CALCULATION OF THE CONSTANTS OF THE THREE-ELECTRODE THERMIONIC VACUUM TUBE.¹

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SYNOPSIS.

Calculation of the Constants of the Three-Electrode Thermionic Vacuum Tube or Audion.—Given the characteristic equation of such a device as

$$I_p = \kappa \left(\frac{E_p}{\mu} + E_g \right)^\eta$$

it is shown that the value of the amplification constant μ is practically independent of the density of the space charge. It therefore follows that μ can be calculated in accordance with well-known principles in electrostatics from the dimensions of the tube. The calculation of μ as well of κ is carried out for four classes of tube structures.

1. The structure with plane-parallel elements in which the filament is symmetrically placed between grids and plates;
 2. The structure with cylindrical anode and grid and a coaxial strand of filament;
 3. The structure with cylindrical anode and grid and several parallel strands of filament arranged on an internal coaxial cylinder;
 4. The structure similar to Class 3 except that the strands of filament lie on an external coaxial cylinder. These calculations result in the following formulas.
- Class 1.* Given α the distance from plate to grid, β the distance from grid to filament, n the number of grid wires per unit length, r the radius of the grid wires, and A the total plate area,

$$\mu = \frac{2\pi\alpha n}{\log_e \frac{1}{2\pi rn}},$$

$$\kappa = 2.33 \times 10^{-6} \frac{A}{\sqrt{\alpha + \beta}} \left(\frac{\mu}{\alpha + \beta(\mu + 1)} \right)^{3/2}.$$

In this and the following values for κ it has been assumed that the exponent η has the value $3/2$. This assumption in general gives good numerical agreement. *Class 2.* Given R_p , R_g , R_f the radii of plate, grid and filament respectively, and l the length of the structure,

$$\mu = \frac{2\pi R_g^2 \left(\frac{1}{R_g} - \frac{1}{R_p} \right) n}{\log_e \frac{1}{2\pi rn}}$$

$$\kappa = 14.65 \times 10^{-6} \frac{l R_p^{1/2} \mu^{3/2}}{[(R_p - R_g) + R_g(\mu + 1)]^{3/2}}.$$

Classes 3 and 4. The expression for μ is that given under Class 2. The expression for κ is

$$\kappa = 14.65 \times 10^{-6} \frac{l R_p \mu^{3/2}}{\sqrt{\pm (R_p - R_f) [\pm (R_p - R_g) \pm (R_g - R_f)(\mu + 1)]^{3/2}}},$$

where the signs are taken in such a way as to make all terms positive.

¹ The following paper was reported upon briefly at the meeting of the American Physical Society held in Philadelphia, October 11, 1919.

THE three-electrode vacuum tube or audion seems destined to play such an important part in the future development of electrical science, both as a laboratory instrument and as a device with ever-increasing industrial applications, that an attempt to establish theoretically the relations between its electrical constants and its structure and dimensions would seem appropriate.

Several years ago when the use of the audion in connection with telephone circuits was first contemplated by the engineers of the Bell System, it was realized that the satisfactoriness of the tube's operation depended to a large extent upon a proper choice of its constants. This led Dr. H. J. van der Bijl of this laboratory to conduct experiments with a view to finding empirical relations between tube dimensions and electrical properties. These relations have been of the greatest value in connection with the subsequent design of tubes for various purposes.

The following paper is an attempt to arrive at a similar set of relations by a theoretical analysis of the tube. The numerical agreement obtained in most cases is good, and is usually within the limits of variation found among a number of tubes of the same specifications.

As van der Bijl has shown² the characteristic of any vacuum tube is of the form

$$I_p = \kappa \left(\frac{E_p}{\mu} + E_g \right)^\eta,$$

where E_p , E_g are the plate and grid voltages respectively and I_p is the space current. The constant μ is known as the amplification constant, and κ is related in a simple way to the internal resistance of the tube. The object of this paper is to deduce relations between these constants and the structure and dimension of any tube. The work is divided into two main divisions, the calculation of μ and κ for the type of structure possessing plane-parallel elements, and the application of these two results to the type with cylindrical elements. There are other tube constants such as input impedance and detection which are of considerable importance, but which lie without the scope of the present paper.

PART I. a.

Calculation of the Amplification Constant μ of a Vacuum Tube with Plane-Parallel Elements.

In attempting to relate the value of μ to the dimensions of a tube, the question of the effect of the "space charge" must first be disposed of. The constant μ measures the relative importance of plate and grid poten-

¹ The following paper was reported upon briefly at the meeting of the American Physical Society held in Philadelphia, October 3, 1919.

² PHYS. REV., 12, 180, 1918.

tials, and since the effect of the space charge will be to divert to itself some of the electric flux which would otherwise go to the filament, it seems reasonable to expect, provided this charge is largely confined to the space between grid and filament, that it will divert about equal fractions of the flux originating at the plate and grid, and thus leave μ unchanged. That this is essentially true is shown by the data given in Table I. These data were taken with a tube which showed a sharp "cut off" in the E_g, I_p characteristic, the μ being determined at various filament currents by measuring the value of E_g necessary to just reduce I_p to zero for two fixed values of E_p .

TABLE I.

Total Emission from Filament.	μ .
1,400 m.a.....	15.0
240	15.2
47	15.6
10	15.8
.75	16.5

Since μ is practically independent of the presence of a space charge, it will be assumed that if ΔE_p and ΔE_g are increments of the plate and grid potentials which cause equal increments in the electric field at the surface of the cathode, the cathode being considered simply as an equi-

Fig. 1.

potential surface and not as a source of electrons, then μ will be the ratio $\Delta E_p / \Delta E_g$.

To calculate this ratio let us consider two infinite conducting parallel planes and a grid of parallel wires between them. We will suppose that the wires are of equal radius and that their radius is small compared

to the distance between adjacent wires or between the grid and either plane. Then the charge on each grid wire may be considered as concentrated along the axis of the wire. Let α and β be the respective distances from the planes to the center of any grid wire, d be the distance between the axes of adjacent wires, and r the radius of any wire. Also let V_1, V_2, V_3 be the potentials as shown in Fig. 1, and let $e_1/d, e_2/d, e_3/d$ be the densities of surface charge, e_2 being the charge per unit length of grid wire. Now let three points P_1, P_2, P_3 be chosen as shown. For convenience they are taken to lie on a perpendicular which passes through the axis of a grid wire. Call $(V_1)_{e_1}$ the potential at P_1 produced by the charge e_1 , $(V_1)_{e_2}$ the potential at P_1 produced by the charge e_2 , etc.

Now let us replace the charges e_1 and e_2 by images of the grid in the two planes, thus obtaining the system of conductors shown in Fig. 1. Then since the potential at any point due to uniform distribution of electricity along a straight line of infinite length is

$$V_P = C_0 - 2e \log \rho,$$

where e is the charge per unit length and ρ is the distance from the line to the point P , we find, after summing over the infinite number of wires, and eliminating the constant C_0 ,

$$\begin{aligned} V_1 - V_2 &= 2e_1A - 2e_3C + 2e_3D, \\ V_2 - V_3 &= -2e_1A + 2e_1B - 2e_3D, \end{aligned} \quad (1)^1$$

where

$$\begin{aligned} A &= \log_e \frac{\pi r}{d \sinh \frac{2\pi\beta}{d}}, \\ B &= \log_e \frac{\sinh \frac{\pi\alpha}{d}}{\sinh \frac{(\alpha + 2\beta)\pi}{d}}, \\ C &= \log_e \frac{\sinh \frac{\pi\beta}{d}}{\sinh \frac{(2\alpha + \beta)\pi}{d}}, \\ D &= \log_e \frac{\pi r}{d \sinh \frac{2\pi\alpha}{d}}. \end{aligned}$$

¹ The equivalent of these equations is to be found in Maxwell's *Electricity and Magnetism*, Vol. 1, page 312, his derivation being based on the properties of conjugate functions.

Or if the ratios $\pi\alpha/d$, $\pi\beta/d$, etc., are greater than 2 we can write without appreciable error,

$$\begin{aligned} A &= -\frac{2\pi\beta}{d} - \log_e \frac{d}{2\pi r}, \\ B &= -\frac{2\pi\beta}{d}, \\ C &= -\frac{2\pi\alpha}{d}, \\ D &= -\frac{2\pi\alpha}{d} - \log_e \frac{d}{2\pi r}. \end{aligned} \quad (2)$$

From equations (1) an expression for μ can readily be found. Suppose that plane (1) is the cathode of a vacuum tube and plane (3) is the anode. Then, solving equations (1) for e_1 , we have, on the basis of the assumption that

$$\mu = \frac{\partial e_1}{\partial V_2} \bigg/ \frac{\partial e_1}{\partial V_3},$$

which is evidently equivalent to the assumption that $\mu = \Delta E_p / \Delta E_g$,

$$\mu = \frac{C}{D - C};$$

or making use of the approximations in equations (2), which are probably sufficiently accurate for practical purposes,

$$\mu = \frac{2\pi\alpha}{d \log_e \frac{d}{2\pi r}}. \quad (3)$$

Letting n represent the number of grid wires per unit length equation (3) becomes

$$\mu = \frac{2\pi\alpha n}{\log_e \frac{1}{2\pi r n}}. \quad (4)$$

It is of interest to note that equation (4) gives μ as being independent of β , the distance between cathode and grid. The curves of Figs. 2 and 3, show the variation of μ with α and n as calculated by means of equation (4), and the plotted points represent experimental values.

PART I. b.

Calculation of the Internal Resistance of a Vacuum Tube with Plane-Parallel Elements.

The calculation of the internal resistance or the current corresponding to any given plate and grid potential, is somewhat complicated at the outset because of the variability of the exponent η in the equation

$$I_p = \kappa \left(\frac{E_p}{\mu} + E_g \right)^\eta. \quad (5)$$

This exponent varies in a given tube with applied voltage, being usually as great as 2 when the effective voltage $(E_p/\mu) + E_g$ is comparable with the potential drop in the filament, and apparently approaching the

Fig. 2.

theoretical value $3/2$ with increasing effective voltage. It also varies from tube to tube. Until the value of η can be foretold with considerable accuracy there is, therefore, little point to attempting great refinement in the theoretical derivation of κ of equation (5); but the following argument leads to a result which, without involving any arbitrary factor, gives a very good idea of the value of the space current which may be expected in any tube for any values of plate and grid voltage.

Imagine two structures each consisting of a plane anode and cathode (non-emitting) which are parallel and of indefinite extent, and let a grid be interposed between the anode and cathode of one structure. Suppose the distance between each anode and cathode to be $\alpha + \beta$ and the distance

between the grid and cathode to be β . Furthermore, let us suppose that the difference of potential E_p' when applied to the structure without the grid produces a certain charge density e on the cathode, and that the difference of potential E_p'' when applied between anode and cathode of the other structure (the grid being held at the same potential as the cathode) produces the same charge density e on that cathode. Since

Fig. 3.

the charge densities on the two cathodes are equal, the fields at their surfaces will also be equal. Now the assumption will be made that the space currents per unit area which the potentials E_p' and E_p'' would produce, were the cathodes capable of indefinitely large emission, would be equal. In justification of this assumption, it may be pointed out that the space charge limits the space current by reducing the field just at the surface of the cathode to practically zero, and hence it is the value of the field very near the cathode rather than at remote points that is most influential in determining the space current. Obviously the potentials E_p' and E_p'' would, even in the presence of space charges, produce practically equal fields at the respective cathodes.

As Child has shown,¹ the current density in a plane-parallel structure

¹ C. D. Child, *PHYS. REV.*, 32, 500, 1911.

without a grid is given by the equation

$$i = \frac{\sqrt{2}}{9\pi} \sqrt{\frac{e}{m}} \frac{E_p^{3/2}}{(\alpha + \beta)^2},$$

which becomes

$$i = 2.33 \times 10^{-2} \frac{E_p^{3/2}}{(\alpha + \beta)^2} \quad (6)$$

if the current is measured in amperes and potential in volts. Now, since the potentials E_p' and E_p'' correspond to equal surface charges on the cathodes, it follows from equation (1) that

$$\frac{E_p'}{E_p''} = \frac{\alpha + \beta}{\alpha + \beta(\mu + 1)}. \quad (7)$$

Hence we find for the space current density in the structure with the grid,

$$I_p = 2.33 \times 10^{-6} \frac{1}{\sqrt{\alpha + \beta}} \left[\frac{E_p}{\alpha + \beta(\mu + 1)} \right]^{3/2}. \quad (8)$$

This equation, though derived for a tube with a plane cathode, has been found to hold without change for the common type with filament cathode, provided the average distance between adjacent strands of filament is less than the distance $\alpha + \beta$. As the distance between adjacent strands is increased above the value $\alpha + \beta$, the measured value of the space current falls more and more below the value given by equation (8). The exact relation which this decrease follows has not as yet been determined. The effect is shown by the figures given for the Type V tube in Table II.

Combining equations (5) and (8) we have as the general expression for space current

$$I_p = 2.33 \times 10^{-6} \frac{A}{\sqrt{\alpha + \beta}} \left[\frac{E_p + \mu E_g}{\alpha + \beta(\mu + 1)} \right]^{3/2} \quad (9)$$

and

$$\kappa = 2.33 \times 10^{-6} \frac{A\mu^{3/2}}{(\alpha + \beta)^{1/2}[\alpha + \beta(\mu + 1)]^{3/2}},$$

in which A is the total area of the anode.

Since the space current in an audion is independent of the separation of the filament strands, within the limits noted above, it follows that it is also practically independent of the diameter of the filament.

It is interesting to note that the values of μ and I_p as given by equations (4) and (9) are independent of the unit of length used. Also it is evident that *any two audion structures, one of which is a large or small scale model of the other, have the same μ and current-voltage characteristics.* The truth of these propositions can readily be shown in other ways.

The extent to which computations based on equations (4) and (9) agree with experimental values is brought out in Table II.

TABLE II.

Average Type E Tube (VT2 of the Signal Corps)

$$\begin{aligned} \alpha &= .466 \\ \beta &= .169 \\ d_g &= .290 \\ r_g &= .010 \\ d_f &= .476 \\ L &= 12.7 \\ A &= 12.0 \end{aligned} \quad \begin{aligned} \mu &= \begin{cases} 6.6 \text{ cal'd.} \\ 6.1 \text{ obs'd.} \end{cases} \\ I_p(300,0) &= \begin{cases} 79.5 \text{ m.a. cal'd.} \\ 69.0 \text{ m.a. obs'd.} \end{cases} \end{aligned}$$

Average Type O Tube.

$$\begin{aligned} \alpha &= .228 \\ \beta &= .169 \\ d_g &= .396 \\ r_g &= .010 \\ d_f &= .476 \\ L &= 12.7 \\ A &= 12.0 \end{aligned} \quad \begin{aligned} \mu &= \begin{cases} 1.97 \text{ cal'd.} \\ 1.9 \text{ obs'd.} \end{cases} \\ I_p(120, -10.7) &= \begin{cases} 70 \text{ m.a. cal'd.} \\ 64 \text{ m.a. obs'd.} \end{cases} \end{aligned}$$

Average Type V Tube.

$$\begin{aligned} \alpha &= .466 \\ \beta &= .169 \\ d_g &= .127 \\ r_g &= .010 \\ d_f &= .792 \\ L &= 5.30 \\ A &= 12.0 \end{aligned} \quad \begin{aligned} \mu &= \begin{cases} 32 \text{ cal'd.} \\ 30 \text{ obs'd.} \end{cases} \\ I_p(100,0) &= \begin{cases} 1.65 \text{ m.a. cal'd.} \\ 1.25 \text{ m.a. obs'd.} \end{cases} \end{aligned}$$

Tube No. 65345.

$$\begin{aligned} \alpha &= .317 \\ \beta &= .317 \\ d_g &= .159 \\ r_g &= .010 \\ d_f &= .635 \\ L &= 17.8 \\ A &= 22.6 \end{aligned} \quad \begin{aligned} \mu &= \begin{cases} 13.6 \text{ cal'd.} \\ 13.0 \text{ obs'd.} \end{cases} \\ I_p(700,0) &= \begin{cases} 112 \text{ m.a. cal'd.} \\ 99 \text{ m.a. obs'd.} \end{cases} \end{aligned}$$

PART II. *a.**Constants of the Cylindrical Type of Tube.*

Tubes of the cylindrical type fall into three classes:

1. Those with cylindrical anode and a single strand of filament along the axis.
 2. Those with several parallel strands of filament arranged on the surface of a cylinder within and concentric with the anode, and whose radius is comparable with that of the anode.
 3. Those similar to class 2 except that the filament is outside of the anode.
- The results obtained in Part I. *a* can readily be extended to yield

about as accurate values for cylindrical tubes. This may be accomplished by use of the transformation known as inversion with respect to a circle. Under inversion both charge and potential remain invariant.

Consider a vacuum tube structure with plane-parallel elements of infinite extent. Let Fig. 4 represent a section of the structure, and imagine this section to be inverted

with respect to a circle as shown.

After inversion the tube elements are arranged around circles which lie tangent to each other at the center of inversion.

Now since both charge and potential have remained the same, it follows that these circles must represent a cylindrical structure with the same constants as the plane-parallel structure. In particular the section aa must have the same constants as the section AA . But the error involved in taking aa to be a section of a

Fig. 4.

structure with its center at the center of inversion is likely to be small. That is, the plane-parallel structure and the cylindrical structure shown by the dotted circles in Fig. 4 will have approximately the same constants. Hence, calling R_f , R_g , R_p the radii of cathode circle, grid circle and anode respectively, R the radius of the circle of inversion, n_g , ν_g , the number of grid wires per unit length in the cylindrical structure and in the plane structure formed by inversion respectively, r_g , ρ_g , the radii of the grid wires respectively, and α the distance between anode and grid in the plane structure, we have by equation (4),

$$\mu = \frac{2\pi\alpha\nu_g}{\log_e \frac{I}{2\pi\nu_g\rho_g}}, \quad (10)$$

in which

$$\begin{aligned} \pm \alpha &= R^2 \left(\frac{1}{R_p} - \frac{1}{R_g} \right), \\ \nu_g &= \frac{R_g^2}{R^2} n_g, \\ \rho_g &= \frac{R^2}{R_g^2} r_g; \end{aligned} \quad (11)$$

or combining equations (10) and (11)

$$\mu = \frac{\pm 2\pi n_g \frac{R_g}{R_p} (R_p - R_g)}{\log_e \frac{1}{2\pi n_g r_g}}. \quad (12)$$

It is evident that the minus sign is to be used when dealing with tubes of class 3.

This argument fits particularly the type of tube in which the grid wires are parallel to the axis; but it seems that for any given spacings of elements and grid wires, it makes little difference whether the grid wires run parallel or perpendicular to the tube axis. Obviously the more nearly equal R_f , R_p , R_g are and the closer the grid mesh, the more accurate the above result will be and it might very reasonably be expected to fail badly in the case of a tube of class 1. Such, however, does not seem to be the case, probably because μ is practically independent of the distance β .

PART II. *b*.

We will now note a certain relationship which exists between plane-parallel and cylindrical structures which contain no grids. In Fig. 5

Fig. 5.

there is represented a section of a cylindrical structure the radius of whose anode, R_p , is large compared with the radius of the cathode R_f (for our present needs R_p will be considered at least ten times as large as R_f). In Fig. 6 there is also shown a section of a plane-parallel structure, the distance between the elements being R_p and their width $2\pi R_p$. Now we know that the space current per unit length (taken perpendicular to the paper) in the plane-parallel structure is

$$\begin{aligned} i &= 2.33 \times 10^{-6} \frac{2\pi R_p V^{3/2}}{R_p^2}, \\ &= 14.65 \times 10^{-6} \frac{V^{3/2}}{R_p}. \end{aligned}$$

But this is also the space current per unit length in the cylindrical structure.¹ Hence the two structures will give equal space currents.

Now imagine a grid added to each structure in such a way that each grid is at the distance R_g from its cathode and that each structure has the same μ . Then they will again have practically equal space currents,

¹ See Langmuir, *PHYS. REV.*, 2, 459, 1913.

and it follows from equation (8) that for the cylindrical structure of

$$\text{Class 1, } I_p = 14.65 \times 10^{-6} \frac{l}{R_p} \left[\frac{(R_p - R_f)(E_p + \mu E_g)}{(R_p - R_g) + (R_g - R_f)(\mu + 1)} \right]^{3/2},$$

in which l is the length of the structure; or, since R_f is practically negligible in comparison with R_p and R_g ,

$$I_p = 14.65 \times 10^{-6} l \sqrt{R_p} \left[\frac{E_p + \mu E_g}{(R_p - R_g) + R_g(\mu + 1)} \right]^{3/2}. \quad (13)$$

Fig. 6.

Similar expressions for Classes 2 and 3 can be found by considering tubes of these types to be built up of sections of Class 1 as shown in Fig. 6. On this basis we obtain the following equations:

Class 2,

$$I_p = 14.65 \times 10^{-6} \frac{R_p l}{\sqrt{R_p - R_f}} \left[\frac{E_p + \mu E_g}{(R_p - R_g) + (R_g - R_f)(\mu + 1)} \right]^{3/2}, \quad (14)$$

Class 3,

$$I_p = 14.65 \times 10^{-6} \frac{R_p l}{\sqrt{R_f - R_p}} \left[\frac{E_p + \mu E_g}{(R_g - R_p) + (R_f - R_g)(\mu + 1)} \right]^{3/2}, \quad (15)$$

where l is again the length of the structure. These equations are independent of the number of strands of filament and may be expected to hold closely only when, upon development as indicated in Fig. 6, the distance between adjacent strands is found to be less than the distance between anode and cathode circles.

Table III. shows the numerical agreement between the above expressions for μ and I_p in several different tubes. All dimensions are again in centimeters and the currents in milliamperes. The number of strands of filament (which, in the tubes given below, were parallel and equally spaced except where noted) is represented by N .

TABLE III.

Class 1, Tube No. 1329. (French Transmitting Tube).

$$\begin{array}{ll}
 R_f = .00635 & \\
 R_g = .2 & \mu = \begin{cases} 6.9 \text{ cal'd.} \\ 7.4 \text{ obs'd.} \end{cases} \\
 R_p = .475 & \\
 d_g = .155 & \\
 r_g = .0125 & I_p(100,0) = \begin{cases} 6.1 \text{ m.a. cal'd.} \\ 4.4 \text{ m.a. obs'd.} \end{cases} \\
 l = 1.5 & \\
 N = 1 &
 \end{array}$$

Class 2, Tube No. 48128.

$$\begin{array}{ll}
 R_f = 1.03 & \\
 R_g = 1.19 & \mu = \begin{cases} 17 \text{ cal'd.} \\ 20 \text{ obs'd.} \end{cases} \\
 R_p = 1.59 & \\
 d_g = .159 & \\
 r_g = 0.127 & I_p(1100,0) = \begin{cases} 610 \text{ m.a. cal'd.}^1 \\ 120 \text{ m.a. obs'd.} \end{cases} \\
 l = 3.18 & \\
 N = 6 &
 \end{array}$$

Class 2 or 3, Tube No. 18380.

$$\begin{array}{ll}
 R_f = 1.35 & \\
 R_g = .783 & \mu = \begin{cases} 2.3 \text{ cal'd.} \\ 3.3 \text{ obs'd.} \end{cases} \\
 R_p = .635 & \\
 d_g = .371 & \\
 r_g = .019 & I_p(700,0) = \begin{cases} 917 \text{ m.a. cal'd.}^2 \\ 613 \text{ m.a. obs'd.} \end{cases} \\
 l = 12.7 & \\
 N = 15. &
 \end{array}$$

Class 3, Tube No. 67962.

$$\begin{array}{ll}
 R_f = .85 & \\
 T_g = .635 & \mu = \begin{cases} 8.6 \text{ cal'd.} \\ 8.9 \text{ obs'd.} \end{cases} \\
 R_p = .508 & \\
 d_g = .254 & \\
 r_g = .33 & I_p(750,0) = \begin{cases} 795 \text{ m.a. cal'd.} \\ 750 \text{ m. a. obs'd.} \end{cases} \\
 l = 10 & \\
 N = 16 &
 \end{array}$$

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¹ In this tube the filament strands were granged in pairs, so that the effect was practically that of three strands. If a diagram of the tube is made it will be seen that in order to make the assumption true on which the calculation of equation (14) is based, the tube should have about fifteen strands of filament. If, therefore, we divide the value of I_p given above by five, we obtain 122 m.a.

² Using the observed values of μ in place of the calculated in computing I_p gives $I_p = 640$ m.a.