the gas molecules. Let  $N_A$  and  $V_A$  represent respectively the number and root-mean-square velocity of the molecules in the compartment  $A$ , and likewise let  $N_B$  and  $V_B$  represent similar magnitudes in regard to the compartment  $B$ . If all the molecules be divided into Joule's six conventional sets, moving parallel and perpendicular to the lamina, we shall have  $\frac{1}{6}N_A V_A$  and  $\frac{1}{6}N_B V_B$  impacts per square centimetre respectively on each side of the lamina. When equilibrium is reached the numbers of molecules passing each way through the orifice must be the same, and hence

$$\frac{1}{6}N_A V_A = \frac{1}{6}N_B V_B.$$

Thus, if  $m$  be the mass of a molecule, the ratio of the pressure in  $A$  and  $B$  is given by

$$\frac{P_A}{P_B} = \frac{\frac{1}{3}N_A m V_A^2}{\frac{1}{3}N_B m V_B^2} = \frac{V_A}{V_B} = \sqrt{\frac{T_A}{T_B}}.$$

There is no difficulty in extending this calculation to the case where the orifice is replaced by a fine bore tube, along which a temperature gradient is maintained, and the step to a porous plate is then simple.

A straightforward explanation can thus be given of the experimental results at low pressures. When, however, with rise of pressure, the mean free path of the molecules becomes comparable to, or less than the size of the pores, difficulties arise, and the simple theory previously given, no longer holds.

An elaborate investigation of the phenomena at such pressures was given by Osborne Reynolds, but it was presented in a form so abstruse that it is doubtful to what extent it cleared the minds of most physicists. Sutherland, for instance, in

his Paper\* remarks that "unfortunately the mathematical form of Reynolds' theory is wearily cumbersome; one gathers that Maxwell found it distasteful, and Fitzgerald ("Phil. Mag.," (5) XI.) describes it as inelegant and unnecessarily elaborate . . . but what appears to me to be the fatal objection to Reynolds' mathematical method, is that it takes the mind away from definite physical concepts of the actual operation of the causes of thermal transpiration and radiometer motion." He goes on to say that the object of his own Paper "is to construct a theory . . . that will fall into line with the current kinetic theory of gases, and keep the physics of the phenomena to the fore."

Sutherland's Paper came, however, at a time when interest in thermal transpiration and cognate phenomena had died down, and it thus escaped sufficient attention. Sir Joseph Larmor in his article on "Radiometer" in the "Encyclopædia Britannica," does not even mention the Paper, and the Danish physicist, Knudsen, in a Paper† on thermal transpiration in its relation to the equilibrium conditions in a gas, does not mention the Paper, and moreover goes over a certain amount of ground that Sutherland had previously traversed.

From the point of view of the author's further research, both Sutherland's and Knudsen's Papers are of considerable importance, and, although both theories have a somewhat similar basis, the methods of development are so different that it has been thought desirable to show how, by a simple, though perhaps rather approximate, calculation based chiefly on Sutherland's methods, we can arrive at and extend Knudsen's results. More especially is this desirable, as the latter are supported by a number of carefully planned experiments.

Consider (as does Knudsen) the case of a tube along which a temperature gradient  $dT/dx$  is maintained. It is necessary to suppose this gradient to be small as otherwise  $\lambda$  the mean free path of the molecules may vary somewhat from point to point, and the calculation then becomes very difficult. It is further necessary to suppose that the gradient is constant over a distance several times  $\lambda$ , that the diameter of the tube is large compared to  $\lambda$ , and, lastly, that any gas currents that occur in the tube are small in magnitude, in order that the temperature may be considered constant over any cross-section.

\* *Loc. cit.*, p. 374.

† "Ann. d. Phys.," XXXI., 1, pp. 205-229, 1910.

Let the tube  $PQ$  (Fig. 2) connect two infinite spaces, and fix attention on a small area  $ds$  of a cross-section  $AB$ . Unless  $ds$  is very near the wall of the tube, molecules can be supposed to arrive at this area from points on the surface of a sphere of radius  $\lambda$ , and they can be divided into two classes—those with a component velocity parallel to the axis of the tube and towards the left, and those with a similar component velocity towards the right. If all the molecules in any section be supposed to have the arithmetic mean velocity  $\Omega$ , and if further, the angle  $\theta$  (measured according to the ordinary convention) represent

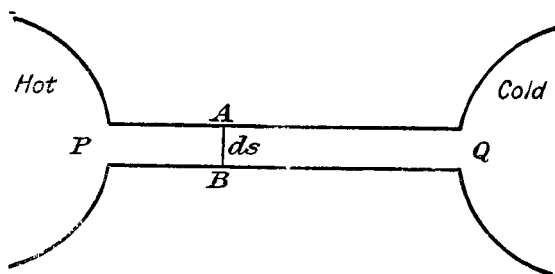


FIG. 2.

the inclination of the direction of motion of any molecule to the axis of the tube, the velocity of this molecule will be  $\Omega + \frac{d\Omega}{dx} \lambda \cos \theta$ , whether it comes from the left or right. Hence,

if from a point  $O$  we draw lines representing the velocities of all the molecules that arrive at  $ds$ , from, say, the left, we shall have a diagram similar to Fig. 3, in which

$$OR = \Omega + \frac{d\Omega}{dx} \lambda, \text{ and } OS = \Omega.$$

Calculation is now much simplified if we assume all the molecules to have the same velocity  $\Omega - \frac{1}{2} \frac{d\Omega}{dx} \lambda$ . This approximation substitutes a hemisphere of radius  $\Omega - \frac{1}{2} \frac{d\Omega}{dx} \lambda$  for the previous curved surface, and is hence more justifiable the smaller the value of  $d\Omega/dx$ . Likewise, we shall assume that if  $N$  be the number of molecules per unit volume at  $ds$ , the

number of molecules per unit volume possessing the velocity

$$\left(\Omega + \frac{1}{2}\lambda \frac{d\Omega}{dx}\right) \text{ is } \frac{1}{2} \left(N + \frac{1}{2}\lambda \frac{dN}{dx}\right).$$

On the basis of these assumptions, it thus appears that on the average the molecules at  $AB$  are made up of those arriving from distances  $\frac{1}{2}\lambda$  to the left and right of  $AB$  respectively. Now it can be shown without difficulty\* that the mean component velocity parallel to the axis of the tube of, say, all the molecules from the left possessing the velocity  $\left(\Omega + \frac{1}{2}\lambda \frac{d\Omega}{dx}\right)$  is given by

$$\frac{1}{2} \left(\Omega + \frac{1}{2}\lambda \frac{d\Omega}{dx}\right),$$

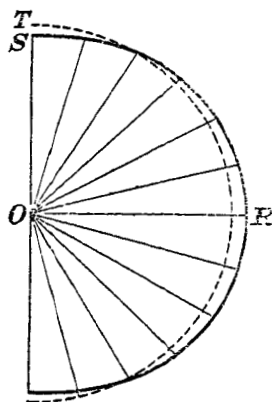


FIG. 3.

and hence the total mass flowing per second through unit area from left to right is

$$\frac{1}{2} \left(N + \frac{1}{2}\lambda \frac{dN}{dx}\right) \left(\Omega + \frac{1}{2}\lambda \frac{d\Omega}{dx}\right) m,$$

where  $m$  is the mass of a molecule, or approximately

$$\frac{1}{2} m \left\{ N \Omega + \frac{1}{2}\lambda \Omega \frac{dN}{dx} + \frac{1}{2}\lambda N \frac{d\Omega}{dx} \right\},$$

since

$$\frac{1}{4}\lambda^2 \frac{dN}{dx} \frac{d\Omega}{dx}$$

is a second order small quantity, and may be neglected.

\* Meyer, "Kinetic Theory of Gases," 2nd ed., p. 83.

Similarly, we could show that the total mass flowing per second through unit area from right to left is given by

$$\frac{1}{4}m\left\{N\Omega - \frac{1}{2}\lambda\Omega\frac{dN}{dx} - \frac{1}{2}\lambda N\frac{d\Omega}{dx}\right\}.$$

Thus, the excess mass flowing from right to left through unit area in unit time is given by

$$-\frac{1}{4}m\left\{\lambda\Omega\frac{dN}{dx} + \lambda N\frac{d\Omega}{dx}\right\},$$

or,

$$-\frac{1}{4}mN\lambda\Omega\left\{\frac{1}{N}\frac{dN}{dx} + \frac{1}{\Omega}\frac{d\Omega}{dx}\right\}.$$

We have assumed that all the molecules have the same velocity  $\Omega$ , but it is more accurate to assume them distributed according to Maxwell's law. In that case we must change  $\frac{1}{4}$  to  $\frac{3}{8}\pi/32$ , and write the previous expression

$$-\frac{3\pi}{32}mN\lambda\Omega\left\{\frac{1}{N}\frac{dN}{dx} + \frac{1}{\Omega}\frac{d\Omega}{dx}\right\}.$$

Since, however,  $p$  the pressure of a gas, is given by

$$p = \frac{\pi}{8}Nm\Omega^2,$$

it follows that

$$\begin{aligned}\frac{dp}{dx} &= \frac{\pi}{8}m\Omega^2\frac{dN}{dx} + \frac{2\pi}{8}Nm\Omega\frac{d\Omega}{dx}, \\ &= p\left\{\frac{1}{N}\frac{dN}{dx} + \frac{2}{\Omega}\frac{d\Omega}{dx}\right\}.\end{aligned}$$

And further, since according to Maxwell  $\eta$ , the coefficient of viscosity of a gas, is given by  $\eta = 0.31mN\lambda\Omega$ , we may write the excess mass moving per second through a tube of radius  $R$  as

$$-\pi R^2\frac{3\pi}{32}\frac{\eta}{0.31}\left\{\frac{1}{p}\frac{dp}{dx} - \frac{1}{\Omega}\frac{d\Omega}{dx}\right\},$$

or as

$$-2.98R^2\eta\left\{\frac{1}{p}\frac{dp}{dx} - \frac{1}{2T}\frac{dT}{dx}\right\},$$

since the temperature  $T$  is proportional to  $\Omega^2$ .

If the spaces at the ends of the tube are infinite this flow, which is uniform over the whole cross section, will continue as long as the temperature gradient is maintained. If, however, the spaces are limited, the flow will continue only until a sufficient pressure is developed on the hot side to

cause an equal flow of gas in the reverse direction. Such a counter-flow of gas, however, can only take place in conformity with the laws of flow in a capillary tube, whereby the velocity, starting from a maximum along the axis, must decrease progressively until the wall of the tube is reached. The result of the superposition of this flow on the uniform flow in the reverse direction will be to give us a gas current near the surface of the tube from the cold to the hot vessel, and a current in the reverse direction along the axis. Between the two there will be a surface of zero velocity. This superposition of two flows is the basis of Sutherland's method, and Fig. 4 is intended to give an idea of the distribution of velocities in this case.

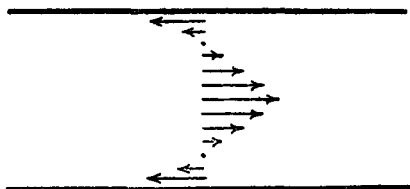


FIG. 4.

The simple formula of Poiseuille gives the mass of gas  $G$  discharged from a tube by a small pressure gradient as

$$G = \frac{\pi}{8} \frac{\rho_1 p R^4}{\eta} \frac{dp}{dx},$$

where  $\rho_1$  is the density of the gas at  $T^\circ\text{C}$ . and under a pressure of 1 dyne per square centimetre. The formula, however, takes no note of the "slip" that occurs at the surface of the tube. This latter is very important at the lower pressures, and hence a more elaborate expression is necessary. Various formulæ have been proposed,\* and they differ in points of detail, but the following serves the present purpose—

$$G = \left\{ \frac{\pi}{8} \frac{\rho_1 p R^4}{\eta} + \frac{4}{3} \sqrt{2\pi \rho_1 R^3} \right\} \frac{dp}{dx}.$$

Equating, therefore, the masses discharged in the reverse directions, we have

$$-\frac{2.98 R^2 \eta}{p} \frac{dp}{dx} + \frac{2.98 \eta}{2T} R^2 \frac{dT}{dx} = \left\{ \frac{\pi}{8} \frac{\rho_1 p R^4}{\eta} + 3.34 \sqrt{\rho_1 R^3} \right\} \frac{dp}{dx}. \quad (1)$$

\* Fisher, "Phys. Rev.," XXIX., p. 325, 1909.



Whence

$$\frac{dp}{dT} = \eta^2 \left\{ 2\eta^2 T/p + 37.1 R\eta \sqrt{\rho_0 T} + 65.1 \rho_0 p R^2 \right\}, \quad (2)$$

where  $\rho_0$  = mass of 1 cubic cm. of gas at  $0^\circ\text{C}$ . under 1 dyne/cm.<sup>2</sup> pressure.

This formula has been deduced on the assumption that the pressure is of a value such that  $R$  is large compared to  $\lambda$ , and it will be remembered that we found the molecules could be considered to come from distances  $\frac{1}{2}\lambda$  from either side of the section  $AB$  in Fig. 2. When the pressure is so low that  $\lambda$  is large compared to the diameter of the tube, this is not so. We may replace the  $\frac{1}{2}$ , however, by a quantity  $k$ , which will be very approximately constant, so long as  $\lambda$  is great.

Hence, instead of equation (1), we must now write

$$-\pi R^2 \frac{3\pi}{32} m N \lambda \Omega \left\{ \frac{2k}{p} \frac{dp}{dx} - \frac{2k}{T} \frac{dT}{dx} \right\} = \left\{ \frac{\pi}{8} \frac{\rho_1 p}{\eta} R^4 + \frac{4}{3} \sqrt{2\pi \rho_1} R^3 \right\} \frac{dp}{dx}.$$

Since, however,  $\lambda$  is by supposition much greater than  $R$ , the equation reduces in the limit to an equation independent of  $k$ , namely,

$$\left\{ \frac{1}{p} \frac{dp}{dx} - \frac{1}{2T} \frac{dT}{dx} \right\} = 0,*$$

which gives us  $\frac{dp}{dT} = \frac{p}{2T}$ .

Now, this is the limit to which the previous formula (2) reduces when  $p$  is made very small. Hence, the formula 2 applies to both high and low pressures. In regard to medium pressures, we should in strictness have to introduce an appropriate value of  $k$ , but it appears that no great error is made by leaving the formula untouched. Hence, the equation

$$\frac{dp}{dT} = \eta^2 \left\{ \frac{2\eta^2 T}{p} + 37.1 R\eta \sqrt{\rho_0 T} + 65.1 \rho_0 p R^2 \right\}$$

may be said to apply approximately to all pressures. The variation of  $dp/dT$  with gas pressure for a tube of definite radius is shown in the accompanying curves. It will be seen

\* Physically, this implies that, in the condition of equilibrium there is no flow—an assumption made in the simple explanation given at the beginning of the Paper.

that with decreasing pressure  $dp/dT$  rises, reaches a maximum and then falls.

It is now necessary to compare these results with those of Knudsen. An outline of his method is somewhat as follows :

A calculation is first made of the traction a small length of tube, along which a temperature gradient is maintained, would experience in the absence of gas currents. In a detailed calculation it is then shown how the traction is modified when the peripheral gas current from the cold vessel discharges as much as the axial current from the hot vessel. He equates the traction thus found to that calculated from the pressure gradient and the cross-section of the tube.

As a result he finds for high pressures, and for medium pressures approached from the high-pressure side,

$$\frac{dp}{dT} = \eta^2 K \left\{ 43.5 R \eta \sqrt{\rho_0 T} + 68.4 \rho_0 R^2 p \right\},$$

where  $K$  varies from 1 at high pressures to  $4/3$  at very low pressures. Likewise, for low pressures and for medium pressures approached from the low-pressure side, he finds that

$$\frac{dp}{dT} = \frac{3}{8} K p \left( 1 + \frac{2R}{\lambda} \right) T,$$

which, when  $K=4/3$  can be reduced to

$$\frac{dp}{dT} = p \eta \left\{ 2T \eta + 32.7 p R \sqrt{\rho_0 T} \right\}.$$

When  $K=1$ , 32.7 is replaced by 43.5—i.e., the value in the previous formula.

Except in so far as Knudsen's results are not combined in a single formula, they are of the same form as those obtained in this work, whilst the discrepancy of the coefficients may be explained chiefly by the variation of  $K$ . The principles laid down by Sutherland would hence appear to form an adequate basis for the more modern work of Knudsen.

The accompanying curves plotted on semi-logarithmic paper so as to secure a greater range, illustrate well the relationship of the various formulæ. At low pressures we see that both agree in making  $dp/dT$ , the pressure difference per degree difference of temperature, proportional to the pressure, and independent of the nature of the gas. Further, at high pressures  $dp/dT$  becomes inversely proportional to the pressure and dependent on the nature of the gas. Both these

results are in accord with the experiments of Osborne Reynolds, to which previous reference has been made.

For the purposes of the experimental verification of his theory Knudsen found it necessary to integrate some of his equations, in order that they might apply when large differences of temperature existed. His results calculated in this way were in good accord with those obtained by his experiments, and considerable confidence may therefore be felt in

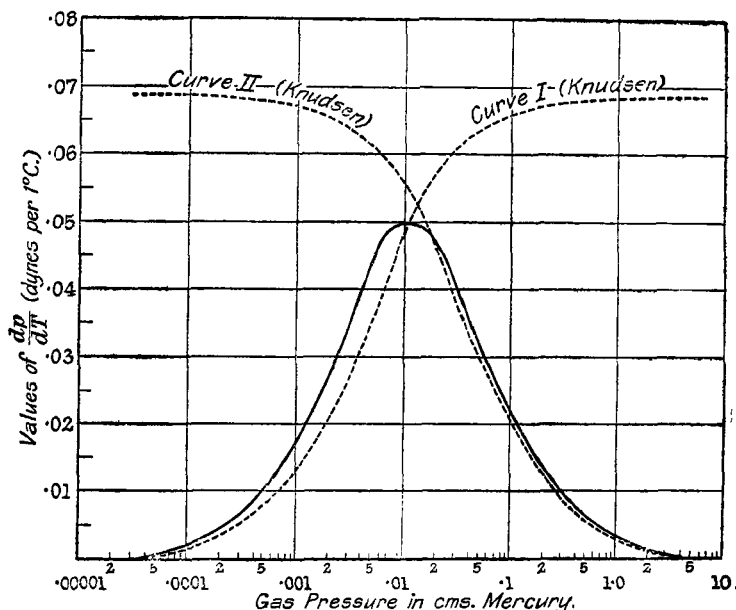


FIG. 5 — THERMAL TRANSPIRATION FORMULE FOR A CYLINDRICAL TUBE OF RADIUS 1 CM.

Full line curve given by  $\frac{dp}{dT} = \eta^2 \left\{ \frac{2\eta^2 T}{p} + 37.1 R \eta \sqrt{\rho_0 T} + 65. \rho_0 p R^2 \right\}.$

Dotted line curve I. given by  $\frac{dp}{dT} = \frac{8}{3} p \left( 1 + \frac{2R}{\lambda} \right) T$  (Knudsen).

Dotted line curve II. given by  $\frac{dp}{dT} = \eta^2 \left\{ 43.5 R \eta \sqrt{\rho_0 T} + 6.84 \rho_0 R^2 p \right\}$  (Knudsen)

them. It is hence satisfactory to find that those developed in this Paper do not differ very materially from those of Knudsen, and for approximate calculations they are thus quite adequate.

Summarising, therefore, we may say that if two vessels containing gas at different temperatures are connected by a

capillary tube, the phenomena vary according to the relation of the mean free path of the molecules to the radius of the tube.

At very low pressures, gas flows from the cold vessel to the hot vessel until a sufficient pressure is developed to check it. If the difference in temperature be small, the pressure difference is proportional to the gas pressure in the two vessels, is independent of the nature of the gas, and is given in fact by

$$dp = \frac{p}{2T} dT.$$

With increasing pressure, the pressure difference rises less rapidly, and eventually reaches a maximum. It then begins to fall off and finally diminishes inversely as the pressure, approximately according to the equation,

$$dp = \eta^2 dT / 65.1 \rho_0 p R^2.$$

In these latter stages a circulation of gas is maintained, and currents flow from cold to hot along the surface of the tube, and from hot to cold along the axis.

### *Summary.*

The process of thermal transpiration, or the establishment of a pressure difference between two vessels connected by a capillary tube and at different temperatures, takes place at all gas pressures. The explanation of the phenomena at low pressures is well known and simple. When, however, the mean free path of the molecules is of the order of, or smaller than, the diameter of the tube, the simple explanation fails and a more elaborate hypothesis is necessary. Reference is made in the Paper to the work of Reynolds, Sutherland and Knudsen. The two latter investigators have proceeded on different lines, but it is shown how Sutherland's original method of treatment can be employed to calculate a formula which is applicable to all pressures, and which is in approximate agreement with the formulæ given by Knudsen for limited ranges of pressure. It is anticipated that the present Paper will be of considerable use in the author's further research.

### DISCUSSION.

Mr. F. J. WHIPPLE observed that the author had adopted the simplification of assuming all the molecules to have the mean velocity. Had he worked out the full treatment on the basis of the Maxwellian distribution of velocities? He stated that in order to bring the formula into accord with the Maxwellian distribution  $1/4$  had to be replaced by  $3\pi/32$ . Was it

easy to show this? He had not followed the author's explanation of the circulation set up in the tube. It was easy to see why the pressure-difference flow should be faster near the axis of the tube than at the walls; but the assumption that the temperature-difference flow should be uniform across the tube seemed somewhat arbitrary.

Dr. H. S. ALLEN was glad to hear the author's tribute to Sutherland. He had come across many cases of neglect of Sutherland's work. For example, the relation between the coefficient of expansion and the specific heat of an element usually attributed to Grönroos, was first published by Sutherland, and there were many similar cases which could be quoted.

Prof. LEES congratulated the author on the simple way in which he had arrived at Knudsen's results.

The AUTHOR, in reply to Mr. WHIPPLE, said the numerical transformation referred to was simply made in conformity with the prevalent custom of replacing  $1/4$  by  $3\pi/32$  in such calculations. As regards the distribution of the temperature-difference flow, the value near the centre is obtained by integration over a hemisphere, while near the edges integration had to be performed over a smaller area—say half a hemisphere. The values obtained did not differ greatly at different parts of the cross-section.