XX.—An Investigation of the Seiches of Loch Earn by the Scottish Lake Survey.

Part III.: Observations to Determine the Periods and Nodes.—Part IV.: Effect of Meteorological Conditions upon the Denivellation of Lakes.—Part V.: Mathematical Appendix on the Effect of Pressure Disturbances upon the Seiches in a Symmetric Parabolic Lake. By Professor Chrystal.

(MS. received July 2, 1908. Read November 16, 1908. Issued separately November 24, 1908.)

PART III.

OBSERVATIONS TO DETERMINE THE PERIODS AND NODES.

Determination of the Periods.

In the first part of the present series of communications * I have given a general discussion of the apparatus and methods of observation employed in the investigation of the seiches of Loch Earn, and have also mentioned some of the errors to which seiche observations are liable, and how they may be avoided or corrected. It may be useful to enumerate here certain of these errors, omitting for the most part those that arise from the construction of the instruments, and can be eliminated by proper design and by preliminary tests. We have the following, arranged more or less in what experience seems to show to be the order of their ultimate importance :—

1. Errors arising from irregularities in the rate of the driving clocks of the fixed limnographs.

2. Errors due to the hygroscopic expansion and contraction of the registering paper in the fixed limnographs.

3. Changes of phase, sudden or gradual, due to disturbances generating seiches of the same period as the one under observation.

4. Irregular displacement or blurring of the turning-points on the limnogram by wind disturbances, trains of surface waves, or seiches of very short period.

5. Uncertainty of the turning-points owing to irregular frictional lagging of the axle in the index limnographs, and also owing to the discontinuity in the reading of these instruments.

6. Displacement of the turning-points by the interference of seiches of different nodality.

7. Errors of the observers' watches.

* *Trans. R.S.E.*, vol. xlv., p. 362, hereafter quoted as *I.S.E.*

In regard to these the following precautions were used :—

1 and 2. Marks were made across the continuous limnograms every twelve hours or oftener, and the corresponding time by the observer's watch, with the date, was written opposite. The rate of the driving clock was relied upon merely for the interpolation between any required turning-point and the nearest time mark. This, of course, involved a great deal of tedious calculation and measurement of the limnograms. To give an idea of the irregularities possible from this source, it may be mentioned that the extreme variation in the number of minutes per millimetre run of the recording paper in the series of observations given in Table II. below was from ·892 on 13th August to ·903 on the following day. For the most part, however, the variation of this number in two or three days did not exceed one or two units in the third decimal place. The effect of the errors arising from the present sources is, of course, much reduced by the method of interpolation followed; and it is still further minimised by taking the weighted mean of a large number of observations.

In order to test for any possible slipping of the paper between the draw-rollers, the distance corresponding to ten revolutions of the clock arbor was always measured in the neighbourhood of the part of the limnogram under consideration. In the series of observations in Table II. the extreme variation of this number was from 651·2 to 653·6. As the variation in a period of several days was often less than a millimetre, and the variation is affected by the hygroscopic as well as the slip error, probably the latter was non-existent.

3. All stretches of the limnogram which contained any sudden change of configuration or other suspicious irregularity were rejected in the determinations of period. In using long and apparently regular series of oscillations, the series was divided into two or more parts, and the periods determined from these parts compared with each other and with the period determined from the whole. As an example, attention may be called to the fine series of 375 uninodal oscillations from 23rd to 27th September (Tables II. and III. below), which was observed both at Picnic Point and near the E. Binode. Alongside of this we may place the set of 95 uninodal oscillations observed on 4th September at the Binode (Table III.). In this last case the times of the initial and final maxima were determined independently by means of an index limnograph, the Sarasin being used merely as a counter.

4 and 5. The errors from these sources were avoided by abstaining from the use of turning-points showing zigzags or unusual flatness, and employing index limnograms taken with a narrow access tube.

6. As already explained in a former communication, errors of this kind were avoided by attending to the configuration period, using turning-points where there was symmetry in the limnogram, and, in the case of index limnograms, by using the process of residuation.*

7. The watches of the various observers were compared at short intervals with my

* *I.S.E.*, p. 382.

own, and the results entered in a time-book, from which correction curves were plotted for reducing all times to the standard of my own watch, the irregularities of which were found to be much less than other sources of error.

In the tables which follow I have grouped together observations made under similar circumstances, and deduced from each a weighted mean, thinking this the most impartial way of treating the observations, with a view both to obtain the best final result and to exhibit the degree of uncertainty of the observations. I have not taken the trouble to calculate the so-called "probable error" in each case, as I have no assured faith in this method for gauging human ignorance.

To facilitate reference to the original material in the archives of the Lake Survey, I have given the date of each observation in the first column of the table. In the second column is given the mean height of the lake-surface above an arbitrary zero on the measuring-staff. In the last column are given either the limits of error, as estimated from the measurement of the limnogram, or else the number of oscillations used in the determination. The weight in taking the mean is assumed to be inversely proportional to the estimated possible error, or directly proportional to the number of oscillations counted.

I. *Observations with Index Limnographs.*

Date. 1905.		Staff. Feet.	T_1. Minutes.	Limits of Error. Minutes.
June	17	1·35	14·42	± ·10
	24	1·55	14·50	± ·10
	26	1·51	14·52	± ·10
	26	1·51	14·60	± ·07
	30	1·50	14·50	± ·06
July	1	1·40	14·46	± ·02
	8	1·35	14·45	± ·10
	10	1·35	14·51	± ·05
	25	1·15	14·59	± ·07
	25	1·15	14·64	± ·20
Aug.	7	2·20	14·58	± ·10
	22	2·40	14·34	± ·10

Weighted mean $T_1 = 14·492$.

II. *Observations with the Waggon Recorder near St Fillans (Picnic Point).*

Date 1905.		Staff. Feet.	T_1. Minutes.	Number of Oscillations.
Aug.	11	2·07	14·64	41
	13	1·95	14·30	19
	14	1·88	14·54	15
	14	1·88	14·60	39
	15	1·82	14·67	10
	15	1·82	14·56	46
	16	1·78	14·57	57
	17	1·72	14·47	49
	18	1·80	14·50	76
	20	2·27	14·55	65
	21	2·20	14·64	15
	22	2·30	14 57	45
	25	2·25	14·56	72
Sept.	2	1·90	14·58	38
	3	1·85	14·63	40
	5	1·83	14·54	71
	6	1·80	14·45	40
	17	2·80	14·49	30
	18	2·48	14·45	36
	18	2·48	14·47	66
	18	2·16	14·47	14
	18	2·16	14·55	30
	18	2·16	14·53	44
	20	2·00	14·57	27½
	23	1·82	14·52	24
	23–27	1·82–1·65	14·52	375
	24	1·80	14·53	79
	25	1·76	14·50	95
	26	1·70	14·53	100
	27	1·66	14·52	101

Weighted mean $T_1 = 14·529$.

III. *Observations with the Sarasin (at low speed) near the E. Binode.*

Date. 1905.		Staff. Feet.	T_1. Minutes.	Number of Oscillations.
Aug.	6	2·25	14·56	41
	7	2·25	14·47	42
	7	2·25	14·51	83
	8	2·15	14·52	75
	9	2·10	14·53	75
	20	2·30	14·62	48
	22	2·40	14·51	56
	22	2·40	14·45	74
	24	2·32	14·54	118
Sept.	1	2·07	14·55	30
	3	1·85	14·52	50
	4	1·87	14·53	95
	24	1 80	14·55	64
	25	1·76	14·49	106
	26	1·70	14·54	100
	23–26	1·82–1·68	14·52	270
	27	1·66	14·51	93

Weighted mean $T_1 = 14·524$.

IV. *Observations with the Sarasin (at higher speed) near the E. Binode.*

Date. 1905.		Staff. Feet.	T_1. Minutes.	Number of Oscillations.
Sept.	9	2·60	14·64	15
	14	2·60	14·54	10
	15	2·45	14·52	45
	16	2·30	14·53	10
	20	2·00	14·44	25
	22	1·87	14·51	50

Weighted mean $T_1 = 14·521$.

V. *Observations with Index Limnographs.*

Date. 1905.		Staff. Feet.	T_2. Minutes.	Limits of Error.
June	17	1·35	8·20	± ·05
July	5	1·36	8·00	± ·03
	8	1·35	8·03	± ·06
	10	1·35	8·07	± ·02
	15	1·23	8·05	± ·08
Aug.	7	2·20	8·02	± ·04
	31	2·10	8·00	± ·05
	31	2·10	8·10	± ·10
Sept.	1	2·07	7·95	± ·10

Weighted mean $T_2 = 8·054$.

VI. *Observations with the Waggon Recorder near St Fillans (Picnic Point).*

Date. 1905.		Staff. Feet.	T_2. Minutes.	Number of Oscillations.
Aug.	11	2·07	8·12	74
	14	1·88	8·08	27
	14	1·88	8·13	70
	15	1·82	8·15	18
	15	1·82	8·07	83
	16	1·78	8·065	103
	17	1·72	8·055	88
	18	1·80	8·06	138
	20	2·27	8·08	117
	21	2·20	8·13	27
	22	2·30	8·10	81
Sept.	2	1·90	8·15	68
	3	1·85	8·12	72
	5	1·83	8·06	128
	6	1·80	8·05	72
	18	2·16	8·10	25
	18	2·16	8·08	54
	18	2·16	8·09	79
	24	1·80	8·085	142
	27	1·66	8·086	169

Weighted mean $T_2 = 8·086$.

VII. *Observations with Index Limnographs.*

Date. 1905.		Staff. Feet.	T_3. Minutes.	Limits of Error.
July	8	1·35	6·00	± ·08
	10	1·35	5·99	± ·05
	27	1·12	5·95	± ·01
Aug.	7	2·20	5·96	± ·08
	30	2·15	5·99	± ·02
Sept.	5	1·80	6·06	± ·08
	5	1·80	6·08	± ·08

Weighted mean $T_3 = 6·004$.

VIII. *Observations with the Waggon Recorder near St Fillans (Picnic Point).*

Date. 1905.		Staff. Feet.	T_3. Minutes.	Number of Oscillations.
Aug.	12	2·07	6·14	36
	13	1·95	5·89	29
	21	2·20	5·91	45
	29	2·22	5·98	36
	29	2·22	6·01	48
	30	2·15	6·07	65
	31	2·10	6·00	60
Sept.	3	1·85	5·98	36
	23	1·82	6·008	58

Weighted mean $T_3 = 6·005$.

IX. *Observations with Index Limnographs.*

Date. 1905.		Staff. Feet.	T_4. Minutes.	Limits of Error.
Aug.	31	2·10	4·03	± ·05
	31	2·10	3·94	± ·10
Sept.	30	?	3·97	± ·04

Weighted mean $T_4 = 3·99$.

X. *Observations with Index Limnographs.*

Date. 1905.		Staff. Feet.	T_5. Minutes.	Limits of Error.
July	8	1·35	3·61	± ·15
	27	1·12	3·45	± ·04
Sept.	2	1·90	3·47	± ·05

Weighted mean $T_5 = 3·48$.

XI. *Observation with an Index Limnograph.*

Date. 1905.	Staff. Feet.	T. Minutes.	Limit of Error.
Sept. 2	1·90	1·36	± ·10

XII. *Observations with the Waggon Recorder near St Fillans (Picnic Point).*

Date. 1905.	Staff. Feet.	T. Minutes.	Number of Oscillations.
Aug. 18	1·90	1·70	50
29	2·22	1·15	11
30	2·20	1·39	84
Sept. 3	1·85	2·88	36

XIII. *Observations with the Sarasin, near E. Binode.*

Date. 1905.	Staff. Feet.	T. Minutes.	Number of Oscillations.
Aug. 17	1·70	1·310	124
19	2·25	1·296	121
Sept. 4	1·87	1·315	100
21	1·90	1·09	20

Weighted mean of first three T = 1·311

XIV. *Comparison of Calculation with Observation.*

ν.	T_ν by H.T.S.	T_ν by Du Boys.	T_ν observed.
1	14·50	17·81	14·52
2	8·14	8·91	8·09
3	5·74	5 94	6·01
4	4·28	4·45	3 99
5	3·62	3·56	3·48–3·60
6	2·93	2·97	2·88
7	...	2·55	...
8	...	2·23	...
9	...	1·98	...
10	...	1·78	1·70?
11	...	1·62	...
12	...	1·48	1·54?
13	...	1·37	1·36?
14	...	1·27	1·31?
15	...	1·19	1 15?
16	...	1·11	1·09?
17	...	1·05	...

During the observations the mean level of Loch Earn varied through a range of nearly 20 inches (over 50 centimetres); but a careful examination of the Tables I.–VIII. does not appear to show any correlation between the depth of the lake and the various periods. It follows that in the case of Earn, within the range of our observations, the periods are independent of the depth.

From the theoretical point of view, there is nothing surprising in the result just arrived at. Let us consider elongated lakes of uniform breadth, and assume that the same normal curve continues to represent the lake-basin when the mean level rises or falls. For a lake whose longitudinal section is a rectangle $T_\nu = 2l/\nu\sqrt{(gh)}$. Hence, since in this case l is constant, as h increases all the periods diminish. If the longitudinal section is parabolic, then $T_\nu = \pi l/\sqrt{\{\nu(\nu+1)gh\}}$.* In this case l is proportional to $\sqrt{h}$; hence all the periods are independent of the depth of the lake. It is easy to see from the analysis in *H.T.S.*, p. 628, that the same is true for a biparabolic lake. If the longitudinal section be rectilinear and symmetrical, shelving at both ends, then $T_\nu = 2\pi l/j_\nu\sqrt{(gh)}$.† In this case l is proportional to h, and j_ν is a mere number depending only on the nodality; hence T_ν is proportional to $\sqrt{h}$—that is to say, all the periods increase when h increases. Generally speaking, we may expect the rise of the mean level in a lake to increase its periods if the rise greatly increases the horizontal surface of the lake; and to decrease the periods, if the rise increases the horizontal surface very little. It appears from the observations of FOREL‡ and EBERT§ that the Lake of Geneva and the Starberger See belong to the latter category; and HALBFASS‖ has found that the Madü See belongs to the former. Loch Earn occupies an intermediate position; the constancy of its periods is therefore an indication that the assumption of a biparabolic normal curve is a good first approximation.

If it be admitted that the periods are independent of the depth, the process of taking means in Tables I.–VIII. is logically unobjectionable; and, as the identity of the uninodal, binodal, and trinodal seiches was fully established by phase observations, it only remains to select what on the whole appear the probable results of observation for T_1, T_2, T_3. Taking all the circumstances into account, I incline to the values

$$T_1 = 14{\cdot}52, \quad T_2 = 8{\cdot}09, \quad T_3 = 6{\cdot}01.$$

The identification of the periods in Tables IX. and X. as quadrinodal and quinquinodal respectively rests merely on the comparatively close agreement of the numbers in each table with each other and with the quadrinodal and quinquinodal periods deduced from the hydrodynamical theory and from Du Boys' formula. No phase observations were available to assist the identification.

* *H.T.S.*, p. 622.

† *H.T.S.*, p. 638.

‡ *Le Léman*, t. ii., p. 122.

§ "Periodische Seespiegelschwankungen beobachtet am Starnberger See," *Sitz. Ber. kgl. bayer. Akad. d. Wiss*, Bd. xxx. (1900), p. 453.

‖ "Stehende Seespiegelschwankungen im Madüsee in Pommern," *Zeitschrift für Gewässerkunde*, Bd. vi., p. 95.

There is still greater uncertainty regarding the periods in Tables XI.–XIII., most of which rest only on a single series. Possibly T = 2·88 is the sextinodal period. It must, however, be borne in mind that the smaller the period the greater is the danger of confusion with progressive wave disturbances, with possible transversal seiches, or even with secondary local oscillations due to indentures in the shore of the lake.

Determination of the Nodes.

The difficulties anticipated * in determining the nodes by direct observation were more than realised in practice. When the range of the seiche is large, there is nearly always a great deal of wind-embroidery of an irregular character, which it is impossible to eliminate either by damping the limnograph or by residuating the limnogram. Also, where the amplitude is small, there is almost always an aperiodic variation of the lake level, probably due to the heaping up of the water on the shallow shore, an effect which will vary with the slope of the beach. The varying slope also affects the range of the seiche to an extent which it would be difficult to calculate with any degree of accuracy. Both these causes introduce uncertainty in the method of observing with index limnographs on two sides of the node where the seiche is found in opposite phases, and then deducing its position by interpolation. A mere null method would scarcely lead to a satisfactory result, unless under exceptional circumstances which did not occur during our observations. Of the many attempts made, only a few led to limnograms which could be utilised; and in every case the curves had to be purified by residuation.

Uninode.—The two best pairs of observations gave almost exactly the same position for the southern end of the uninode, and led to the conclusion that it lies about 105 yards west of the position given in my paper on the Calculation of the Periods and Nodes of Lochs Earn and Treig. † This is precisely what was expected, as the actual normal curve (*C.*, p. 825) rises above the assumed biparabolic curve on the west and falls below it on the east of the calculated position of the uninode. It would be useless to calculate what the amount of divergence ought to be; because the uncertainty of one of these determinations, as shown by the observations themselves, is ±65 yards, and of the other ±129 yards, the latter being more than the divergence itself. The exact agreement of the two determinations is probably an accident.

Eastern Binode.—Two determinations agreed almost exactly in placing the Eastern Binode about 117 yards west of the calculated position; but the uncertainty of these determinations was ±94 yards in the one case and ±59 yards in the other.

Western Binode.—The best pair of observations gave a position for the southern end of the Western Binodal line 305 yards west of the calculated position. A divergence in this direction was to be expected from the shape of the true normal curve in the neighbourhood; but the amount is somewhat surprising. There can be little

* *Trans. R.S.E.*, vol. xli., p. 850.

† Henceforth referred to as "*C.*"

doubt of the correctness of the observation, because it was confirmed by another pair of observations, one made almost exactly at the position above indicated, the other 250 yards farther east. The latter gave on residuation a well-marked binodal seiche, the former none that could be recognised.

Eastern Trinode.—The best observation available places the southern end of the Eastern Trinodal line 88 yards west of the calculated position. The uncertainty of the determination, however, exceeds 120 yards; so that we cannot say for certain whether the actual trinode is really west or east of the calculated position; from the shape of the normal curve we should expect a considerable divergence to the east.

Middle Trinode.—Unfortunately, the observations for the determination of the Middle Trinode were rendered useless by casual wind-disturbances.

Western Trinode.—No observations of sufficient accuracy are available.

PART IV.

EFFECT OF METEOROLOGICAL CONDITIONS UPON THE DENIVELLATION OF LAKES.

General Character of the Seiches on Loch Earn.

Owing to the comparatively regular shape of its basin and the fact that the depth is considerable compared with the length, the seiches on Loch Earn are very regular and very persistent. Also, probably because its longest axis is more or less parallel to the paths of the major and minor atmospheric disturbances,* Earn is very rarely free from seiches. During 1070 hours, from 10th August to 28th September, the Waggon recorder at Picnic Point was almost constantly in action; yet only $2\frac{1}{2}$ hours of calm † were recorded. During 1350 hours, from 12th October to 7th December, while the Waggon recorder was in action at Lochearnhead, there were in all about 90 hours of calm. Of these, 81 hours were made up by continuous stretches of 21^h, 37^h, and 23^h on 4th, 16th, and 20th November.

The greatest ranges observed in August and September were 79 mm., 66 mm., 73 mm., 55 mm., 55 mm., 63 mm., on 19th and 21st August and 3rd, 7th, 8th, and 9th September. Only one very exceptional range was observed between 12th October and 7th December, viz. 55 mm. on 7th December.

The range of the seiche at St Fillans is usually over 10 mm. A rough estimate showed that during the 1070 hours of observation at Picnic Point the range of the seiche was over 30 mm. during 214 hours; and during the 1350 hours at Lochearnhead it was over 30 mm. during 57 hours only. It follows that, whether we test by hours of calm, by hours of excess over 30 mm., or by occurrence of exceptional ranges, the period

* See my paper, "On the Theory of the Leaking Microbarograph, etc.," *Proc. R.S.E.*, vol. xxxviii., p. 454 (1908).

† *I.e.* whole range of seiche less than 2 mm.

from 12th October to 7th December showed much less seiche activity than the period from 10th August to 28th September.

In more or less settled weather, by far the commonest seiche configuration on Earn is a uninodal and binodal dicrote.* This varies between the two extremes where the binodal on the one hand and the uninodal on the other are scarcely noticeable; but the seiche in our observations was hardly ever either purely uninodal or purely binodal. In these seiches the 5–9– configuration period caused by the interference of the uninodal

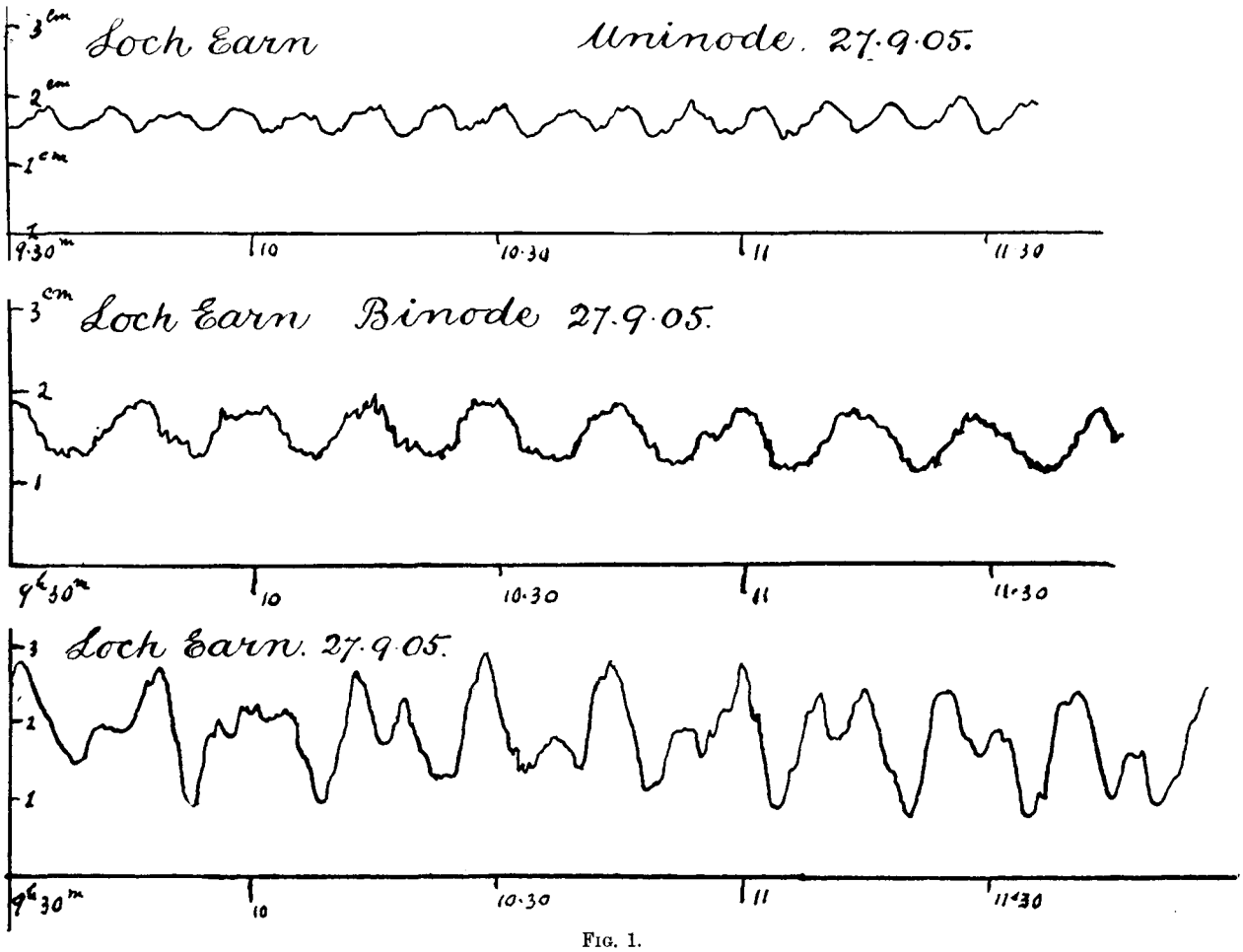


FIG. 1.

and binodal components is usually reproduced with the most beautiful regularity, sometimes for a whole day or even longer. For example, in the seiche observed at Lochearnhead from 16th to 22nd October 1905, which lasted about $6\frac{1}{2}$ days, say for 127 configuration periods, only six of these periods were found too short by one uninodal, and three too long by the same amount. It is probable that the gradual change of phase accompanying the rise and fall of the amplitudes of the components more than com-

* For brevity, in what follows we shall denote such a seiche by "UB-dicrote." Similarly, "UBT-tricrote" would mean a tricrote seiche with uninodal, binodal, and trinodal components; and we shall occasionally denote the amplitudes (half ranges) of these components by U, B, T respectively.

pensated for the fact that 9/5 is not so close an approximation to T_1/T_2 as is the sixth convergent, 70/39.

In times of storm or even moderate wind there is of course a strong embroidery of various kinds; but usually the UB-dicrote configuration can be seen through all the confusion, and it soon becomes the prominent feature when the weather begins to settle.

At this point we may indicate how the lake can be made to analyse its own seiches. Fig. 1 shows three simultaneous limnograms, the lowest one taken at the Picnic Point, about 480 yards from the eastern end of the lake, the middle one taken near the binode, the uppermost one taken near the uninode. All three are somewhat embroidered by the wind, but the St Fillans seiche is a UB-dicrote, the middle one a nearly pure uninodal, and the uppermost one a nearly pure binodal. The figure is at once an interesting confirmation of FOREL's theory and a verification of the approximate accuracy of the mathematical theory of Loch Earn regarded as a biparabolic lake.*

COMPARISON OF EARN WITH TAY AND LUBNAIG.

The seiches on Loch Tay present the strongest possible contrast to the seiches of Earn. No clear dicrote or other easily recognisable configuration is ever seen. Often it is not easy even to recognise the uninodal seiche. The contrast may be partly realised by looking at the two pairs of limnograms in fig. 2. In the first pair is seen the beginning of the long seiche on Earn above mentioned, alongside of the simultaneous seiche on Tay—which was the most regular one found on that lake between 4th October and 9th November 1905. The second pair is the end of the long seiche on Earn with the simultaneous one on Tay, whose irregularity is typical.

As yet our knowledge of the seiches and meteorological conditions of Loch Tay is not sufficient to enable us to explain this difference; but we may point out here that Loch Tay is relatively a shallower lake than Earn; it is more crooked; and the relation of its axial line to the path of the minor atmospheric disturbances is different.

This divergence of conditions occurs in an exaggerated form in the case of Lubnaig, which is very shallow, has a very irregular basin, and lies across the path of the atmospheric disturbances. Accordingly, we found only four cases in which we could recognise a definite seiche in Lubnaig, having a period of about 24 min.; and in each case only a few undulations could be counted. One of these seiches is shown in fig. 3. During the rest of the six weeks of observation nothing was found but wind-embroidery and sub-permanent wind denivellation, such as would be naturally expected in a shallow lake. About this negative result there seems to be little room for doubt,

* The following method of roughly analysing a dicrote seiche which is tolerably pure and shows the 5/9 configuration may be mentioned here. If y_0 be the *minimum minimorum* of the ordinates of the limnogram, y_1, y_2, y_3 the ordinates at distances $\frac{5}{8}T_1$, $\frac{5}{4}T_1$, and $\frac{5}{2}T_1$ from y_0, and if A be the ordinate of mean level, U and B the amplitudes (semi-ranges) of the uninodal and binodal components, then $A=\frac{1}{2}(y_0+y_3)=y_2$, $U+B=\frac{1}{2}(y_3-y_2)$, $U-B=\sqrt{2}(y_1-A)=1{\cdot}414(y_1-A)$.

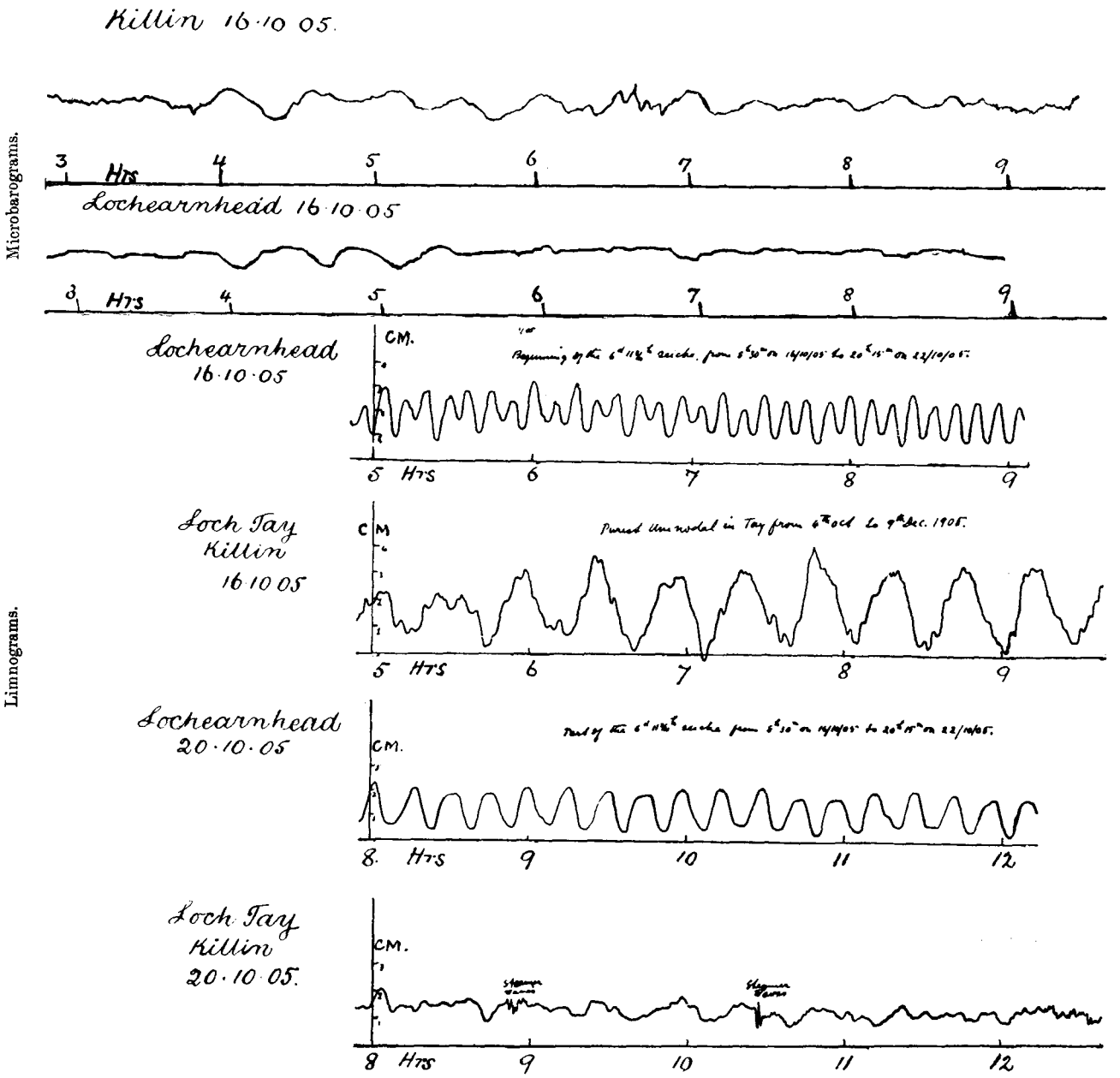

Microbarograms.
Killin 16·10·05.
Hrs
Lochearnhead 16·10·05
Lochearnhead 16·10·05
CM.
Loch Tay Killin 16·10·05
Lochearnhead 20·10·05
Loch Tay Killin 20·10·05.
Limnograms.

FIG. 2.

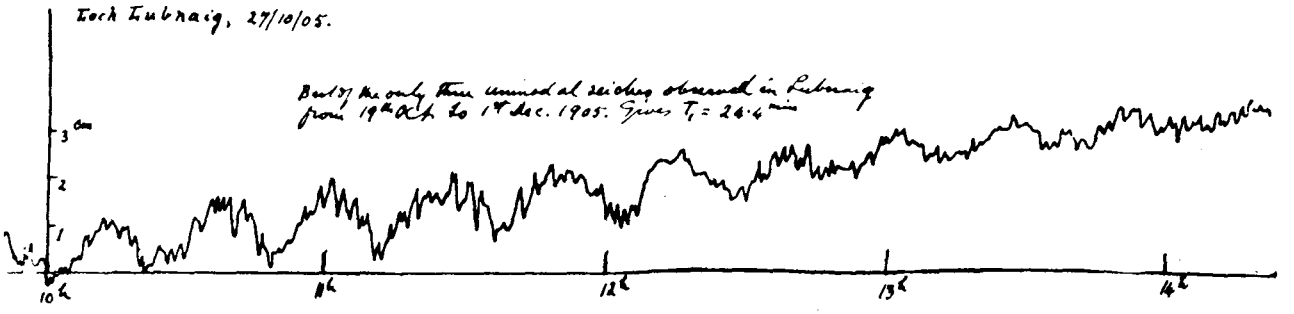

Loch Lubnaig, 27/10/05.

FIG. 3.

as the indications of the converted Sarasin limnograph were controlled by occasional observations with the much more delicate index limnograph.

For our disappointment in Tay and Lubnaig we find consolation in the beautiful seiche behaviour of Earn, which we regard as a small but elegant daughter of Geneva, the great mother of seiches.

Origin of Seiches.

Forel and his followers, Du Boys, Von Cholnoky, and others, have discussed the causes of seiches; and recently Endrös, in his important memoir on the Chiemsee, has confirmed the conclusions of his predecessors, and added some fresh details of great interest. In what follows we shall not advance anything of great novelty; but there are two points of interest that may be worthy of the reader's notice. In the first place, the use of the Dines-Shaw microbarograph enabled us to follow continuously the minute variations of the atmospheric pressure with an ease and certainty hitherto unattainable.* Also, in an appendix to this memoir the mathematical theory of the effect of pressure disturbances of various kinds on an ideal lake, of form not very remote from Earn, has been worked out, so as to show that the usually assigned cause of seiches, viz. the minor local fluctuations of the barometric pressure, is in reality sufficient to cause the disturbances observed, and is not a negligible quantity on ordinary lakes, such as the tidal action of the moon can be shown to be.†

Regarding those causes of seiches which have never yet been proved to be other than accidental, it may be of interest to record the fact that during our observations, viz. on 21st September 1905, at $23^h\ 33^m$, we were favoured with what Dr C. Davison,‡ in a paper on the Ochil Earthquakes, calls a "principal earthquake." The estimated duration of the shock was 3·4 sec. Some of the members of my family who heard it, took it for the rumble of a train passing at an unusual hour on the opposite side of Loch Earn. The centre of disturbance seems to have been about 19 miles S. 39° E. of St Fillans; and the normal to Dr Davison's isoseismal 4 makes an angle of about 63° with the axis of Loch Earn.

At the moment the Waggon recorder was not working; but the converted Sarasin at the binode was running at high speed (158 mm. per hour) and giving a smooth trace. The circumstances were as favourable as could be conceived for showing any seiche disturbance due to the earthquake; but none can be identified. There is, of course, no reason to expect that the rapid oscillations of ordinary earthquakes could cause seiches. Still, negative evidence in special cases is not without value; because in exceptional cases, such as the Lisbon earthquake of 1755, seiches have been produced, and we do not as yet know the reason why.

* A separate account of the observations with the microbarographs has been published in the *Proceedings* of the Society, vol. xxviii., p. 437 (1908).

† Except in very large lakes, such as Erie. See Endrös, *Petermanns Geogr. Mittheilungen*, 1908, Heft ii., p. 16.

‡ *Quart. Jour. Geol. Soc.*, lxiii., p. 366 (1907).

Observers are now agreed that the development of seiches usually accompanies local disturbances of the barometric pressure whose duration if they are transitory, or period if they are periodic, does not differ greatly from the period of the seiche in question. Our observations on Earn fully bear out this conclusion. Disturbance on the microbarogram is always accompanied by disturbance on the limnogram, although the magnitudes do not always correspond. Sometimes a violent disturbance on the microbarogram is accompanied by a moderate or slight disturbance on the limnogram; and occasionally the disturbance on the limnogram is much greater than might at first sight be expected from the microbaric disturbance. The mathematical theory (and indeed common sense apart from recondite theory) indicates the reason for this. If an increase of pressure operates on one half of a symmetric parabolic lake during half the period of the uninodal seiche while the water in that half is falling, it will evidently work the whole time towards increasing the amplitude of the seiche. Also, if there were to be increase of pressure for half-periods alternately on the two sides of the uninode, always tending to drive the water in the direction in which it was going, it is obvious that a very small increase might end by producing a very large seiche. As a matter of fact, we do find occasionally a considerable rise of seiche when the microbarogram is comparatively smooth; but in such cases a closer examination usually shows a faint undulation with a period not very different from that of the seiche which is generated.

On the other hand, if we suppose our increase of pressure to act on one half of the parabolic lake during the whole period of the uninodal seiche, or if it is distributed equally on both sides of the uninode, it is easy to see that the final result in altering the range of the seiche will be nil, however long the increase of pressure may act.

Absence of microbaric disturbance is accompanied by absence of seiche disturbance; that is to say, either there is no seiche at all, or an existing seiche continues unaltered. Under these circumstances the limnograms from Earn are of great beauty. As an example, we may mention a record taken by the converted Sarasin near the binode from 23rd to 27th August. This shows a regular uninodal seiche with an average range of 6 to 7 millimetres, which continued for over 89 hours. During all that time the microbarogram shows only very slight disturbance—faint undulations, occasionally periodic. The range of the seiche is not absolutely constant, but sometimes rises and sometimes falls gradually, the minimum being, say, 4·5 mm. and the maximum 8 mm. (corresponding to 7·7 mm. and 13·6 mm. at St Fillans). There is nowhere any sudden change of phase.

Examination of the limnograms shows that seiches may be generated "suddenly," *i.e.* attain their full range in one or two oscillations, or may be generated "gradually," *i.e.* the full range may be attained only after a considerable number of oscillations.

Among the causes that might generate seiches suddenly we may consider the following:—

1. The sudden release of a static denivellation of the whole lake-surface, due to the progression of the general system of the atmospheric isobars.

2. Sudden release of a denivellation caused by the transport of water from one end of the lake to the other by a wind which has blown in one direction for a time and then fallen calm or reversed its direction.

3. A sudden denivellation in one part of the lake due to very rapid flooding.

4. A sudden denivellation due to a heavy fall of rain, snow, or hail over a part of the lake. This might be partly static, *i.e.* due merely to the gravitation of the precipitated water; or it might be partly dynamic, *i.e.* due to the impact of the precipitated water.

5. Sudden alteration of the atmospheric pressure, due to the passage over parts of the lake of a local atmospheric disturbance (squall), such as is indicated by a disturbance on the microbarogram.

6. The impacts of wind-gusts on the lake-surface.

Among causes that might be expected to generate seiches gradually may be mentioned:—

7. The action over portions of the lake-surface of small fluctuations of the barometric pressure which happen to synchronise more or less nearly with some of the seiche periods of the lake.

8. Action similar to last of fluctuation in the velocity and pressure of the wind, as shown in the anemogram.

1. *Effect of the Progression of the General System of the Isobars.*—In order to form an idea of the potency of cause 1, let us take an extreme case. The greatest gradient noticed on the weather charts for August and September 1905 was 2·5 mm. of mercury, *i.e.* 34 mm. of water, in about 30 sea miles. Taking the length of Earn as 6 miles, this would give a difference of pressure between the two ends of 6·8 mm. of water. At a distance of about 50 miles on the chart the gradient had fallen by about one-fifth. If we take an extreme supposition, viz. that the system of isobars travelled with a velocity of 30 (mile/hour) in the direction of maximum gradient, which we further assume to be in the axis of Loch Earn, then the decrease of pressure difference in an hour would be $6{\cdot}8 \times 3/25$. A variation of this kind (if we suppose the gradient uniform over Earn) can only generate the uninodal of Earn,* the period of which we may take roughly to be 15^{m}. If now we suppose the time of action to be the most favourable, viz. $7\frac{1}{2}^{m}$, and the increase of the gradient to be uniform in time, then, by the result given in the Appendix on p. 513, the increment in the range of the uninodal seiche is

$$6{\cdot}8 \times 3/25 \times 16 = {\cdot}051 \text{ mm.}$$

An alteration of this amount would of course be invisible on our limnograms. It seems hopeless, therefore, to look for an explanation of ordinary seiches in the variations of the general system of isobars shown in the daily weather charts.

2. *Effect of Wind Denivellation.*—It is well established by the researches of Sir

* See Part V., p. 513.

JOHN MURRAY that a wind which has prevailed for some time causes transport of the water of a lake in the direction in which the wind is blowing; and the observations of VON CHOLNOKY on Lake Balaton show that in shallow lakes this wind denivellation may be considerable, and that its sudden release may give rise to seiches.

After a long and careful examination of our limnograms, we have arrived at the conclusion that this kind of denivellation is very small on Loch Earn under ordinary circumstances, and is rarely an effective cause of seiches. It is, however, not easy to judge of this matter. When the wind is light, the effect is very small, and cannot be separated from the denivellations due to precipitation and evaporation, and to variations in the barometric gradient. When the wind is high it is usually accompanied by considerable fluctuations of the barometric pressure, or by rainfall, or by both; and again the difficulty of separating the causes arises. That wind denivellation should be small on Earn is not surprising, for, looking at the ratio of its depth to its length, we must classify it as a deep lake; and in such lakes, as is now well known, the return undercurrent readily forms, and prevents the accumulation of wind denivellation.

It may be of interest to record one or two of the more striking cases which were examined.

On 18th August at $16^h\ 46^m$ the microbarograph showed a nearly uniform increase of 8·8 mm. (water) in the atmospheric pressure in about half an hour. At $17^h\ 5^m$ the wind had fallen dead calm, and so continued for about 25^m. Then in 12^m to 13^m it rose to a mean velocity of 25 (mile/hour), with an extreme of 38. Thereafter a mean velocity of 18 to 20 was maintained for over three hours. The squall at $17^h\ 30^m$ was very violent. One of my boys was out in a boat on the lake, and saw a large solitary wave travel up from west to east. The water was calm in front, but very rough behind; and after the wave came the strong wind. He estimated the height of this wave at 2 feet; but, as he was badly scared by the difficulty experienced in navigating his boat, he most likely exaggerated.

The effect of this remarkable wind-squall on the limnogram is comparatively slight. There is a rise of 11 mm. or so, which took over two hours to develop. There is some increase, but not much, of the characteristic wind embroidery, and a considerable disturbance in the phases of the somewhat irregular UB-dicrote which had prevailed before the squall. But there is no very marked permanent increase in the general range of the seiche.

From 22^h on the 18th there is a great increase in the range of the seiche, till it reaches 79 mm. about $4^h\ 30^m$ on the 19th (see fig. 4)—the greatest range we observed on Earn. This increase was clearly due to the barometric disturbances indicated by the microbarograph, which at the moment of maximum range had a period of about 12^m.

It is specially notable that the seiche was not further increased by the great increase of wind between 5^h and 7^h. The maximum of wind follows the maximum of seiche in this case, as in several others that were closely examined. As the velocity

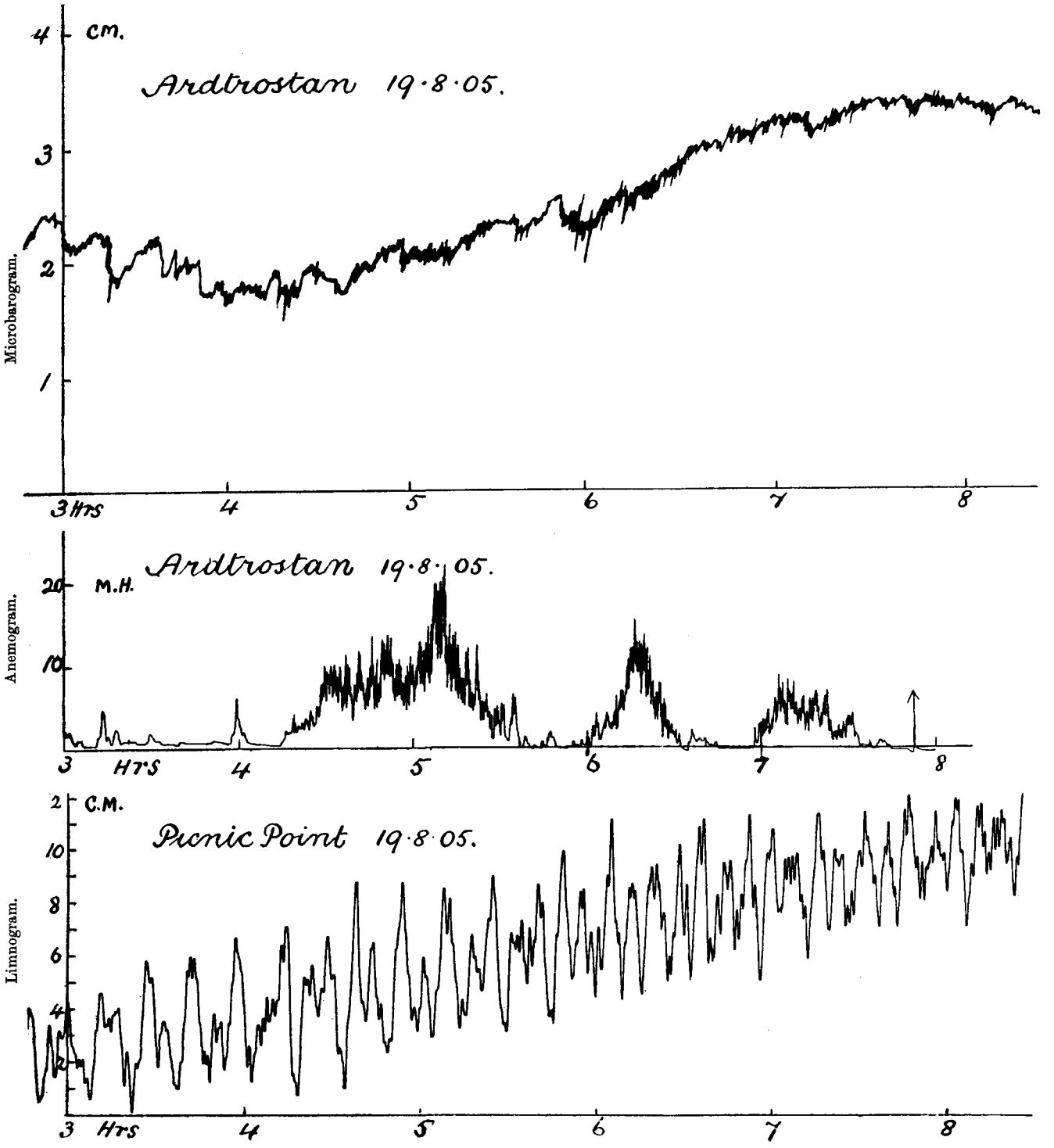
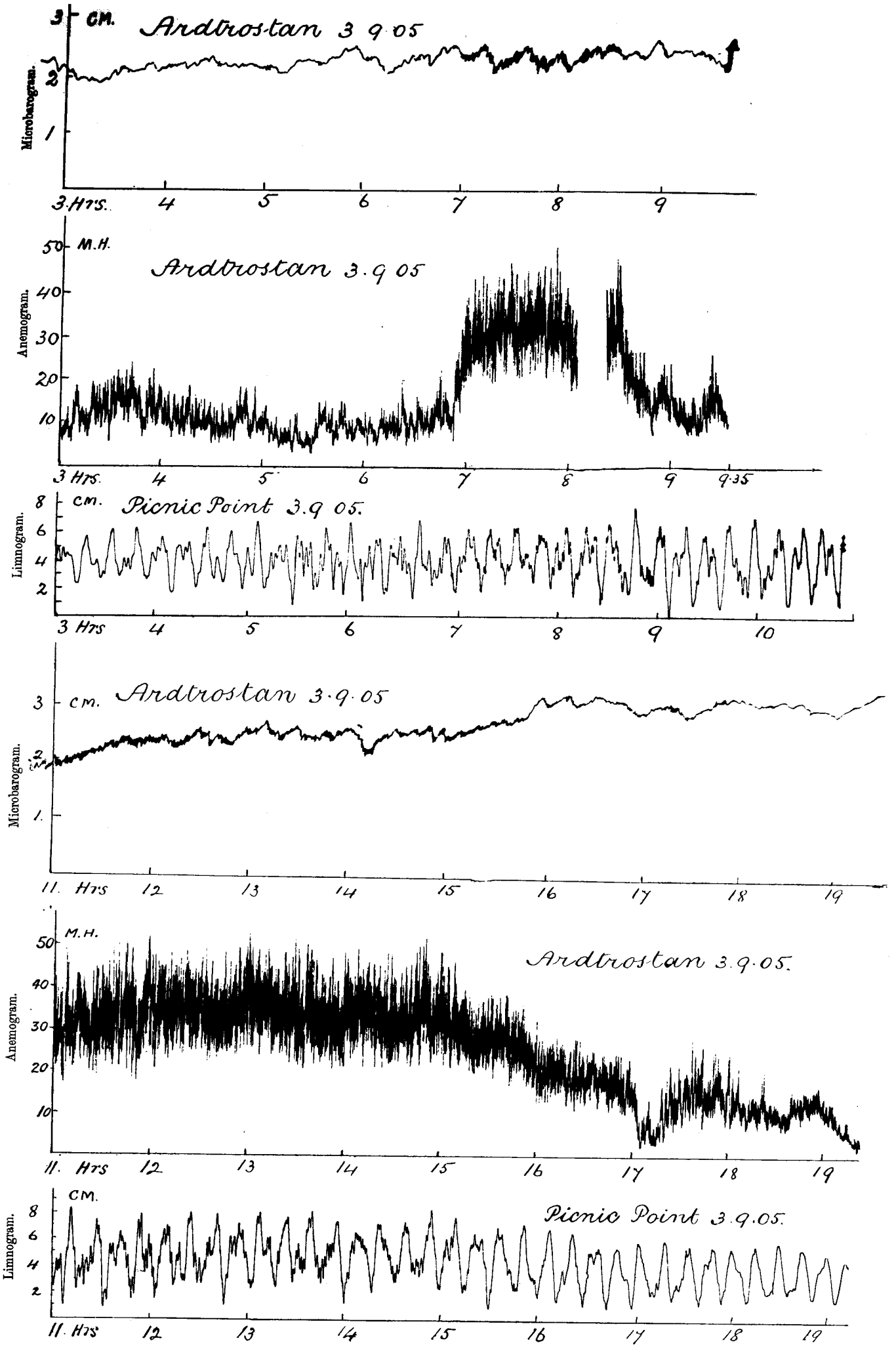


Fig. 4.

of the wind ranged from 15 to 35 (mile/hour), this can hardly have been due to the fact that the anemometer was nearer the leeward than the windward end of the lake.


Ardtrostan 3 9 05
CM.
Microbarogram.
3 Hrs.
Ardtrostan 3.9 05
M.H.
Anemogram.
Picnic Point 3.9 05.
Limnogram.
Ardtrostan 3.9.05
Ardtrostan 3.9.05.
Picnic Point 3.9.05.

The seiche of 3rd September 1905 (fig. 5) is interesting because it was accompanied by the strongest gale experienced during the two months of observation.

For some hours before midnight the wind had been very light, and at 2^h it was practically calm. About $2^h\ 37^m$ the wind began to rise; and in an hour it had reached a mean velocity of about 15 (mile/hour). The velocity fluctuated between 6 and 15 till 7^h, when a very sudden rise began. By $7^h\ 30^m$ the average velocity had risen to 35, with extremes of 45 to 50. About $8^h\ 30^m$ there was a sudden drop to about 25, then a more gradual drop to 10 at $9^h\ 20^m$. After that the gale rose again to a mean velocity of 35 to 40, with extremes occasionally reaching 53. After lasting four hours, the gale began to abate about 15^h; and then fell more or less uniformly to calm about 20^h, there being two rather sudden lulls at 17^h and $19^h\ 20^m$.

Throughout the whole of this time the microbarogram is much disturbed. During the strongest parts of the gale it shows the characteristic wind blurring, and throughout there are fluctuations of various periods: *e.g.* 7·2′ at 2^h, 5·6′ at $4^h\ 30^m$, 13·6′ at $8^h\ 30^m$, 17′ at 16^h.

Till about midnight there had been a fairly regular UB-seiche with a small trinodal component, the total range of the whole being about 31 mm. Soon after midnight, that is, more than $2\frac{1}{2}$ hours before the wind began to rise, the limnogram begins to show serious disturbance. This disturbance becomes strongly marked at 5^h, when the total range of the seiche reaches 60 mm.; and there is a strong development of seiches of higher nodality, in particular of one having a period of about $2{\cdot}9^m$.

At 7^h, when the wind suddenly rises into a gale, there is no very marked change in the seiche. But between $8^h\ 30^m$ and 9^h there is an increase in the total range from 56 mm. to 78 mm., due no doubt to the simultaneous microbaric disturbance, which has a period of about $13{\cdot}6^m$. After this the seiche tends to settle down into a UB-dicrote, strongly embroidered with higher components while the gale lasts. It is worthy of note that at 14^h, *i.e.* 7 hours after the gale commenced, the mean level of the lake at Picnic Point has only risen about 6 mm. About 16^h there is a decrease in the total range of the seiche from 64 mm. to 51 mm. This may be due partly to the drop in the wind, but much more probably to the simultaneous microbaric disturbance, which has a period of about 17^m, and would strongly affect the uninodal component of the seiche.

The range of the disturbance on the microbarogram was a little under 2 mm.; and our data from the triangle of microbarographs showed that it travelled along the lake with a velocity of 53 miles an hour. For rough purposes and for convenient calculation we may take 48 instead of 53; and suppose the period of the pressure disturbance and also of the uninodal period to be 15^m, and the circumstances as to phase to be the most favourable possible. The formula (57) of the mathematical Appendix to this memoir then gives for the addition to the amplitude of the uninodal in 15^m $\partial k_1 = \frac{3}{4}\partial p = 3$ mm. The effect after two undulations will therefore be 6 mm., that is, an alteration of 12 mm. in the range of the seiche, which, as it happens, is within a millimetre of the value observed.

3. *Case in which a Seiche was probably caused by a Flood.*—Fig. 6 shows the limnogram, taken near the binode, of a seiche disturbance beginning at $16^h\ 9{\cdot}6^m$ on 4th August

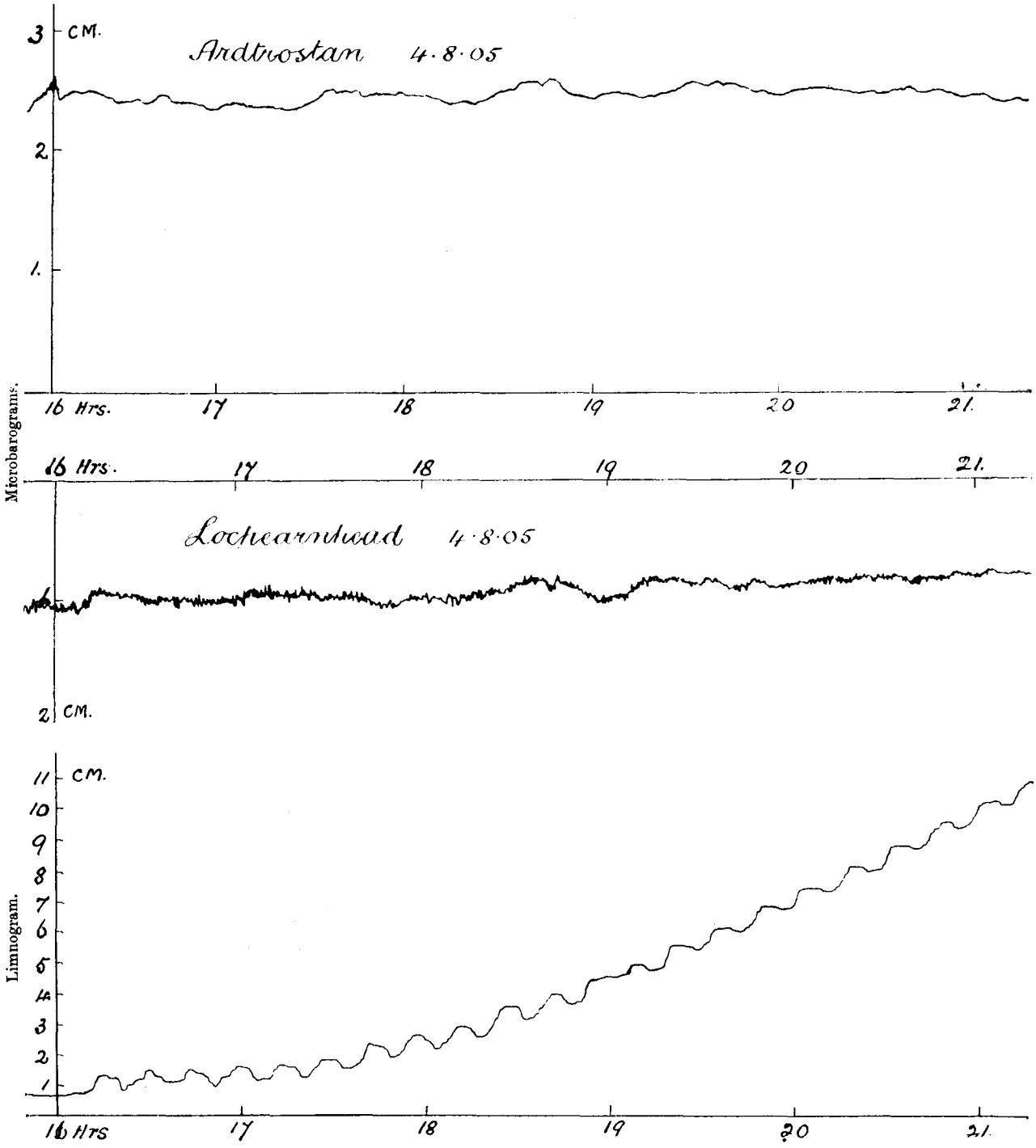


FIG. 6.

1905. The upward slope is due to a sudden rise in the lake caused by heavy rain. On the 3rd there had been ·96 in. of rain, and on the 4th 2·03 in., the greatest rainfall

observed during August and September. The limnograms taken at the uninode and Picnic Point are similar, except that the former shows merely a feeble binodal seiche, while the latter has a well-marked trinodal superposed on the uninodal seiche.

The wind on the 4th was light and easterly ; but a well-marked barometric depression, travelling with a velocity of about 18 (mile/hour), passed in a direction towards N. 15° E., probably a little to the west of Loch Earn, the centre being nearest about $0^h\ 52^m$ on the 5th.

The microbarograph at Ardtrostan shows a somewhat gradual drop of 2 mm., followed by a sharp rise of 4 mm. between $15^h\ 44^m$ and $16^h\ 3^m$.

It does not appear that either the passage of the main depression or the minor fluctuation attending it could have caused the sudden initial rise shown on the limnogram at $16^h\ 9{\cdot}6^m$. Both of these causes would indeed have worked, if at all, in the opposite direction.

We are therefore driven to the probable conclusion that the uninodal seiche was caused by the flood. A glance at the map shows that the area—Glen Beich, Glen Ogle, Glen Droma, Glen Ample, and Glen Voirlich—which drains into the western half of Loch Earn, much exceeds that—Glen Tarken, Allt an Fionn, and Finglen—which drains into the eastern half. It appears from the limnogram that for some time after the flood commenced the level of the whole lake was rising at the rate of ·32 mm. per minute. In half the period of the uninodal seiche this would give us a rise of 2·3 mm. If we suppose this flood at the very beginning to be thrown only on the western half of the lake, we have a disturbance equivalent to an increase of atmospheric pressure of 4·6 mm. of water. Acting during half the uninodal period, this, according to the calculation given in Part V., p. 503, would produce uninodal and trinodal seiches having extreme amplitudes of 6·8 mm. and 2·8 mm. If the first incidence of the flood were concentrated on, say, the western quarter of the lake-surface, the resultant seiche would of course be still greater. The rise shown at the binode was actually about 5·5 mm., which corresponds to an extreme amplitude for the uninodal seiche of 9·4 mm. It is therefore quite possible that the seiche may have been wholly due to the sudden flood on the western half of Loch Earn, and there appears to be no other way of accounting for it.

4. *Effect of Rainfall.*—In order to obtain an idea of the effect of heavy rainfall in causing a seiche, let us suppose a cloudburst to fall on the eastern half of Loch Earn (idealised into a symmetric parabolic lake). If σ denote the rainfall in centimetres per second, v the velocity of the rain-drops as they reach the surface of the lake, p the pressure at time t after the shower begins, then we have

$$\begin{aligned} p &= \sigma(v+gt) \quad \text{(dyne/cm.}^2\text{)} \\ &= \sigma v/g + \sigma t \quad \text{(gm./cm.}^2\text{)}; \end{aligned}$$

or, if we measure the pressure in millimetres of water,

$$\begin{aligned} p &= 10\sigma v/g + 10\sigma t \\ &= q + rt, \quad \text{say.} \end{aligned}$$

Let us suppose that the shower begins when the uninodal seiche culminates, and that it lasts for half the uninodal period. Then, if ∂k_1 denote the alteration in the amplitude of the uninodal seiche at the end of the lake, we get, from formula (46) of Part V.,

$$\partial k_1 = \tfrac{3}{2}q + \tfrac{3}{4}r\mathrm{T}_1,$$

where $\frac{3}{2}q$ is due to the impact, and $\frac{3}{4}r\mathrm{T}_1$ to the static effect of the precipitated water.

To take an extreme case,* let us put $\sigma = \cdot 3/60 = \cdot 1/30$, $v = 700$. Then, taking $\mathrm{T}_1 = 15 \times 60$ as a round number, we get $q = \cdot 024$, $r = 1/30$. Hence

$$\partial k_1 = \cdot 036 + 22\cdot 5 = 23 \text{ mm., say.}$$

The result would therefore be a seiche having a range of 46 mm. It will be noticed that the effect arising from the impact, viz. ·036, is negligible.

The conclusion thus arrived at bears out the inference of ENDRÖS † regarding the effect of a rainfall of 7 mm. during 20^{m} upon the 43^{m} seiche of the Chiemsee. Such a fall on one half of a parabolic lake having a 40^{m} period would generate a uninodal seiche having a range of 10·5 mm.

We have little doubt that in some of the cases, to be cited presently, the precipitation played an important part; but the observations of SHAW and DINES on the effect of passing rain-clouds in raising the barometric pressure tend to place difficulties in the way of separating the effect of precipitation from the barometric pressure proper. It would appear that the pressure to which the lake reacts so delicately is equal to the pressure before the rain has fallen, that is, while it is still in the cloud in the form of vapour; but the matter requires and deserves further investigation.

5. *Effect of Squalls.*—On 11th August, 8^{h} to 9^{h}, a prolonged depression on the microbarogram is associated with a prolonged elevation on the limnogram. The release of this denivellation caused a considerable uninodal seiche (see fig. 7).

The embroidery on the two limnograms is interesting. It has the same period, $\mathrm{T} = 1\cdot 15^{\mathrm{m}}$, at the binode and at the Picnic Point.

The limnogram taken at the binode on 21st August between 11^{h} and 17^{h} gives a good illustration of the effect of the passage of well-marked disturbances, whose whole duration (including a positive and a negative phase) was not very different from the period of the uninodal seiche. The binodal limnogram in fig. 8 is a photograph from a very rough tracing which had to be made, because at the moment the Sarasin pen was out of order, and was replaced by a pencil which gave only a very faint trace.

It will be observed that there are two well-marked increases of range, one about $12^{\mathrm{h}}\ 15^{\mathrm{m}}$, the other about $14^{\mathrm{h}}\ 30^{\mathrm{m}}$. These are connected with two sharp V-shaped disturbances on the microbarograms, the first of which is preceded by a couple of undulations whose period is not very different from 15^{m}.

The velocities of propagation of the two disturbances in the direction of the axis of

* See HANN, *Lehrbuch der Meteorologie* (1906), pp. 270, 275.

† *Seeschwankungen beobachtet am Chiemsee* (1903), p. 103.

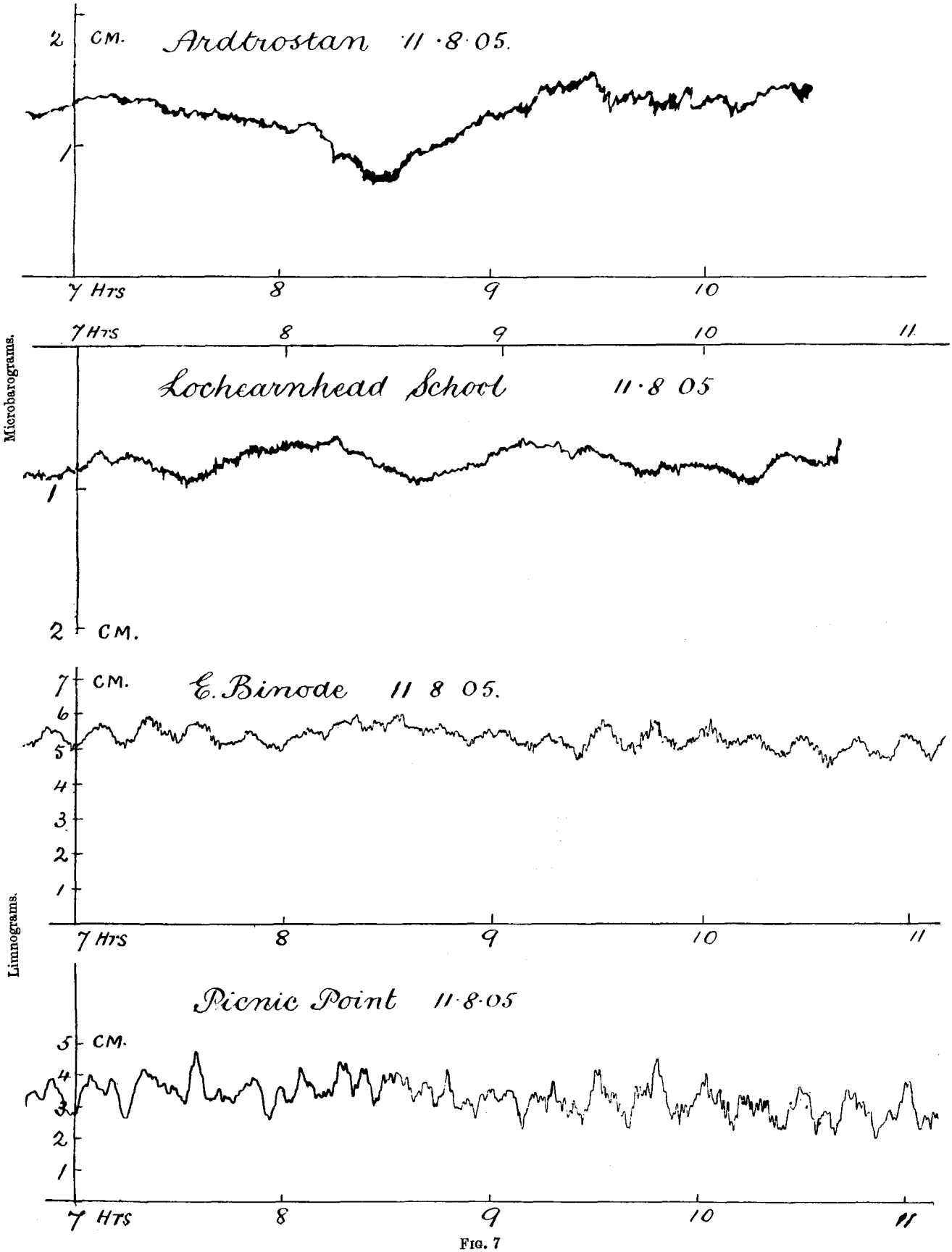

Ardtrostan 11·8·05.
2 CM.
1
7 Hrs
8
9
10
Microbarograms.
Lochearnhead School 11·8 05
1
2 CM.
11
E. Binode 11 8 05.
7 CM.
6
5
4
3
2
1
Limnograms.
Picnic Point 11·8·05
5 CM.
4
3
2
1

Fig. 7

the lake were about 28 and 22 (mile/hour) respectively. To obtain some idea of the effectiveness of such disturbances, we may take this velocity to be v, and such that $v\mathrm{T}_1 = 2a$, where T_1 is the uninodal period, and $2a$ the length of Earn. If we denote the pressure disturbance by

$$f(w, t) = \tfrac{1}{2}\alpha \left\{ 1 - \cos \frac{n_1}{v}(vt - a(1+w)) \right\},$$

where $\alpha = 8$ mm., say, and $f(w, t) = 0$ when $vt - a(1+w) >$ or $< 2\pi$, then formula (46) of Part V. gives

$$4\partial k_1/3\alpha = \int_{-1}^{+1} dw\, w \int_{a(1+w)/v}^{\mathrm{T}_1} dt\, n_1 \sin n_1 t \left\{ 1 - \cos \frac{n_1}{v}(vt - a(1+w)) \right\}$$
$$= -\cos 2\theta + \frac{5 \sin 2\theta}{2\theta} - \frac{2(1 - \cos 2\theta)}{\theta^2} + \frac{\pi(1 + \cos 2\theta)}{\theta} - \frac{\pi \sin 2\theta}{\theta^2},$$

where ∂k_1 is the increment of the amplitude of the uninodal seiche at the end of the lake,* and $\theta = 2\pi a/v\mathrm{T}_1$.

In the case supposed $\theta = \pi$, and we get

$$\partial k_1 = 3\alpha/4 = 6 \text{ mm.}$$

Owing to the strong embroidery on the binodal limnogram, it is very difficult to estimate the actual increment of the seiche amplitude at either of the two dis-

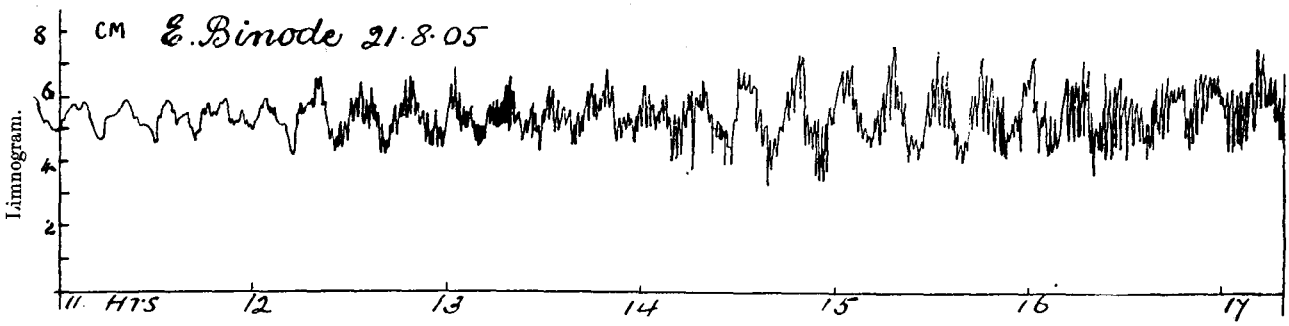


FIG. 8.

continuities; but it is clear that the results of calculation and observation are of the same order of magnitude.

It is interesting to note that the very strong short-period embroidery that blurs the binodal limnogram was almost totally absent on the limnogram taken at Picnic Point. During the day the wind had been variable in direction from south-east to south-west, gusty but never very high. The surface waves on the lake were not high during any part of the day; they came from the east in the morning, and from the west in the evening.

On the 7th of September, about $8^{\mathrm{h}}\ 30^{\mathrm{m}}$, occurred the greatest barometric fluctuation of short duration which we observed.† The extreme range was 19·3 mm. (Aq.), the

* *I.e.* 1·7 times the amplitude at the binodal limnograph.

† For further details, see my paper, *Proc. R.S.E.*, vol. xxviii., p. 457.

total duration about half an hour. It came from E. 56° N. with a velocity of propagation of 19 (mile/hour), the velocity along the lake being about 30.

As will be seen from fig. 9, the effect was to increase the total range of the seiche from about 18 mm. to 50 mm., and to generate a strong BT-dicrote. It is worthy of remark that the rise in the wind follows about an hour after the barometric disturbance. To the spiky anemogram which then follows corresponds a strongly embroidered seiche,

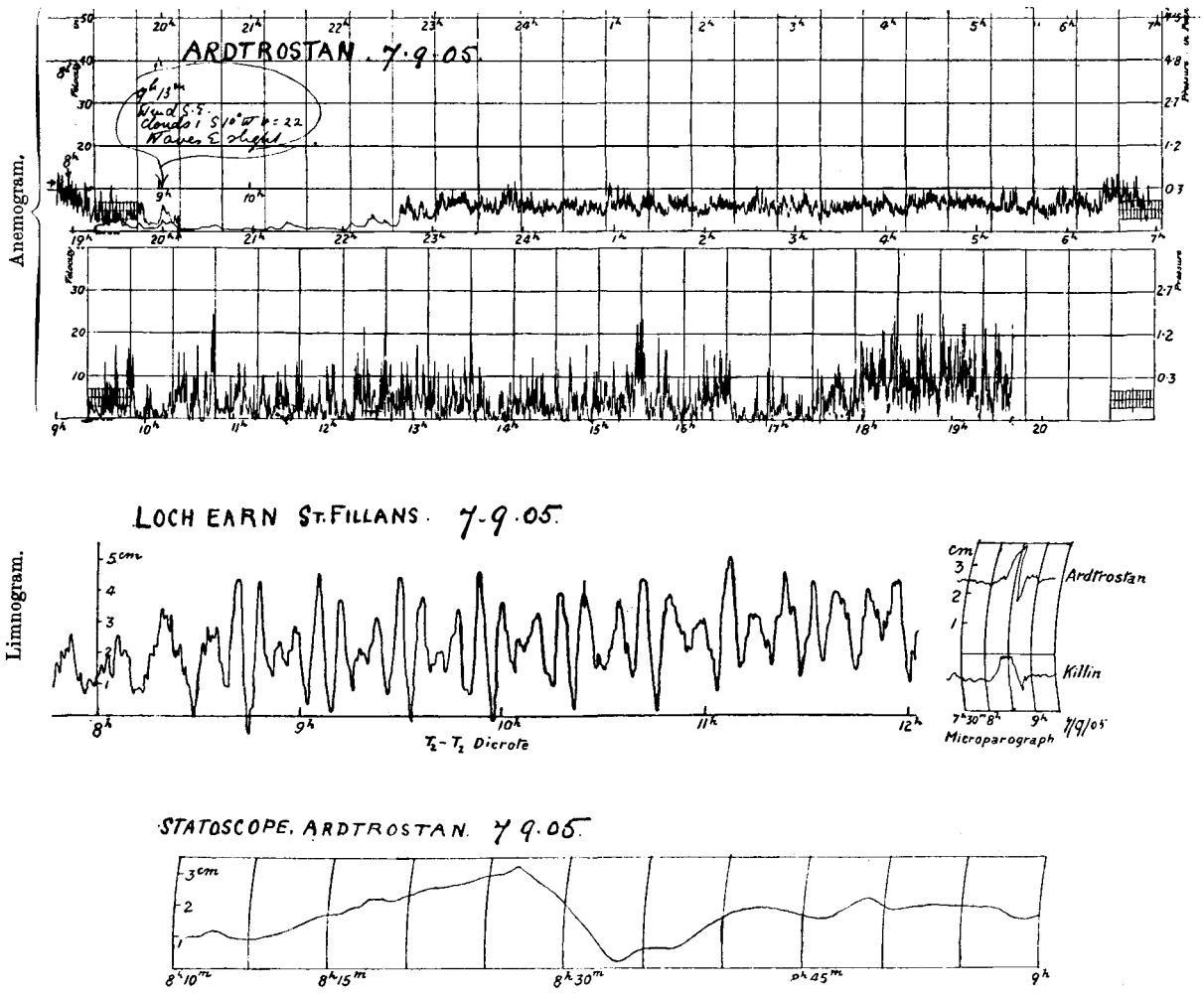


FIG. 9.

which shows no increase in maximum range. I have tried, but unsuccessfully, to find a period in the anemogram corresponding to that of the seiche-embroidery, viz. $T = 1{\cdot}5^m$ to $1{\cdot}6^m$.

On 8th September, between 16^h and 17^h, a well-marked barometric disturbance, having a range of 3 mm. to 4 mm. (Aq.), caused a change of phase in the previously existing UB-seiche, and also a considerable increase of range. This UB then persisted for nearly 24^h, until about $13^h\ 15^m$ on the 9th September its configuration was utterly destroyed by the great barometric disturbance shown in fig. 10. This disturbance

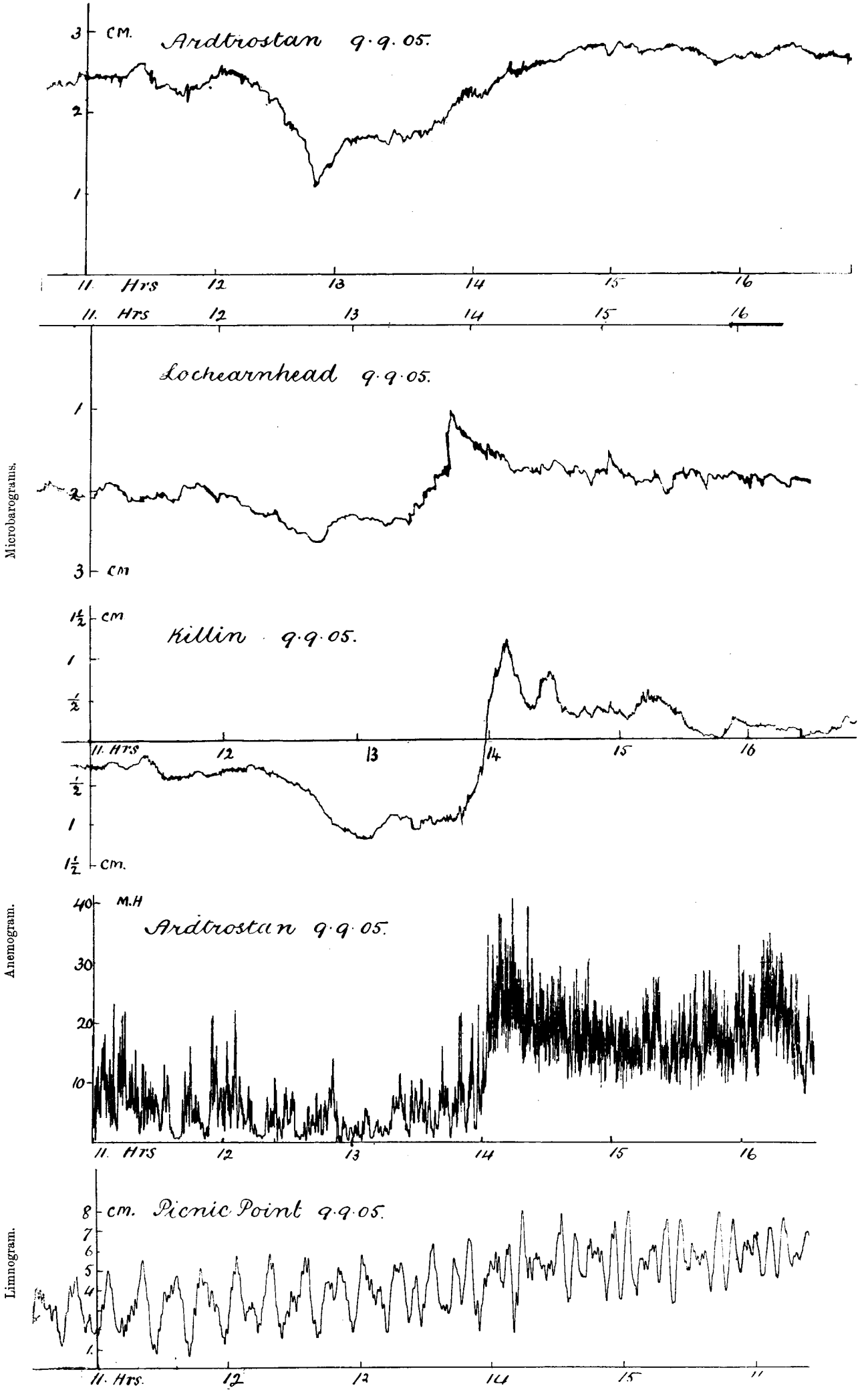

Ardtrostan 9.9.05.
cm.
Hrs
Lochearnhead 9.9.05.
Microbarograms.
Killin 9.9.05.
Ardtrostan 9.9.05.
M.H
Anemogram.
Picnic Point 9.9.05.
Limnogram.

Fig. 10.

lasted nearly two hours, and caused a maximum depression of 14 mm. (Aq.); it came from W. 62° S. to W. 67° S., and travelled with a velocity of 17 to 22 (mile/hour), *i.e.* with a velocity of 33 to 51 along the lake. The sections of the disturbance at Killin and Lochearnhead on the one hand and at Ardtrostan on the other were very different. The minimum was rounded and pretty flat at the two former places, but cuspidal at the latter. Again, at Killin and Lochearnhead the minimum was followed by a sharp-pointed maximum, with an almost perpendicular rise; while at Ardtrostan the recovery after the minimum is very gradual, and there is only a little wart corresponding to the peaks at the other two stations.

It is interesting to notice that the minimum of the disturbance, although it destroys the configuration of the UB, and generates one of the best-marked BT-dicrotes that we observed, yet produces no great change in the total range of the seiche. It does produce a small rise of level at Picnic Point of 7 mm. to 11 mm. This is confirmed by the limnogram taken at the binode, where at that time the Sarasin was running at high speed (160 mm. per hour). This shows a rise of level of 5 mm., and a diminution in the range of the uninodal seiche of about 8 mm.

At 14^h 13^m there is a sudden rise of level of about 14 mm., evidently due to the intense action of the maximum of pressure developed towards the western end of the lake, which there is nothing to counterbalance on the eastern part. It is after this point that the new BT configuration becomes conspicuous. As will be seen from the anemogram, the barometric and seiche disturbances at 14^h were associated with a very sudden rise in the average wind velocity from 5 to 25 (mile/hour).

The Glen Ogle Storm.—It seems unnecessary further to multiply instances of the connection between abrupt barometric and seiche disturbances. I shall therefore conclude this part of my report by describing one of the most remarkable observations we made on Loch Earn.

On the 23rd August, after a dead calm during the night and heavy rain in the early morning, at 8^h 20^m there was a light breeze, W., 5 (mile/hour). There was low cumulus on the hills to E. and N.E.; but there was bright sunshine, and the clouds (3) in general were high. The main drift was from S.E.; but there was a mackerel formation apparently moving in a different direction; also a mare's-tail showed to S.W.

The waves were running from W.—a slight swell diversified by oil bands, which were seen at intervals throughout the day.

From 8^h 20^m to 12^h 30^m the wind was light, fluctuating with a rough period of 1^h. At 12^h 30^m there was a sudden gust of 15 (mile/hour). After that the wind rose somewhat, and fluctuated for about 5 hours between 0 and 13 (mile/hour) mean velocity. It was unusually gusty, and at 14^h 5^m an extreme velocity of 25 (mile/hour) was registered. At this moment a black rain-cloud came down Glen Ogle, and reached over the western part of the lake as far as Ardvoirlich, where it stopped.

At 14^h 50^m there came on a sudden rain-shower, the wind being then W. by S. After this there was rain at intervals till 20^h 18^m, an especially heavy shower at 17^h 20^m.

At $20^h\ 18^m$ the wind was W.S.W. and variable.

At $14^h\ 7^m$ a microbaric disturbance passed Ardtrostan, travelling with a velocity of 15 (mile/hour) from W. 60° N. (36 along the lake).

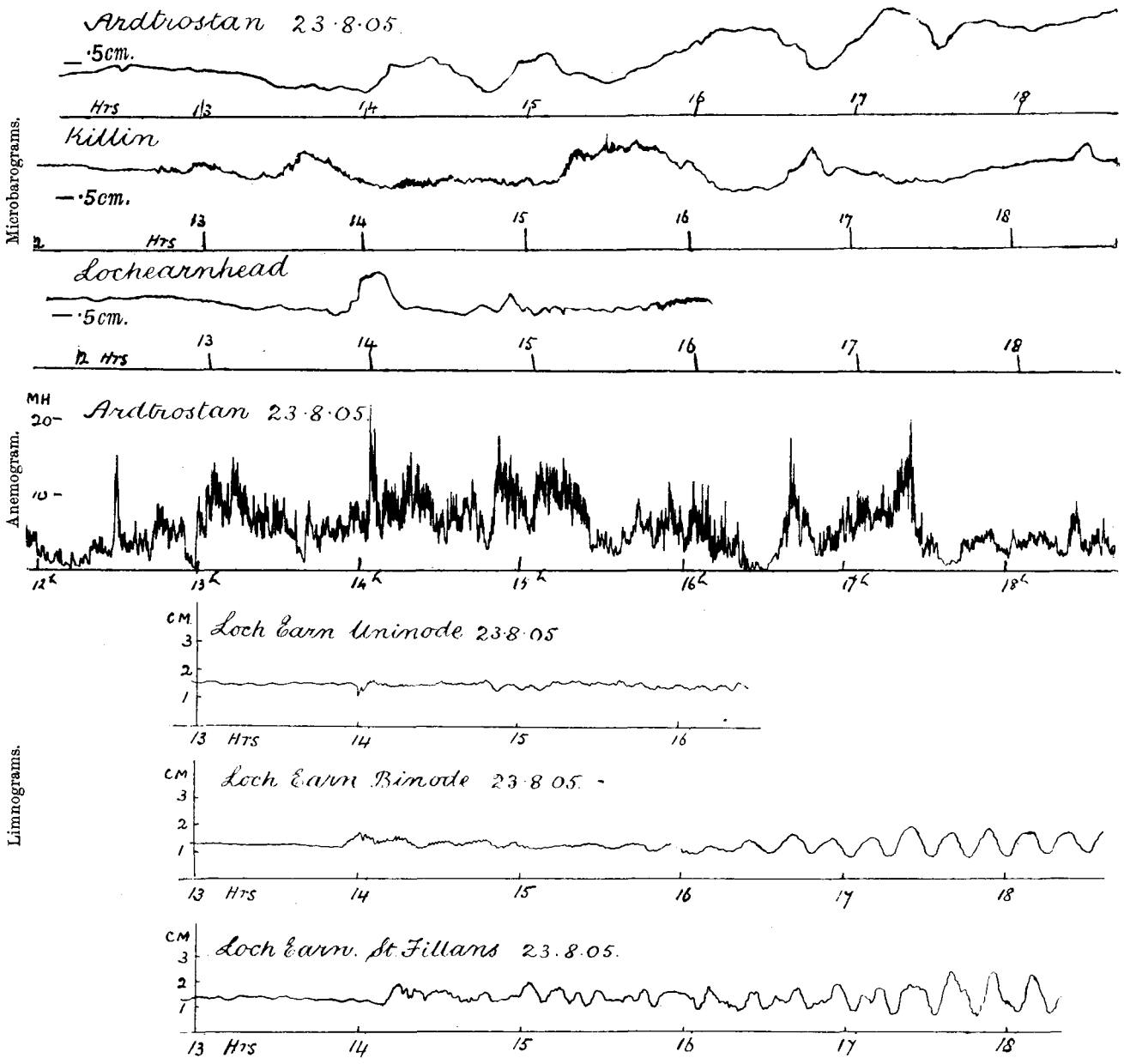


FIG. 11.

One of the Lake Survey staff was looking at the uninodal limnograph, and saw it record the sharp depression shown in fig. 11, just as the squall came up. For some time before, the limnographs at the uninode, binode, and Picnic Point had been drawing almost straight lines. The seiche weather had, in fact, been the calmest known in our two months of observation.

The maximum depression (4 mm.) at the uninode and the maximum elevation (5 mm.) at the binode were nearly simultaneous, the latter apparently following about $1\frac{1}{2}^m$ after the former. Unfortunately, owing to the irregularity of the clock at the uninode, certainty on this point is not attainable.

It seems clear that an abrupt elevation of the surface travelled along the eastern part of the lake. The first rise began at the binode at $13^h\ 55{\cdot}31^m$; and at the Picnic Point at $14^h\ 5{\cdot}24^m$, that is, $9{\cdot}93^m$ later. The first maximum (5 mm.) is seen at the binode at $14^h\ 1{\cdot}05^m$; and at Picnic Point at $14^h\ 10{\cdot}57^m$, that is, $9{\cdot}52^m$ later. The velocity of propagation of the first rise would thus be 6·0 (mile/hour), and of the first maximum 6·3 (mile/hour); and it is interesting to notice that by the time the wave has reached Picnic Point a shallow minimum has developed in front of the maximum. If the wave had travelled as a solitary long wave, it would have taken only about 7^m to travel from the binode to Picnic Point.

After the wave reached St Fillans it seems to have been reflected backwards and forwards between the ends of the lake, at first with a good deal of irregularity; but gradually it developed the characteristics of a regular dicrote seiche. There are two points (easily seen on the binodal limnogram), viz. 16^h and $17^h\ 20^m$, where the range of the seiche was suddenly increased, evidently by barometric disturbances which occurred at these times. The increase at $17^h\ 20^m$ may have been partly due to the heavy shower.

At $17^h\ 20^m$ on the 23rd the dicrote is fully developed ($2U = 11{\cdot}5$, $2B = 7{\cdot}0$). It retains its character, with gradually decreasing range, until a little before 24^h on the 24th. About that time the microbarograph at Killin shows disturbances with periods $T = 10^m$, $T = 15{\cdot}6^m$; and there is an alteration of the UB from $2U = 3{\cdot}7$ mm., $2B = 1{\cdot}7$ mm. to $2U = 11{\cdot}4$ mm., $2B = 1{\cdot}0$ mm. The dicrote then remains steady until 22^h on the 25th, when it undergoes a sudden disturbance, which rapidly destroys its configuration. This sudden disturbance and the almost total destruction of the seiche about 5 hours later are difficult to explain by the meteorological conditions, unless they were due to variations of the wind.

6. *Effect of the Impact of Wind-Gusts.*—Inasmuch as a wind velocity of 10 (mile/hour) is calculated to produce a pressure of about 1·5 mm. (Aq.) by direct impact on a small area, it is reasonable to expect that the impact of wind-gusts, especially in the case of lakes enclosed by high hills, may at times cause seiches. There are, however, various difficulties in obtaining data on the subject. It is difficult to determine the angle of impact of the wind-blasts. Then it is uncertain whether the wind ever falls at the same angle and at the same time over large parts of the surface of a lake. The appearance of the lake-surface on windy days very often suggests the contrary. What we frequently see are patches of wind disturbance progressing over the lake-surface with varying velocities.

Then again it is difficult to separate the effect of wind impact from the disturbances of the ordinary barometric pressure which always accompany high winds.

It has not been possible to deduce any definite results from our observations under the present head.

7. *Effect of Periodic Fluctuations of the Atmospheric Pressure.*—Our observations afford many examples of this cause of seiches. It must, however, be understood that strictly periodic fluctuations of the barometric pressure of short period rarely if ever occur. We often find, however, fluctuations extending over an hour or two in which

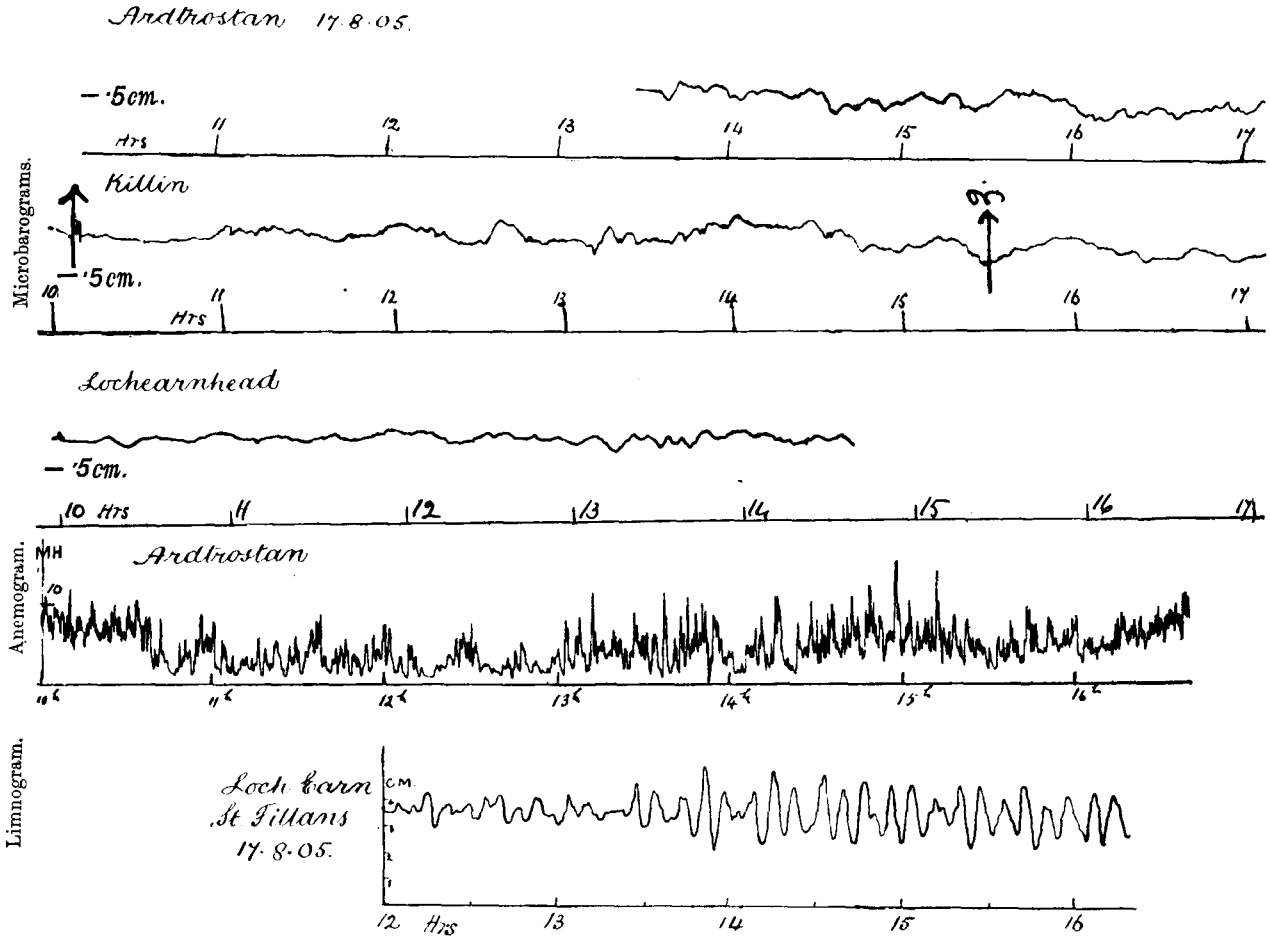


FIG. 12.

the undulations are approximately of equal length; and still oftener we find two or three consecutive undulations of approximately the same length. Such fluctuations we shall describe in what follows as periodic; and by the "period" is meant the average of the intervals between the passage of corresponding phases (say maxima) of two successive undulations at the same point.

It follows from theory, and is confirmed by observation, that a periodic disturbing cause is most effective when its period is not very different from that of the seiche in question. In practice, however, the disturbing effect is considerable even if there is considerable divergence between the two periods. It should also be noticed that, even theoretically, if we consider only one or a limited number of oscillations, and neglect

the viscosity, the maximum effect does not correspond to exact equality of the two periods.*

In what precedes we have already given some examples of the effect of a periodic disturbing agency; we shall now add a few more.

17*th August.*—Between 13^h and 14^h (see fig. 12) a strong binodal seiche with a trinodal component is worked up by resonance with the pressure disturbance shown on the

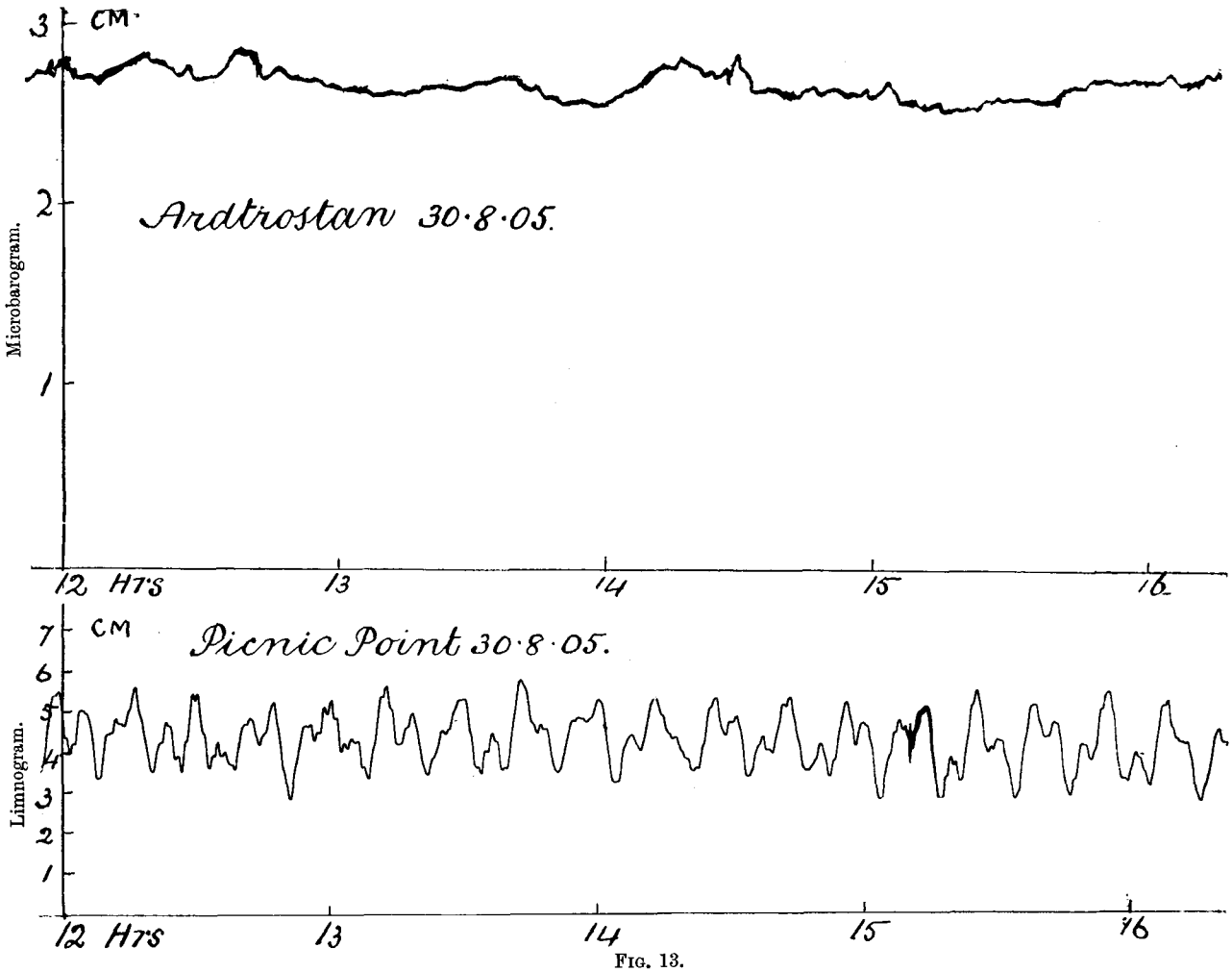


FIG. 13.

microbarograms, which has a period of 6^m to 8^m, well marked at Killin and Lochearnhead, less clearly at Ardtrostan.

21*st August* (fig. 8).—At 1^h 11^m a BT-dicrote appears; and the microbarogram at Lochearnhead shows an 8^m period; probably its scale is too small to show the 6^m period. At 5^h 37^m the binodal component is much strengthened by a microbaric disturbance having a period of 8^m to $8{\cdot}8^m$, which is very clearly seen at Killin. The uninodal seiche is also present, due no doubt to the periods of 15^m to 13^m observed at Lochearnhead.

30*th August.*—Fig. 13 shows the kind of microbaric disturbance which pro-

* See Part V., p. 514.

duces a UBT-tricrote seiche. This was the best example obtained during our observations.

4th September, 0^h–8^h.—Fig. 14 shows a case where a strong trinodal component was introduced into a UB-dicrote by microbaric disturbances having periods of $6{\cdot}1^m$, $8{\cdot}0^m$, $8{\cdot}2^m$, $5{\cdot}1^m$.

6th September.—From 2^h to 10^h there was a perfectly smooth UB-dicrote, part of which is reproduced in fig. 4, vol. xlv., p. 366 of the *Transactions*. At first the uninodal component decreases and the binodal increases. Thus, at 3^h, 2U = 11·7 mm., 2B = 13·3 mm.; at 5^h 30^m, 2U = 3·9 mm., 2B = 15 mm. The microbarogram shows

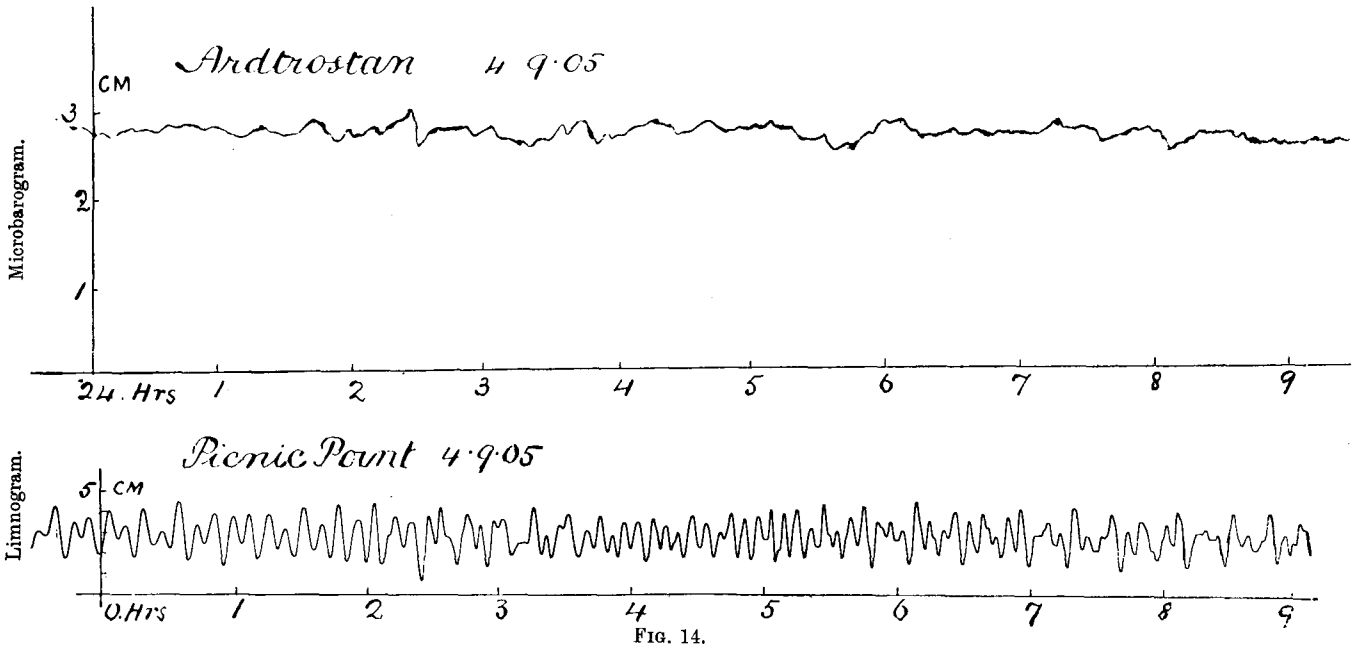


FIG. 14.

periods of $10{\cdot}7^m$ about 4^h, and $9{\cdot}4^m$ about 6^h, which were no doubt responsible for this gradual alteration of the seiche.

16th September.—About 6^h (see fig. 15) a succession of four very regular waves of barometric disturbance, having a period of $13{\cdot}3^m$, generated a very regular UB-dicrote, which lasted about 15^h. The uninodal component gradually diminished, as will be seen from the following measurements:—

Hour.	2U.	2B.
h.	mm.	mm.
Ca 10	22·9	4·5
,, 16	17·9	4·1
,, 21	9·8	4·2

16th October, 4^h–9^h (see fig. 2, above).—A very interesting example, showing both positively and negatively the effect of a periodic barometric disturbance, is

obtained by contrasting the limnograms taken simultaneously on Loch Tay at Killin, and on Loch Earn at Lochearnhead. The period of the microbaric disturbance is about 29^m; and it will be observed that it greatly increases the uninodal seiche of

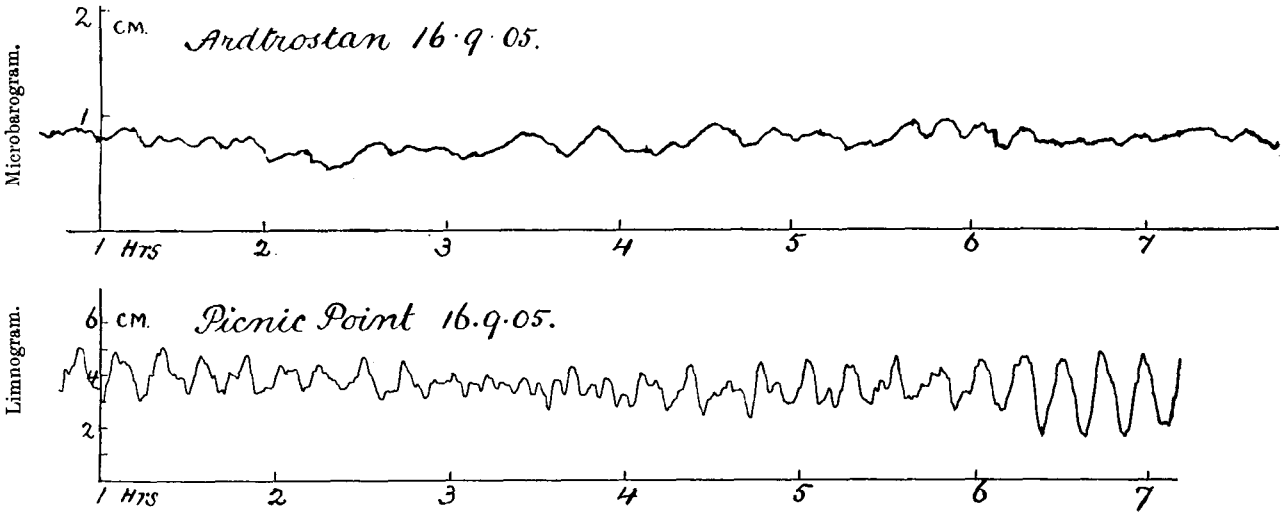


FIG. 15.

Tay, the period of which is about $28{\cdot}4^m$. Indeed, the uninodal thus produced was the best we found on Tay. On the other hand, this strong barometric disturbance produces little or no effect on the smooth UB-dicrote which was in progress on Earn, because the periods of its components are $14{\cdot}52^m$ and $8{\cdot}09^m$.

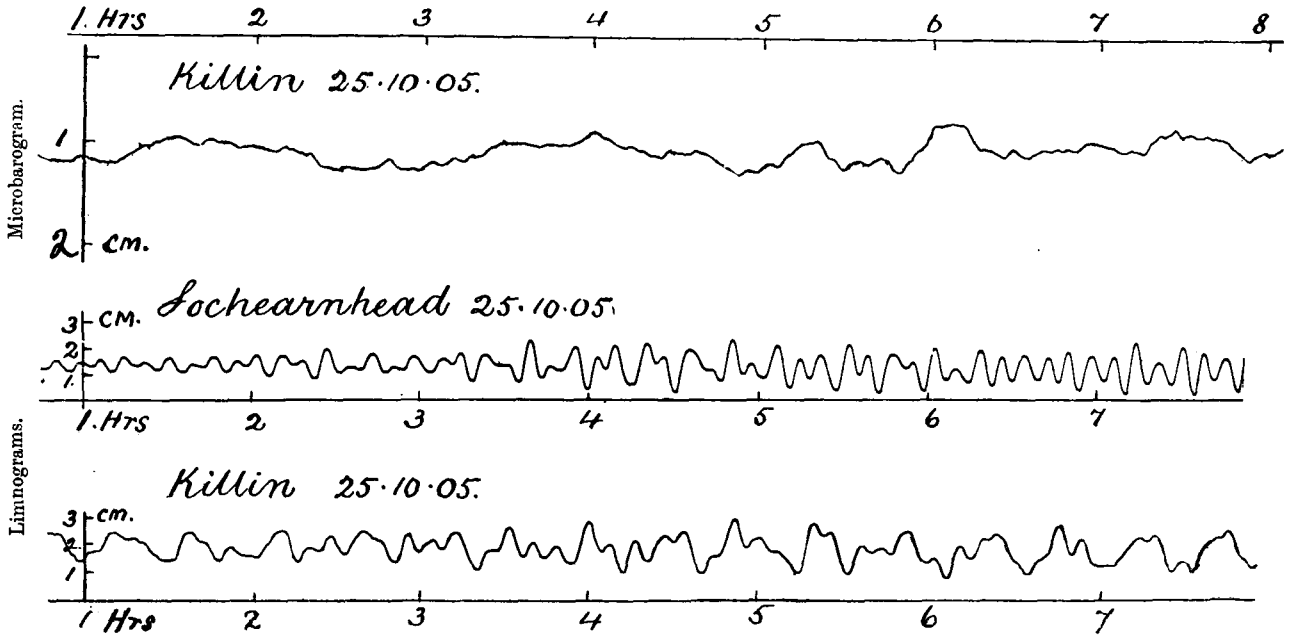


FIG. 16.

As further examples of this kind, we may also mention the following observations:—

25th October, 3^h–8^h (fig. 16).—Microbaric disturbances of period $7{\cdot}5^m$ to $8{\cdot}0^m$ brought out the binodal of Earn ($T_2 = 8{\cdot}09^m$) and the quadrinodal of Tay ($T_4 = 8{\cdot}6^m$).

7th December, ca. $14^h\ 30^m$ (fig. 17).—About this hour was observed the greatest total range of seiche found on Tay, viz. 80 mm. At that moment the range on Earn, which at 8^h had been as much as 55 mm., was only about 25 mm. The explanation of this is doubtless to be found in the well-marked period of $26{\cdot}5^m$ shown in the microbarogram between 12^h and 16^h.

14th August.—Fig. 18 shows an interesting case of the gradual development of a UB-dicrote seiche. The anemogram shows a fall of wind during this development;

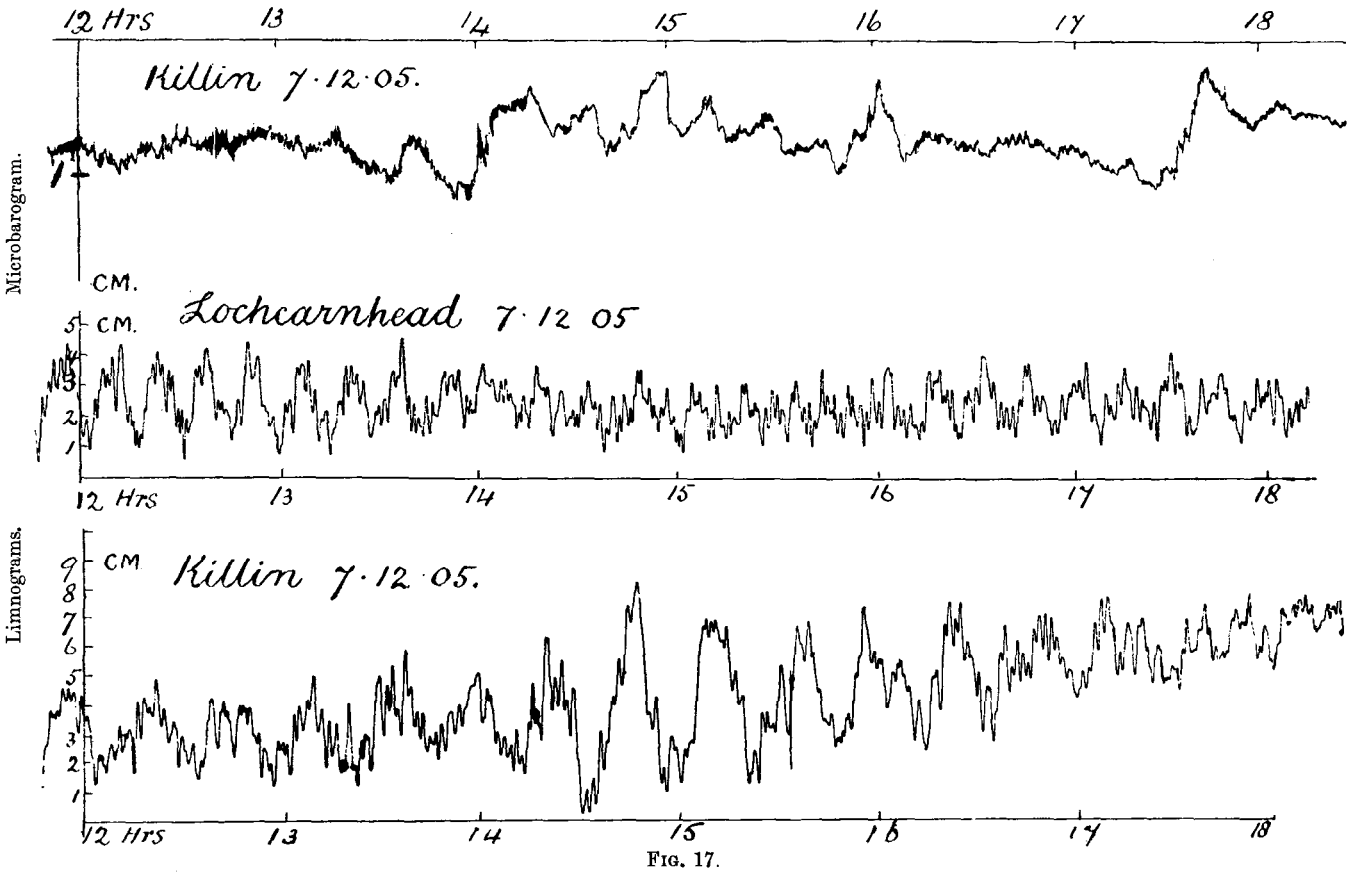


Fig. 17.

but it seems to have been too gradual to be the effective cause of the seiche. There can be little doubt that the true cause was a periodic microbaric disturbance, which is very faintly indicated in the microbarograms taken at Ardtrostan and Lochearnhead.

The present is one of many examples found in the course of our observations which prove that a lake-surface is much more sensitive to minor fluctuations of the atmospheric pressure than any barometric apparatus hitherto constructed.

We might produce many more examples; but probably the above are sufficient to establish that the synchronism of quasi-periodic disturbances of the atmospheric pressure with the seiche-periods of a lake are a frequent cause of seiches.

It is true that the resonance experiments which Nature performs in her own rough laboratory have not the nice exactitude of those we devise and carry out in a physical

institute. But then it is not the way of Nature to flaunt her beauties before the unappreciative, or to press the secret principles of her action upon the attention of the unreflecting.

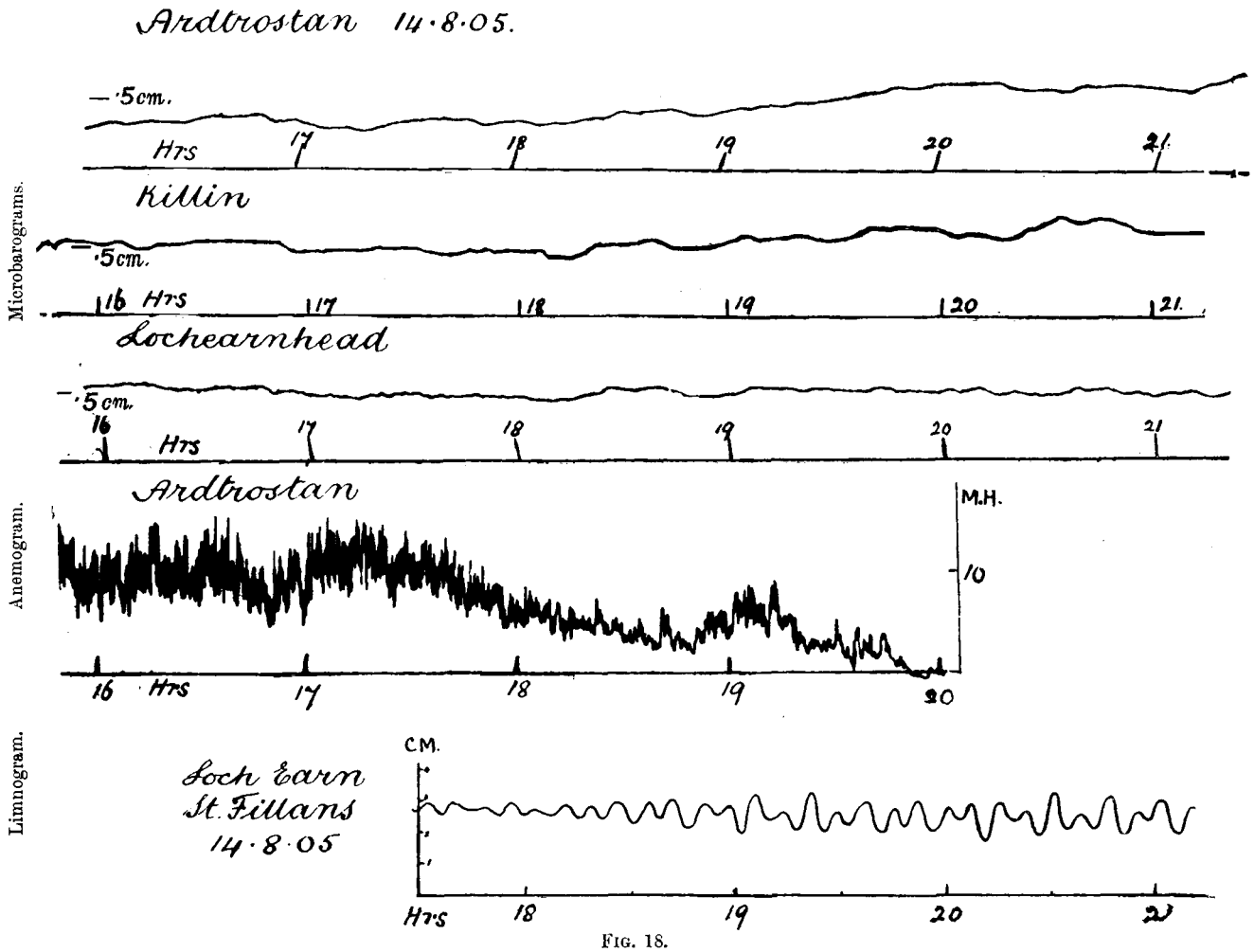


FIG. 18.

LABORATORY EXPERIMENTS ILLUSTRATING THE ORIGIN OF SEICHES.

In a lecture* given at the Royal Institution the writer showed some experiments with a miniature parabolic lake, to illustrate the nature and origin of seiches, which seem worthy of mention here. The trough used was 12 feet long, about $2\frac{3}{4}$ inches wide, and 12 inches deep. It was fitted with a parabolic wooden bottom, and filled with slightly coloured water to a depth of about $7\frac{3}{4}$ inches.

Following a method used by Messrs WHITE and WATSON in their interesting experiments on this subject,† we found it possible, by stirring horizontally with a small paddle at the theoretical positions of the nodes and with the corresponding

* See *Proc. Roy. Inst.*, Friday, May 17, 1907.
† *Proc. R.S.E.*, vol. xxvi., p. 146 (1906).

period, regulated by means of a metronome, to generate uninodal, binodal, and trinodal seiches easily visible to a large audience.

This method of generating seiches probably does not correspond to anything observable under ordinary circumstances in a lake; * but the experiment is interesting in view of the important discovery recently made by the Japanese observers, † that the secondary oscillations in many of the bays on the coast of Japan are seiches, having a node at the mouth and a loop at the bottom of the bay. These oscillations, which are sometimes of considerable range, are apparently due to resonance with comparatively inconspicuous undulations in the external oceanic swell, the periods of which are equal to some of the natural periods of the bay.

It was also possible, by means of a trough like that above described (length, 5 feet; breadth, 4 inches; depth, 5 inches; depth of water, 3 inches), to illustrate the generation of seiches in an ordinary lake by periodic variations of the surface pressure. By laying a sheet of tin on the top of the trough, an air-channel was formed over the surface of the water. Through this channel air could be blown by means of a Blackman's fan, and, by working a slider timed by the metronome, the air-current could be made intermittent. When the whole of the surface was covered over by the sheet of tin, the effect of the current, whether steady or intermittent, was merely to generate a train of progressive surface waves, the motion due to which was shown, by dropping in a stream of red ink, to be confined to a stratum of the water near the surface. But, when only half the length of the miniature lake was covered in, an intermittent current having the proper period generated a uninodal seiche. When a strip of tin dipping into the water at the end of the covering sheet just over the middle of the water was used to block the air-current, after a few alterations of the blast the amplitude of the generated seiche was such as to cause the water to splash over the ends of the trough.

In like manner, by covering in the tank up to the theoretical position of the binode, a binodal seiche was generated, the parabolic surface of which at its culmination had about the same curvature as the parabolic bottom of the trough.

By using streams of red ink from a pipette, it was easy to demonstrate to the audience the essential nature of the seiche-motion, and to contrast it with the essentially different motions which characterise the progressive and the solitary wave.

On the Vibrations which cause the Embroidery on the Limnogram.

To the oscillations of a lake-surface having a period of less than 2^m, which under certain circumstances cause a regular or irregular embroidery on the limnogram, Forel gave the name of vibrations. The complete explanation of these vibrations can hardly

* Endrös, however, has given examples in point, in some cases of constricted lakes, where a seiche in one part forces a seiche of the same period in another part.

† "Secondary Undulations of Oceanic Tides," by Honda, Terada, Yoshida, and Isitani, *Journ. of the College of Science, Imperial University, Tokio*, vol. xxiv., p. 1 (1908).

be said to have been given as yet. They are, however, of great interest, because there is some reason to believe that in part at least they reflect in miniature the action of the causes which produce the storm-waves of the ocean, our knowledge of which is still far from complete, although they are of such vital importance to seafaring men.

Inasmuch as our first object was to determine as accurately as possible the seiche periods and the positions of the nodes of Loch Earn, the limited time at our command was allotted and our apparatus disposed mainly for these two purposes; and it was not until near the end of our observations, after the extemporisation of the statolimnograph, that much attention was given to the vibrations of the lake. We cannot, therefore, pretend to offer much towards a final solution of the problem of the vibrations; but we may record a few observations which seem to enhance the interest of the question, and may ultimately prove useful in its final solution.

The embroidery caused by these vibrations, as may be seen by comparing the figures of this memoir and figs. 5, 6, 7, 10, 11, 12, 16, 17, 18 of the previous memoir of this series,* varies considerably in form, and may be regular or irregular according to circumstances. It must also be remembered, as was long ago pointed out by FOREL, that, owing to the damping effect of the well and access tube, each limnograph reproduces more or less of these vibrations according to its adjustment. The statolimnograph, used with a wide access tube, owing to the very small inertia of its moving parts, is best adapted for this purpose. Compare, for example, figs. 4 or 5 of Part II. with fig. 11 of Part I.

Although occasionally the embroidery continues regular for a considerable time, and appears to have a perfectly definite period and constant or at least slowly varying range, as a rule its configuration changes rapidly, and any regularity is transient. This makes it very difficult to analyse it into harmonic components, even if analysis into a finite number of such components were possible.

In our observations the maximum range of the vibrations varied from 0 to 21 mm.; an average value might be about 6 mm. At times the range of the vibrations (*e.g.* fig. 19) exceeded the range of the seiche, so that the former quite obscured the latter.

The periods observed showed much less variation. In the limnograms taken with the Waggon recorder and Sarasin instruments, the period ran from $1{\cdot}3^m$ to 2^m; in the statolimnograms, from $\cdot 42^m$ to $\cdot 79^m$. It must be remembered, however, that in the latter the short-period embroidery obscures that of longer period; and in the former the vibrations of shortest period are damped out. For the ordinary limnograms the average of the periods might be put at $1{\cdot}47^m$. The period that actually occurred oftenest in the cases we happened to examine was $1{\cdot}5^m$.

The embroidery was never observed unless there had been sufficient wind to cause progressive surface waves; and it subsided at once when these waves disappeared. The observations of HALBFASS, ENDRÖS, and others show that it is usually more marked when the limnogram is taken at the leeward end of the lake: it may be very marked there

* See *Trans. R.S.E.*, vol. xlv., pp. 366, 369, 370, 380, and 383.

and almost or altogether absent at the windward end. It also depends on the amount of shelter at the point of observation.

In most cases the occurrence of embroidery is accompanied by the characteristic wind blurring on the microbarogram, or else by fluctuations of very short period and very small range. In some cases the fluctuations could be counted; and in one or two their period seemed to coincide with the period of the lake vibrations. The sensibility of the microbarographs used and the number of interpretable cases were not sufficient, however, to justify any general conclusion.

Attempts were made to connect the periods of the lake vibrations with the periods of the wind fluctuations, as indicated on the anemogram, but without success, possibly owing to the fact that the time scale of the anemograph was so short that it was impossible to count the wind fluctuations with any certainty.

The simultaneous limnograms taken on Earn and Tay during October and November 1905 were examined to see whether there was any connection between the vibrations on the two lakes pointing to a common atmospheric cause. It was found that the average of the maximum ranges and of the periods was much the same for both lakes; but there seemed to be no connection between the occurrence of a particular range or a particular period in the two. The range might be high in both lakes and the periods different; or the periods nearly the same and the ranges different; or there might be vibrations of considerable range on one of the lakes, and none, or only the merest tremor, on the other.

Several suggestions have been or may be made regarding the nature of these lake vibrations.

1. They might be longitudinal seiches of very high nodality. This was the suggestion put forward tentatively by FOREL, after trying in vain every other explanation that occurred to him.

If the period of $1{\cdot}47^{m}$ were due to a longitudinal seiche, the number of the nodes would be 12 or 13. It is easy, by regarding Earn as a symmetrical rectilinear lake,* to calculate roughly the positions of the nodes. It would therefore be possible, by means of careful experiments with two or more self-registering instruments, such as the statolimnograph, to obtain positive or negative evidence regarding the truth of the hypothesis that the vibrations are wholly or partially plurinodal longitudinal seiches.

In the present state of our knowledge the balance of evidence seems to be against this hypothesis. A plurinodal seiche is a simultaneous oscillation of the whole lake. If, therefore, a vibration were a plurinodal seiche, it should be apparent simultaneously at both ends of the lake; whereas we know that it may be present at either end and apparently absent at the other. Also, if it be a plurinodal seiche, it should be present simultaneously at nearly opposite points on the two sides of the lake. We made repeated attempts to detect correlations of phase, by stationing observers on the two sides, and signalling the maxima or minima of the vibrations, but were quite unable to

* See my memoir on the "Hydrodynamical Theory of Seiches," *Trans. R.S.E.*, vol. xli., p. 639 (1905).

establish either coincidence or opposition of phases. We also made observations with the statolimnograph at a point opposite the limnograph near the eastern binode, while the latter was running at high speed (2·96 mm. per minute). Not only were there no apparent coincidences of phase, but the binodal limnograph showed a well-marked vibration whose period was $1{\cdot}35^m$, while the best-marked period of the embroidery on the statolimnogram was $\cdot 44^m$ to $\cdot 47^m$.*

2. The vibrations might be transversal seiches of the lake. In a former memoir I expressed some doubt whether seiches of this kind could be stable in an elongated lake. In an elaborate and most interesting review of our present knowledge of the seiche periods of lakes in general, recently published,† Dr ENDRÖS has stated that he has, by means of phase observations, definitely established the existence of a transversal seiche of period $1{\cdot}56^m$ in the Tachinger See, and shown that both it and the seiche between Morges and Evian, observed by FOREL and suspected by him to be transversal, as well as certain other cases of the same phenomenon, agree very well with the hydrodynamical theory. My doubt on this matter must therefore be abandoned. Dr ENDRÖS' view is that only part of an elongated lake takes part in the transversal oscillation, and that the establishment of a cross seiche is favoured by the existence of bays on the two sides of the lake, the ends of which determine the axis of the seiche. This view is strongly supported by the results of the Japanese observers regarding secondary tidal oscillations in the bays of the coast of Japan, to which we have already referred.

There remain, however, two difficulties as regards Loch Earn. I have calculated by means of the parabolic approximation the periods of the cross seiches for various breadths of Earn, and find values which average $1{\cdot}85^m$, the smallest being $1{\cdot}83^m$, the greatest $2{\cdot}30^m$. The section at the eastern binode, where the observations above referred to were made with the statolimnograph and the Sarasin limnograph, is very nearly parabolic in shape, and the period there would be $1{\cdot}9^m$ or more, which exceeds any of the periods observed in the embroidery by more than any likely error, either of observation or calculation.

Then there is the further fact, already mentioned, that no correspondence of phase could be detected, although it was anxiously looked for, and indeed at first expected.

3. Another cause of the embroidery of the limnogram may possibly be found in progressive surface waves and wave groups.

Everyone is aware that the effect of a persistent wind, which has blown for some time along a lake-surface, is to produce a progressive train of waves travelling down the wind. The height and also the length of these waves depend on the "fetch," *i.e.* the length of water over which the wind has blown, as well as on its velocity. The range and the wave-length both increase as we go "down the wind," until at last the wave-crests break and "white horses" are formed. Then a sort of dynamical equilibrium is

* See fig. 19, where the statolimnograms in question are reproduced.

† *Petermanns Geog. Mittheilungen*, Heft ii., 1908.

established, and the range and wave-length increase no longer, unless the waves run into shallow water. This progressive surface wave motion may persist for a considerable time (in the ocean for days) after the wind has fallen, in the form of swell; and it may be propagated into regions where there has been no wind. In ordinary circumstances, owing to the continual variation in the strength of the wind, and in the case of lakes probably also to reflections from the shores, at any particular moment not one train of waves is generated, but many of slightly differing wave-length and differing phases. These trains interfere and cause a succession of wave maxima, commonly called "wave groups."

Several observations of the periods of surface waves and wave groups on Loch Earn were made by counting the waves or wave maxima which passed a given point in a certain time. This is easy in the case of the wave maxima; not so easy in the case of the single waves, which have a bewildering habit of losing themselves by running into and through each other and through the maxima. Still, the results were fairly concordant. The observations were made at the eastern binode and at the Picnic Point during westerly winds of various kinds. The average of the periods for single waves was $\cdot 035^m$, the smallest and greatest values being $\cdot 024^m$ and $\cdot 045^m$. The most usual value of the period for the groups was $\cdot 5^m$ to $\cdot 66^m$, the least and greatest values observed being $\cdot 33^m$ and $1\cdot 17^m$. For the single waves ranges of 6 in. to 12 in. were common; but on one stormy day ranges of 2 ft. to 3 ft. were observed.

From a set of observations made at my request by Mr JAMES MURRAY on Loch Tay, the following data were calculated for that lake. T is the period, λ the wave-length, and v the velocity of propagation for the single waves; T_g, λ_g, v_g, the corresponding magnitudes for the wave groups. The observations were made at Killin, when there was no wind, on swell coming in from the lake and running in water 13 ft. 6 in. to 12 ft. deep.*

For Single Waves.

$T = \cdot 017^m$, $\lambda = 18$ ft. to 25 ft., $v = 18$ to 25 (ft./sec.) = 12 to 17 (mile/hour).

For Larger Groups of 4 *to* 6 *Maximum Waves.*

$T_g = \cdot 5^m$ to $\cdot 75^m$, $\lambda_g = 252$ ft. to 283 ft., $v_g = 8\cdot 4$ to $6\cdot 3$ (ft./sec.) = $5\cdot 7$ to $4\cdot 3$ (mile/hour).

For Smaller Groups of 2 *to* 3 *Maximum Waves.*

$T_g = \cdot 083^m$ to $\cdot 17^m$, $\lambda_g = 42$ ft. to 63 ft., $v_g = 8\cdot 4$ to $6\cdot 3$ (ft./sec.) = $5\cdot 7$ to $4\cdot 3$ (mile/hour).

It is obvious that the single waves could not cause the ordinary and most prominent periods in the embroidery, which run from about $\cdot 5^m$ to $1\cdot 5^m$; but there is no doubt that they cause the thickening or blurring of the limnogram which usually appears when the wind is high. On the other hand, the periods of the wave groups are nearly coincident with some of the more prominent periods of the embroidery. Part of this

* The velocity of a "long wave" in which would be about 20 (ft./sec.).

embroidery may therefore be due to wave groups; but more observations are required to settle the matter beyond doubt.*

4. In a paper "On the Relation between the Velocity of the Wind and the Dimension of Oceanic Waves, with an Explanation of the Waves of Longer Period on Open Coasts,"† Professor BÖRGEN has suggested that the secondary tidal oscillations and waves of unusually long periods occasionally observed on open coasts, where the circumstances do not seem to justify the assumption of a seiche, may be due to difference and summation waves (whose theoretical existence arises from the non-applicability of the theory of the linear superposition of small motions), after the analogy of the difference and summation tones of HELMHOLTZ. It is quite possible that some such explanation may apply in part to lake vibrations; but we have no evidence to produce for or against such a hypothesis.

5. Towards the end of our survey of Loch Earn, we made some observations with the statolimnograph (unfortunately we had time to make only a few) which point to yet another explanation of some part, especially the more irregular part, of the embroidery on the limnograph.

In fig. 19 are placed together two statolimnograms, which were taken in close succession at two stations near to each other on the northern shore of Loch Earn, during a moderate westerly breeze [mean velocity 12 to 16 (mile/hour), extreme velocity occasionally 24 (mile/hour)]. The upper one was taken in a sheltered bay to leeward of the delta of the Glentarken Burn, the lower about 100 yards farther west, to windward of the delta. The bay was comparatively calm, disturbed only by the swell propagated into it from the wind waves rolling outside. The difference between the two limnograms is very striking. The maximum range of the embroidery to windward is much greater, and the pattern is much more irregular and complicated. What

* It is much to be desired that further observations should be made on the period, wave-length, and velocity of propagation of single waves and wave groups, in lakes, on sea-coasts, and in the open sea. Sailors have many opportunities for such observations; and physicists might devote some attention to the matter, when they take an open-air vacation from the ardent pursuit of the electron.

It is curious how ignorant we still are regarding some of the most important hydrodynamical phenomena, notwithstanding something like a century and a half of continued researches, both mathematical and experimental. We know very little, for example, regarding the action by which the wind increases the range and the length of the waves as we pass to windward.

We are told,[1] and it is easy to understand, that a wind whose velocity is greater than the velocity of progression of a train of waves must increase their range; but what is the explanation of the increase of wave-length? Observations, some of which are mentioned below, have strongly suggested the following as the *modus operandi*:—The dynamic instability of the surface after the wind has reached a certain velocity leads to the generation of wave trains of slightly varying length and phase. These trains interfere and produce wave maxima The wind, so long as it travels faster than the wave maxima, will increase the range of the waves near the maxima more than elsewhere. Thus the periodically occurring wave maxima will be elevated into independent wave trains no longer resolvable into the previous harmonic components. Thus a new train of progressive waves will be formed of considerably greater mean range and mean wave-length than before, but of slightly differing ranges and wave-lengths. These again will interfere, and through the action of the wind generate other trains of still greater mean range and mean wave-length; and so on, until the process is stopped by the breaking of the wave crests. This is merely a speculation, without sufficient basis, either theoretical or experimental; but the subject seems to call for investigation, and its practical importance is undeniable.

† *Annalen der Hydrographie und maritimen Meteorologie*, Heft i., Jan. 1890.

[1] See Lamb's *Hydrodynamics*, p. 569 (1906).

remains to leeward has much the same prominent periods as we observe to windward, viz. ·4^{m} to ·5^{m}; but it is obvious that the intervening promontory has screened off a great

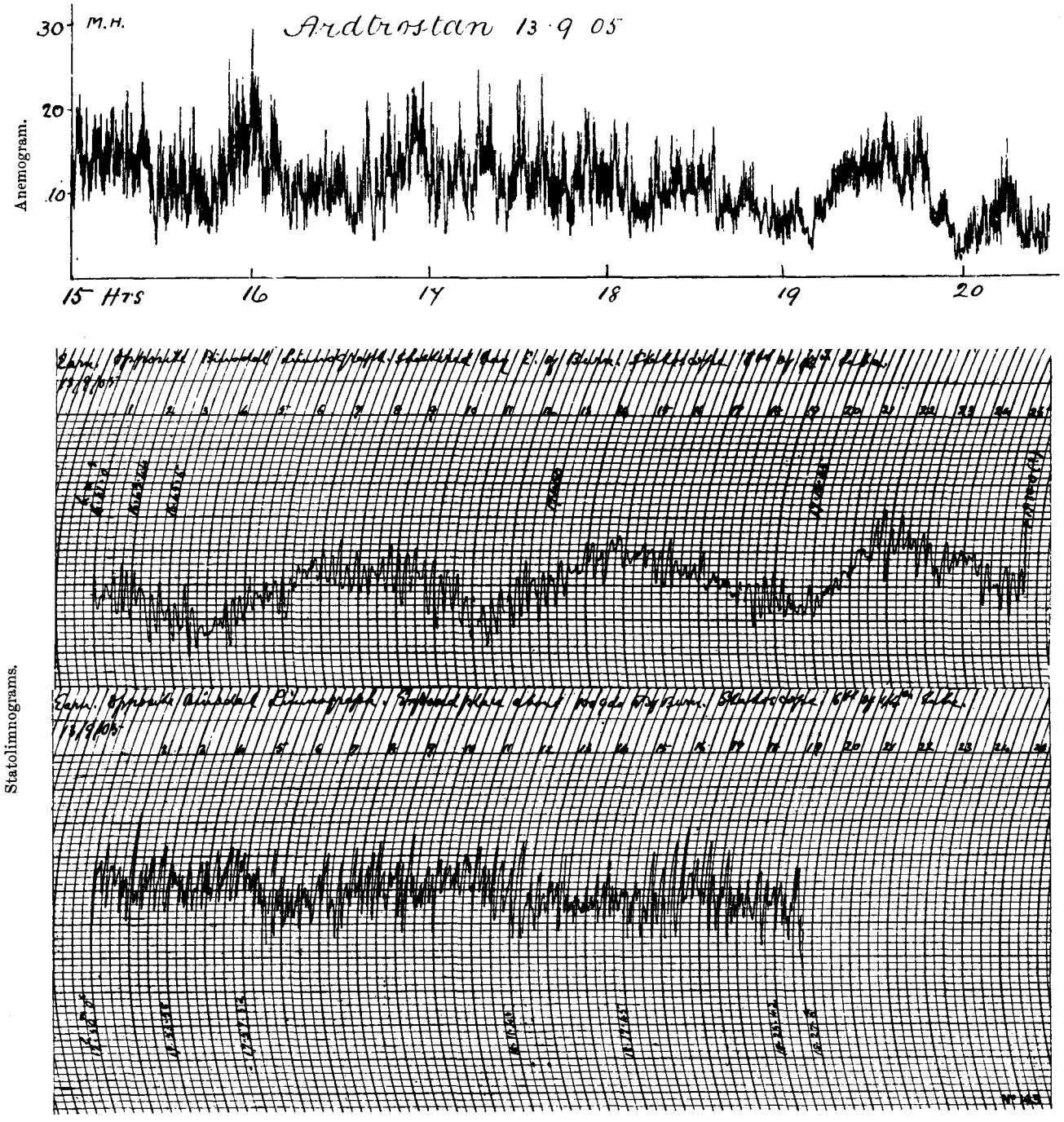


FIG. 19.

part of the vibrations. The part thus screened off could only consist of surface waves of short length, and could not consist either of longitudinal or of transverse seiches.

Again, I often watched the statolimnograph slowly inscribing indentations, such as

those which are so marked in the lower limnogram in fig. 19, and noticed over and over again that it would set one down in an interval of total or comparative calm. On looking to windward when this happened, a black line would be seen on the water some distance off, indicating a coming wind-squall; then presently would be heard the rustle of the wind in the trees overhead; and the increased prattle of the waves among the pebbles on the beach would show that the squall had reached the observer. In short, the lake vibration had gone before, and the wind had followed after. The explanation seems to be that the squall exerts a horizontal traction on the water and causes a drift current. By and by this current becomes greater than the compensating return current underneath. Thus a hump (or a group of waves) is raised on the surface, which is propagated in the water with a speed usually exceeding the velocity of the wind in a moderate breeze. We have here, in fact, in small a phenomenon with which sailors are familiar

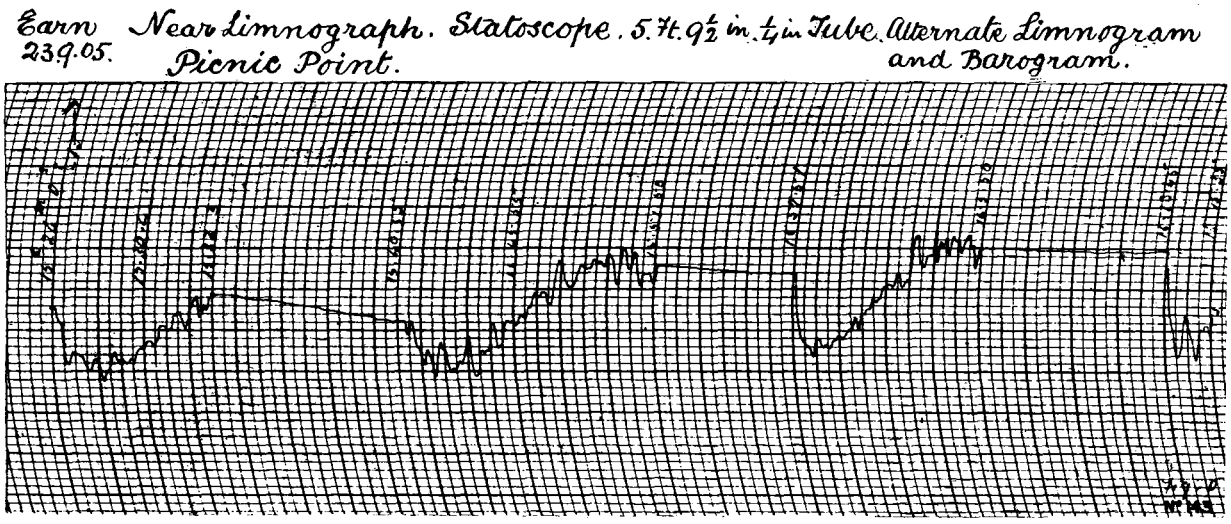


FIG. 20.

on a large scale, when they point to the long swell which records or presages a distant storm at sea.

I obtained a striking confirmation of this view in the course of an observation planned to test a totally different hypothesis. I had supposed that the vibrations might be due to some extent to simultaneous abrupt or periodic disturbances of the atmospheric pressure. As explained in Part I. of this report,* the statolimnograph can be used in rapid alternation as a limnograph and as a microbarograph. Fig. 20 shows the result of an observation of this kind.† The limnogram is deeply embroidered; the microbarogram is all but straight. Since the sensitiveness of the Richard statoscope is fifteen to twenty times that of a mercury barometer, the ordinate of the microbarogram represents the air-pressure on a larger scale than a water barometer. If we allow for the damping effect of the well and access tube on the half-minute vibrations, we shall

* *Trans. R.S.E.*, vol. xlv., p. 368 (1906).

† Another is given in Part I. of this report, *Trans. R.S.E.*, vol. xlv., p. 370, fig. 12 (1906).

be under the mark if we admit that the statolimnograph magnified the range of these vibrations three times. The obvious conclusion is that there was no disturbance of the atmospheric pressure of an order sufficient to cause directly the embroidery observed on the limnogram. It follows that it must have been due to some cumulative atmospheric cause whose action originated at a distance from the observers, and I am inclined to look for this cause in the surface waves, solitary or periodic or quasi-periodic, caused by the heaping action of the wind. It is, of course, obvious that such action as this would be screened off by a promontory or an island, and would be most marked at the windward end of a lake. This cause was suggested, under the name of *Windstau*, by ENDRÖS in his classical memoir on the complicated seiches of the Chiemsee, which has done so much to enlarge our knowledge of lake oscillations.

Before concluding this part of my report, I must say a word or two in recognition of the services of those who assisted us in our seiche survey.

In the first place, thanks are due to Mr LAURENCE PULLAR, whose generosity furnished the greater part of the money required for our undertaking.

We have also to thank the Government Grant Committee of the Royal Society of London, for giving us a grant for the hire and instalment of meteorological apparatus; and Mr W. N. SHAW, Director of the Meteorological Office, for much sympathy, which went the length of a visit to Loch Earn, and even to manual assistance in the erection of some of our instruments.

To the constructive skill and practical scientific capacity of Mr JAMES MURRAY we owed the overcoming of many of our early difficulties, and also some of our best observations.

Messrs P. WHITE and W. WATSON, who were specially attached to the Lake Survey for our work during August and September 1905, worked throughout with the greatest zeal and good judgment. Most of the meteorological observations and nearly all of the later observations with the index limnographs were made by them.

Mr MACDONALD, Schoolmaster, and Mr THORNTON, Postmaster, Lochearnhead, and also the Postmistress at St Fillans, kindly assisted us by taking charge of barographs; and the Stationmasters at Lochearnhead and Balquhidder did their best to give us correct time.

The proprietors of the shores of Loch Earn most courteously allowed us to instal our fixed instruments in the most suitable places. In particular, Colonel STEWART of Ardvoirlich not only allowed us to put up our limnographs on his property, but permitted us to cut wood for the staging, and furnished us with information regarding the daily rainfall at Ardvoirlich.

The photographic reproduction of the figures in this report was in many cases difficult, owing to the faintness of some of the records, in all cases tedious, owing to the troublesome adjustment of scale. The patient intelligence devoted to this part of the work by the firm of HISLOP & DAY deserves the highest praise.

Finally, I must apologise to the Trustees of the Lake Survey and to those who assisted in the Seiche Survey for the long delay in the completion of the report. This delay has been due to the heavy pressure of unavoidable professional and public duties that has fallen upon me during the three years that have passed since the pleasant autumn when we worked together upon Loch Earn.

PART V.

MATHEMATICAL APPENDIX ON THE EFFECT OF PRESSURE DISTURBANCES UPON THE SEICHES IN A UNIFORM PARABOLIC LAKE.

Estimation of the Effect of Pressure Disturbances on the Seiches in a Symmetric Parabolic Lake of Uniform Breadth.

1. In what follows I shall use the method of *Normal Co-ordinates* introduced by Lord Rayleigh,* to which reference was made in my memoir on the Hydrodynamical Theory of Seiches, § 21.†

With very slight and obvious modifications, the notation employed is the same as in the memoir just referred to, and, to make the results approximately applicable to Loch Earn, it may be supposed that the length, $2a$, of the symmetric parabolic lake is 6 miles, say 10^6 cm., and the maximum depth 270 feet, say 8000 cm. Unless the contrary is indicated, C.G.S. units are used throughout.

Then we have, if ξ and ζ be the horizontal and vertical displacement at time t of a particle on the surface of the lake,

$$(1-w^2)\xi = u = -\Sigma a k_\nu \cos n_\nu(t-\tau_\nu)Q_\nu(w) \quad . \quad . \quad . \quad . \quad (1),$$

$$\zeta = +\Sigma k_\nu \cos n_\nu(t-\tau_\nu)Q'_\nu(w) \quad . \quad . \quad . \quad . \quad (2),$$

where $n_\nu^2 = gh\nu(\nu+1)/a^2$, $w = x/a$; k_ν is the extreme amplitude of the ν-nodal seiche corresponding to $x = +a$, *i.e.* to $w = +1$; and $Q_\nu(w)$ is a solution of the equation

$$(1-w^2)Q''_\nu(w) + \nu(\nu+1)Q_\nu(w) = 0 \quad . \quad . \quad . \quad . \quad (3),$$

which vanishes when $w = \pm 1$, and is such that $Q'_\nu(1) = 1$.

It is convenient for our present purposes to use the forms of the Seiche Functions for which $c = \nu(\nu+1)$, given by Dr Halm,‡ viz.—

$$Q_\nu(w) = \frac{1}{2^\nu \nu!}\frac{d^{\nu-1}}{dw^{\nu-1}}(w^2-1)^\nu \quad . \quad . \quad . \quad . \quad (4),$$

$$Q'_\nu(w) = \frac{1}{2^\nu \nu!}\frac{d^\nu}{dw^\nu}(w^2-1)^\nu \quad . \quad . \quad . \quad . \quad (5).$$

* *Theory of Sound*, vol. i., § 87 (1877).
† *Trans. R.S.E.*, vol. xli. (1905).
‡ *Ibid.*, p. 660 (1905).

The following table gives the values of $Q_\nu(w)$ and $Q'_\nu(w)$ for the first five values of ν:—

ν	$Q_\nu(w)$	$Q'_\nu(w)$
1	$\frac{1}{2}(w^2-1)$	w
2	$\frac{1}{2}(w^3-w)$	$\frac{1}{2}(3w^2-1)$
3	$\frac{1}{8}(5w^4-6w^2+1)$	$\frac{1}{2}(5w^3-3w)$
4	$\frac{1}{8}(7w^5-10w^3+3w)$	$\frac{1}{8}(35w^4-30w^2+3)$
5	$\frac{1}{16}(21w^6-35w^4+15w^2-1)$	$\frac{1}{8}(63w^5-70w^2+15w)$

It will be observed that $Q'_\nu(w)$ is the zonal harmonic of the νth order; so that $Q_\nu(\pm 1)=0$, $Q_{2p}(0)=0$, $Q_{2p-1}(0)=(-1)^p 1\cdot 3 \ldots (2p-3)/2^p p!$; and $Q'_\nu(1)=1$, $Q'_\nu(-1)=(-1)^\nu$, $Q'_{2p-1}(0)=0$, $Q'_{2p}(0)=1\cdot 3 \ldots (2p-1)/2\cdot 4 \ldots 2p$.

If now we put

$$\left.\begin{aligned}\phi_\nu &= ak_\nu \cos n_\nu(t-\tau_\nu),\\ &= A_\nu \cos n_\nu t + B_\nu \sin n_\nu t\end{aligned}\right\} \qquad (6),$$

we may write the general equations which represent the motion of the lake in the case where the atmospheric pressure is uniform

$$h(1-w^2)\xi = u = -\Sigma\phi_\nu Q_\nu(w) \qquad (7),$$

$$\zeta = +\sum \frac{\phi_\nu}{a} Q'_\nu(w) \qquad (8);$$

and $\phi_1, \phi_2, \ldots, \phi_\nu, \ldots$, infinite in number, may be regarded as the *normal co-ordinates* of the *motion* in LORD RAYLEIGH'S sense of the phrase.

If $\mathfrak{T}$ be the kinetic and $\mathfrak{V}$ the potential energy in the case just supposed, we have

$$\mathfrak{T} = \tfrac{1}{2}\int_{-a}^{+a} dx h(1-x^2/a^2)\dot{\xi}^2,$$

$$= \frac{a}{2h}\int_{-1}^{+1} \frac{dw}{1-w^2}\{\Sigma\dot{\phi}_\nu Q_\nu(w)\}^2,$$

$$= \frac{a}{2h}\int_{-1}^{+1} \frac{dw}{1-w^2}\Sigma\dot{\phi}_\nu^2\, Q_\nu^2(w);$$

that is,

$$\mathfrak{T} = \tfrac{1}{2}\Sigma a_\nu \dot{\phi}_\nu^2 \qquad (9),$$

where

$$a_\nu = \frac{a}{h}\int_{-1}^{+1} \frac{dw Q_\nu^2(w)}{1-w^2} \qquad (10)$$

Since the co-ordinates $\phi_1, \phi_2, \ldots$ are normal, the products $\dot{\phi}_r\dot{\phi}_s$ do not appear in the expression for $\mathfrak{T}$.

Also, the zero configuration being the lake at rest, we have

$$\mathfrak{V} = +\tfrac{1}{2}g\int_{-a}^{+a} dx\zeta^2,$$
$$= +\frac{g}{2a}\int_{-1}^{+1} dw\{\Sigma\phi_\nu Q'_\nu(w)\}^2,$$
$$= +\frac{g}{2a}\int_{-1}^{+1} dw\Sigma\phi_\nu^2 Q'_\nu(w)^2,$$

since the co-ordinates are normal,

$$= +\tfrac{1}{2}\Sigma b_\nu\phi_\nu^2 \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where

$$b_\nu = \frac{g}{a}\int_{-1}^{+1} dw Q'_\nu(w)^2 \quad . \quad . \quad . \quad . \quad . \quad (12).$$

By a well-known property of the zonal harmonic, $\int_{-1}^{+1} dw Q'_\nu(w)^2 = 2/(2\nu+1)$.

Hence

$$b_\nu = \frac{2g}{(2\nu+1)a} \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Since the hypothesis of long waves involves the neglect of the squares and products of $\phi_1, \phi_2, \ldots; \dot{\phi}_1, \dot{\phi}_2, \ldots$ in the equations of motion, the Lagrangian equations for the motion of the lake reduce to

$$\frac{d}{dt}\left(\frac{\partial\mathfrak{T}}{\partial\dot{\phi}_\nu}\right) + \frac{\partial\mathfrak{V}}{\partial\phi_\nu} = 0 \quad (\nu = 1, 2, \ldots) \quad . \quad . \quad . \quad . \quad (14);$$

that is to say

$$a_\nu\ddot{\phi}_\nu + b_\nu\phi_\nu = 0 \quad (\nu = 1, 2, \ldots) \quad . \quad . \quad . \quad . \quad (15).$$

Since (15) must be satisfied by $\phi_\nu = ak_\nu \cos n_\nu(t - \tau_\nu)$, we must have

$$a_\nu = b_\nu/n_\nu^2,$$
$$= \frac{2a}{h\nu(\nu+1)(2\nu+1)} \quad . \quad . \quad . \quad . \quad . \quad (16).*$$

2. *Effect of a Uniform Excess of Pressure ∂p over a Part of the Lake.*—Let us now suppose that an excess of pressure of ∂p (measured in cm. of water) extends from the point corresponding to $w = \lambda$ to the point corresponding to $w = \mu$, and that this excess begins at $t = 0$ and ends at $t = T$.

In the analysis everything will be as before, except that there will be an addition to the potential energy of

$$g\partial pa\int_\lambda^\mu dw\zeta = g\partial p\int_\lambda^\mu dw\Sigma\phi_\nu Q'_\nu(w).$$

* It follows, of course, that $\int_{-1}^{+1}\frac{dwQ_\nu^2}{1-w^2} = \frac{2}{\nu(\nu+1)(2\nu+1)}$, which may be readily verified independently.

Hence we shall now have

$$\mathfrak{V} = \tfrac{1}{2}\Sigma b_\nu \phi_\nu^2 + \Sigma e_\nu \phi_\nu \quad . \quad . \quad . \quad . \quad . \quad (17),$$

where b_ν has the same value as before; and

$$e_\nu = g\partial p\{Q_\nu(\mu) - Q_\nu(\lambda)\} \quad . \quad . \quad . \quad . \quad . \quad (18).$$

The equations for the motion of the lake from $t=0$ to $t=T$ are now

$$a_\nu \ddot{\phi}_\nu + b_\nu \phi_\nu + e_\nu = 0 \quad (\nu = 1, 2, \ldots) \quad . \quad . \quad . \quad . \quad (19);$$

and from $t=T$ onwards

$$a_\nu \ddot{\phi}_\nu + b_\nu \phi_\nu = 0 \quad (\nu = 1, 2, \ldots) \quad . \quad . \quad . \quad . \quad (20),$$

as before, with the condition that the values of ϕ_ν and $\dot{\phi}_\nu$ must be continuous when $t=T$.

Since our equations of motion are all linear, and (15) are linear and homogeneous, any admissible solution of (15) may be added to any solution of (19). It will therefore be convenient first to find the integral equations of motion corresponding to our supposed disturbance operating upon a lake initially wholly at rest. If we superpose upon this motion that represented by the equations (6), (7), (8), we shall obtain the integral equations of motion (after $t=T$) corresponding to our disturbance when it operates on a lake in which the initial motion is given by

$$\left.\begin{aligned} \zeta &= \Sigma k_\nu Q'_\nu(w) \cos n_\nu \tau_\nu, \\ \dot{\zeta} &= -\Sigma k_\nu n_\nu Q'_\nu(w) \sin n_\nu \tau_\nu \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (21).$$

The general solutions of (19) and (20) are

$$\phi_\nu = A'_\nu \cos n_\nu t + B'_\nu \sin n_\nu t - B_\nu/b_\nu \quad (\nu = 1, 2, \ldots),$$

and

$$\phi_\nu = A''_\nu \cos n_\nu t + B''_\nu \sin n_\nu t \quad (\nu = 1, 2, \ldots),$$

where A'_ν, B'_ν, A''_ν, B''_ν are constants to be determined by the conditions that ζ and $\dot{\zeta}$ shall vanish when $t=0$, and be continuous when $t=T$.

We thus get

$$\zeta = \Sigma f_\nu (1 - \cos n_\nu t) Q'_\nu(w) \quad . \quad . \quad . \quad . \quad . \quad (22),$$

where

$$\frac{e_\nu}{ab_\nu} = f_\nu = -\tfrac{1}{2}(2\nu+1)\partial p\{Q_\nu(\mu) - Q_\nu(\lambda)\} \quad . \quad . \quad . \quad . \quad (23),$$

when $0 \leqslant t \leqslant T$; and

$$\left.\begin{aligned} \zeta &= \Sigma f_\nu\{-(1 - \cos n_\nu T) \cos n_\nu t + \sin n_\nu T \sin n_\nu t\} Q'_\nu(w), \\ &= 2\Sigma f_\nu \sin\left(\frac{n_\nu T}{2}\right) \sin n_\nu (t - \tfrac{1}{2}T) Q'_\nu(w) \end{aligned}\right\} \quad . \quad . \quad . \quad (24),$$

when $t \geqslant T$.

From the second form of the equation (24) it follows (as is otherwise obvious) that, *cæteris paribus*, the disturbing effect on the ν-nodal seiche is greatest when $T = \pi/n_\nu$, *i.e.* when T is half the period of the ν-nodal seiche.

3. *General Case.*—If now we suppose that initially the extreme amplitudes of the

component seiches are k_1, k_2, ... and the phases τ_1, τ_2, ... , we get by superposition the general solution for the motion when $t \geqslant T$:

$$\left.\begin{aligned}\zeta &= \Sigma[k_\nu \cos n_\nu(t-\tau_\nu) + f_\nu\{-(1-\cos n_\nu T)\cos n_\nu t + \sin n_\nu T \sin n_\nu t\}]Q'_\nu(w),\\ &= \Sigma[\{k_\nu \cos n_\nu\tau_\nu - f_\nu(1-\cos n_\nu T)\}\cos n_\nu t + \{k_\nu \sin n_\nu\tau_\nu + f_\nu \sin n_\nu T\}\sin n_\nu t]Q'_\nu(w),\\ &= \Sigma k'_\nu \cos n_\nu(t-\chi_\nu)Q'_\nu(w);\end{aligned}\right\} \quad . \quad (25),$$

where

$$\left.\begin{aligned}\tan n_\nu\chi_\nu &= \frac{k_\nu \sin n_\nu\tau_\nu + f_\nu \sin n_\nu T}{k_\nu \cos n_\nu\tau_\nu - f_\nu(1-\cos n_\nu T)},\\ \tan n_\nu(\chi_\nu - \tau_\nu) &= \frac{2f_\nu \sin\left(\frac{n_\nu T}{2}\right)\cos n_\nu(\tau_\nu - \frac{1}{2}T)}{k_\nu + 2f_\nu \sin\left(\frac{n_\nu T}{2}\right)\sin n_\nu(\tau_\nu - \frac{1}{2}T)};\\ k'^2_\nu &= k_\nu^2 + 2k_\nu f_\nu\{\cos n_\nu(\tau_\nu - T) - \cos n_\nu\tau_\nu\} + 2f_\nu^2(1-\cos n_\nu T),\\ &= k_\nu^2 + 4k_\nu f_\nu \sin\left(\frac{n_\nu T}{2}\right)\sin n_\nu(\tau_\nu - \tfrac{1}{2}T) + 4f_\nu^2 \sin^2\left(\frac{n_\nu T}{2}\right)\end{aligned}\right\} \quad . \quad . \quad (26).$$

The maximum value of k'_ν for a given value of T is therefore given by taking $\sin n_\nu(\tau_\nu - \frac{1}{2}T) = \pm 1$. We then have

$$k'_\nu = k_\nu + 2\left|f_\nu \sin\left(\frac{n_\nu T}{2}\right)\right|.$$

The phase disturbance is then zero.

4. *Particular Cases.*—Suppose $T = \pi/n_1 = \frac{1}{2}T_1$, $\tau_1 = 0$, $\lambda = 0$, $\mu = +1$, which gives us a disturbance extending over the positive half of the lake, and beginning when the uninodal seiche is at its maximum extreme amplitude and lasting for half the uninodal period.

Then

$$f_1 = -\tfrac{3}{4}\partial p, \quad f_2 = 0, \quad f_3 = +\tfrac{7}{16}\partial p, \quad f_4 = 0.$$

Hence the seiches of even nodality are all unaffected; and, if we denote the uninodal, trinodal, etc. components by ζ_1, ζ_3, ... , we have

$$\begin{aligned}\zeta_1 &= (k_1 + \tfrac{3}{2}\partial p)Q'_1(w)\cos n_1 t;\\ \zeta_2 &= k_2 Q'_2(w)\cos n_2(t-\tau_2);\\ \zeta_3 &= \left\{k_3 - \tfrac{7}{8}\partial p \sin\left(\frac{n_3\pi}{2n_1}\right)\right\}Q'_3(w)\cos n_3(t-\tau_3),\end{aligned}$$

where we have determined τ_3 so that the disturbance of the trinodal component shall be the greatest possible.

Since $n_3/n_1 = \sqrt{(c_3/c_1)} = \sqrt{6} = 2{\cdot}450$, and $\sin(n_3\pi/2n_1) = {\cdot}687$, we have in the case of maximum disturbance

$$\zeta_3 = (k_3 + {\cdot}601\partial p)Q'_3(w)\cos n_3(t-\tau_3).$$

It is unnecessary to set down more formulæ, but the following table will give an idea of the effect of various kinds of uniform pressure disturbances on the uninodal, binodal, and trinodal seiche components. B and B′ denote the two binodes, U the uninode, T the trinode, and A the positive end of the lake. ∂k_ν denotes the increase

of the extreme amplitude of the ν-nodal seiche in the case where the phase is such that the increase (due to the disturbance of pressure ∂p lasting for a time T) is a maximum.

T	Extent.	$2f_1/\partial p$	$2f_2/\partial p$	$2f_3/\partial p$	$\partial k_1/\partial p$	$\partial k_2/\partial p$	$\partial k_3/\partial p$
$\frac{1}{2}T_1$	UA	−1·500	0	·875	1·500	0	·601
$\frac{1}{2}T_2$	BA	−1·000	− ·962	·388	·787	·962	·309
,,	UB	− ·500	·962	1·264	·394	·962	1·006
,,	B′B	0	−1·925	0	0	1·925	0
$\frac{1}{2}T_3$	TA	− ·600	− ·775	−·700	·359	·695	·700
,,	UT	− ·900	− ·775	1·575	·538	·695	1·575

5. *Disturbance caused by a suddenly generated Distribution of Pressure, given by the Law* $\partial p = f(w)$, lasting from $t = 0$ *to* $t = T$.—The only supposition we shall make is that $f(w)$ is expansible in a series of zonal harmonics; so that we have

$$\partial p = \Sigma q_\nu Q'_\nu(w) \quad . \quad . \quad . \quad . \quad . \quad (27),$$

where

$$q_\nu = \tfrac{1}{2}(2\nu+1)\int_{-1}^{+1} dw f(w) Q'_\nu(w).*$$

The addition to the potential energy $\mathfrak{V}$ due to this distribution of pressure is given by

$$ga\int_{-1}^{+1} dw \partial p \zeta = g\int_{-1}^{+1} dw \{\Sigma q_\nu Q'_\nu(w)\}\{\Sigma \phi_\nu Q'_\nu(w)\}.$$

Now, since $\int_{-1}^{+1} dw Q'_\mu(w) Q'_\nu(w) = 0$ if $\mu \neq \nu$, we have

$$ga\int_{-1}^{+1} dw \partial p \zeta = \Sigma e_\nu \phi_\nu,$$

where

$$e_\nu = g q_\nu \int_{-1}^{+1} dw \{Q'_\nu(w)\}^2,$$
$$= 2gq_\nu/(2\nu+1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

It follows that the formulæ (22), (24), (25), (26) are all applicable to the general case now under discussion, the only difference being that f_ν is now given by the equation

$$f_\nu = -\frac{e_\nu}{ab_\nu} = -\frac{2gq_\nu}{2\nu+1}\Big/\frac{2g}{2\nu+1} = -q_\nu = -\tfrac{1}{2}(2\nu+1)\int_{-1}^{+1} dw f(w) Q'_\nu(w) \quad . \quad . \quad (29).$$

* See WHITTAKER'S *Modern Analysis*, ch. x., § 128.

In particular, the maximum disturbance of amplitude for the ν-nodal seiche is given by

$$\partial k_\nu = \left| 2q_\nu \sin \frac{n_\nu T}{2} \right| \quad . \quad . \quad . \quad . \quad . \quad (30).$$

It is of special interest to notice that the disturbance of the ν-nodal seiche is due solely to the νth harmonic term in the zonal harmonic expansion of $f(w)$. It follows that a disturbance of pressure which is proportional to $Q'_\nu(w)$ affects the ν-nodal component of the lake oscillation and leaves all the others wholly unaltered.

6. *Example* 1.—Consider the effect of a uniform gradient of pressure suddenly generated over the whole length of the lake, and causing a difference ∂p between the two ends. Such a disturbance will be represented by $\frac{1}{2}\partial p w$; *i.e.* by $q_1 Q'_1(w)$, where $q_1 = \frac{1}{2}\partial p$. This will give $\partial k_2 = 0$, $\partial k_3 = 0$, etc., and

$$\partial k_1 = \left| \partial p \sin \left(\frac{n_1 T}{2} \right) \right|,$$

if we suppose the uninodal seiche in such a phase when the disturbance commences that the maximum effect is produced.

The greatest effect of all results when $T = \pi/n_1 = \frac{1}{2}T_1$; *i.e.* when the disturbance lasts during half the uninodal period. We have then

$$\partial k_1 = \partial p .$$

It thus appears, as a result of our analysis, that a uniform pressure gradient established over the whole of a symmetric parabolic lake can only generate or destroy a pure uninodal seiche. If the oscillation of the lake have any other components, they are unaffected. This conclusion, which might have been expected *a priori*, seems to confirm the soundness of the assumptions on which we have based the present theory.

7. *Example* 2.—If we suppose the pressure disturbance given by $\frac{1}{2}\partial p(3w^2 - 1) = q_2 Q'_2(w)$ $(q_2 = \partial p)$, which gives a parabolic distribution with a turning-point at the middle of the lake, and suppose the disturbance to catch the seiche in the most favourable phase, *i.e.* at a maximum, when the disturbing pressure tends to drive the water in the direction which it would follow if undisturbed, and if we further suppose the disturbance to last for half the binodal period, then we get

$$\partial k_2 = 2q_2 = 2\partial p$$

for the increase of the extreme amplitude of the binodal seiche, all the other seiche components being unaltered.

8. *Example* 3.—In like manner we see that the pressure disturbance which generates a pure trinodal seiche in a symmetric parabolic lake must have the cubic distribution $\frac{1}{2}q_3(5w^3 - 3w)$; and so on.

9. *Effect of a Disturbance of Pressure which varies both in Space and in Time.*—Let us suppose that the pressure, measured in centimetres of water, at time t, at any point $w(= x/a)$ of the parabolic lake, is given by

$$\partial p = f(w, t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

Availing ourselves of the principle of superposition, as heretofore, we can build up the general solution now required by adding together the contributions to ζ due to all the different elements of the lake-surface, and all the different elements of time at each element of surface.

Setting aside in the meantime the initial seiche motion, and calculating merely the part $d^2\zeta$ of ζ due to the disturbance of pressure, corresponding to $\mu \leqslant w \leqslant \mu + d\mu$ and $\mathrm{T} \leqslant t \leqslant \mathrm{T} + d\mathrm{T}$, we see at once from (23) and (24) that

$$d^2\zeta = \tfrac{1}{2}\Sigma(2\nu+1)\mathrm{Q}'_\nu(w)\mathrm{Q}'_\nu(\mu)n_\nu \sin n_\nu(\mathrm{T}-t)f(\mu, \mathrm{T})d\mu d\mathrm{T}.$$

If, therefore, we suppose the disturbance to last from $t=0$ to $t=\mathrm{T}$, we get, if $t \geqslant \mathrm{T}$, the following expression for the contribution to ζ due to the disturbance of pressure :

$$\partial\zeta = \tfrac{1}{2}\Sigma(2\nu+1)\mathrm{Q}'_\nu(w)\int_{-1}^{+1} d\mu \mathrm{Q}'_\nu(\mu)\int_0^{\mathrm{T}} d\mathrm{T} n_\nu \sin n_\nu(\mathrm{T}-t)f(\mu, \mathrm{T}) \quad . \quad . \quad . \quad (31).$$

This last equation will give the required disturbance in any particular case. We have merely to give the proper determination to the function $f(\mu, t)$ and carry out the two integrations. It may be noted that in general the integral $\int_0^{\mathrm{T}} d\mathrm{T} n_\nu \sin n_\nu(\mathrm{T}-t)f(\mu, \mathrm{T})$ will be a function of μ.

If the pressure-disturbance have the form of a wave steady in shape and propagated with a uniform velocity v, then instead of $f(\mu, t)$ we may write the more specialised function $f\{a(1+w)-vt\}$.

10. *Special Case of a Sudden Rise of Pressure* ∂p, *propagated with uniform velocity* v, *starting at the negative end of the lake at* t = 0, *and ceasing all over the lake at* t = T.—First suppose $v\mathrm{T} > 2a$, so that every point of the lake is sooner or later affected. In this case, it is obvious that after T has reached the value $2a/v$, the disturbance contemplated has no longer any effect on the seiche motion, beyond an increase of the pressure everywhere by the amount ∂p. We may therefore suppose $\mathrm{T} = 2a/v$, and our formulæ will be applicable for $t \geqslant 2a/v$.

Since at any particular point $w=\mu$ the pressure is undisturbed until $t = a(1+\mu)/v$, and thereafter is raised by ∂p until $t=\mathrm{T}$, the determination of $f(\mu, t)$ in this case is

$$f(\mu, t) = 0 \text{ for } t < a(1+\mu)/v;$$
$$f(\mu, t) = \partial p \text{ for } t \geqslant a(1+\mu)/v.$$

Hence (31) gives

$$\partial\zeta = \frac{\partial p}{2}\Sigma(2\nu+1)\mathrm{Q}'_\nu(w)\int_{-1}^{+1} d\mu \mathrm{Q}'_\nu(\mu)\int_{a(1+\mu)/v}^{\mathrm{T}} d\mathrm{T} n_\nu \sin n_\nu(\mathrm{T}-t),$$

$$= \frac{\partial p}{2}\Sigma(2\nu+1)\mathrm{Q}'_\nu(w)\int_{-1}^{+1} d\mu \mathrm{Q}'_\nu(\mu)[\cos n_\nu\{a(1+\mu)/v - t\} - \cos n_\nu(\mathrm{T}-t)].$$

Or, since $\int_{-1}^{+1} d\mu Q'_\nu(\mu) = 0$ in all cases,

$$\partial\zeta = \frac{\partial p}{2} \Sigma(2\nu+1)Q'_\nu(w) \int_{-1}^{+1} d\mu Q'_\nu(\mu) \cos n_\nu\{a(1+\mu)/v - t\} \quad . \quad . \quad . \quad (32).$$

If, therefore, we denote the ν-nodal component of the disturbance by $\partial\zeta_\nu$, we have

$$\partial\zeta_\nu = (A'_\nu \cos n_\nu t + B'_\nu \sin n_\nu t)Q'_\nu(w) \quad . \quad . \quad . \quad . \quad (33),$$

where

$$\left.\begin{aligned} 2A'_\nu/(2\nu+1)\partial p &= \int_{-1}^{+1} d\mu Q'_\nu(\mu) \cos \frac{n_\nu a}{v}(1+\mu) \\ 2B'_\nu/(2\nu+1)\partial p &= \int_{-1}^{+1} d\mu Q'_\nu(\mu) \sin \frac{n_\nu a}{v}(1+\mu) \end{aligned}\right\} \quad . \quad . \quad . \quad (34).$$

Next suppose $v\mathrm{T}<2a$. Then the pressure disturbance does not reach farther than $x = v\mathrm{T} - a$, that is $w = v\mathrm{T}/a - 1$; and we have

$$f(\mu, t) = 0, \text{ if } t > \mathrm{T},\ \mu > v\mathrm{T}/a - 1;$$
$$f(\mu, t) = \partial p, \text{ if } a(1+\mu)/v \leqslant t \leqslant \mathrm{T},\ -1 \leqslant \mu \leqslant v\mathrm{T}/a - 1.$$

Hence (31) now gives

$$\begin{aligned} \partial\zeta &= \frac{\partial p}{2} \Sigma(2\nu+1)Q'_\nu(w) \int_{-1}^{v\mathrm{T}/a-1} d\mu Q'_\nu(\mu) \int_{a(1+\mu)/v}^{\mathrm{T}} d\mathrm{T} n_\nu \sin n_\nu(\mathrm{T}-t), \\ &= \frac{\partial p}{2} \Sigma(2\nu+1)Q'_\nu(w) \int_{-1}^{v\mathrm{T}/a-1} d\mu Q'_\nu(\mu)[\cos\{a(1+\mu)/v - t\} - \cos n_\nu(\mathrm{T}-t)] \quad . \quad . \quad (32'). \end{aligned}$$

It follows that, if $t>\mathrm{T}$, then

$$\partial\zeta_\nu = (A'_\nu \cos n_\nu t + B'_\nu \sin n_\nu t)Q'_\nu(w) \quad . \quad . \quad . \quad . \quad (33'),$$

where now

$$\left.\begin{aligned} 2A'_\nu/(2\nu+1)\partial p &= \int_{-1}^{v\mathrm{T}/a-1} d\mu Q'_\nu(\mu) \cos n_\nu \frac{a(1+\mu)}{v} - Q_\nu\left(\frac{v\mathrm{T}}{a} - 1\right) \cos n_\nu \mathrm{T}, \\ 2B'_\nu/(2\nu+1)\partial p &= \int_{-1}^{v\mathrm{T}/a-1} d\mu Q'_\nu(\mu) \sin n_\nu \frac{a(1+\mu)}{v} - Q_\nu\left(\frac{v\mathrm{T}}{a} - 1\right) \sin n_\nu \mathrm{T} \end{aligned}\right\} \quad . \quad . \quad (34').$$

Superposing now the initial motion given by (6) and (8), we get for the ν-nodal component of ζ

$$\left.\begin{aligned} \zeta_\nu &= [\{k_\nu \cos n_\nu\tau_\nu + A'_\nu\} \cos n_\nu t + \{k_\nu \sin n_\nu\tau_\nu + B'_\nu\} \sin n_\nu t]Q'_\nu(w), \\ &= k'_\nu \cos n_\nu(t - \chi_\nu)Q'_\nu(w), \\ \text{where} \quad k'^2_\nu &= k_\nu^2 + 2k_\nu(A'_\nu \cos n_\nu\tau_\nu + B'_\nu \sin n_\nu\tau_\nu) + A'^2_\nu + B'^2_\nu; \\ \tan n_\nu\chi_\nu &= \frac{k_\nu \sin n_\nu\tau_\nu + B'_\nu}{k_\nu \cos n_\nu\tau_\nu + A'_\nu}; \\ \tan n_\nu(\chi_\nu - \tau_\nu) &= \frac{B'_\nu \cos n_\nu\tau_\nu - A'_\nu \sin n_\nu\tau_\nu}{k_\nu + A'_\nu \cos n_\nu\tau_\nu + B'_\nu \sin n_\nu\tau_\nu}. \end{aligned}\right\} \quad . \quad . \quad (35).$$

The values of A'_ν and B'_ν are given by (34) or (34′), according as $v\mathrm{T}>$ or $<2a$.

If the square of ∂p be negligible, so that $\partial k_\nu = k'_\nu - k_\nu$ and $\partial\tau_\nu = \chi_\nu - \tau_\nu$ are both small, we have

$$\left.\begin{aligned}\partial k_\nu &= A'_\nu \cos n_\nu\tau_\nu + B'_\nu \sin n_\nu\tau_\nu \\ \partial\tau_\nu &= (B'_\nu \cos n_\nu\tau_\nu - A'_\nu \sin n_\nu\tau_\nu)/k_\nu n_\nu\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (36).$$

The maximum value of ∂k_ν for different values of τ_ν corresponds to $\tan n_\nu\tau_\nu = B'_\nu/A'_\nu$; and under these circumstances $\partial\tau_\nu = 0$, and $\partial k_\nu = (A'^2_\nu + B'^2_\nu)^{\frac{1}{2}}$.

11. As an example of the application of the above formulæ, let us consider the effect on the uninodal seiche of a sudden rise of pressure ∂p which begins at the negative end of the lake at $t = 0$.

Putting $Q'_1(\mu) = \mu$, and $Q_1(\mu) = \frac{1}{2}(\mu^2 - 1)$, we get

$$\left.\begin{aligned}A_1 &= -\tfrac{1}{2}(2\nu+1)\partial p\Theta, \\ B_1 &= +\tfrac{1}{2}(2\nu+1)\partial p\Phi;\end{aligned}\right.$$

where, if $\theta = n_1 a/v = 2\pi a/vT_1$,

$$\left.\begin{aligned}\Theta &= \frac{1-\cos 2\theta}{\theta^2} - \frac{\sin 2\theta}{\theta}; \\ \Phi &= \frac{\sin 2\theta}{\theta^2} - \frac{1+\cos 2\theta}{\theta},\end{aligned}\right\} \text{ when } vT > 2a:$$

and

$$\left.\begin{aligned}\Theta &= -\left\{\frac{1-\frac{1}{2}(n_1T)^2}{\theta^2} + \frac{n_1T}{\theta}\right\}\cos n_1T - \left\{\frac{n_1T}{\theta^2} - \frac{1}{\theta}\right\}\sin n_1T + \frac{1}{\theta^2}; \\ \Phi &= \left\{\frac{1-\frac{1}{2}(n_1T)^2}{\theta^2} + \frac{n_1T}{\theta}\right\}\sin n_1T - \left\{\frac{n_1T}{\theta^2} - \frac{1}{\theta}\right\}\cos n_1T - \frac{1}{\theta},\end{aligned}\right\} \text{when } vT < 2a. \quad . \quad (37).$$

It will be observed that in general Θ and Φ are functions of v alone when $vT > 2a$; but functions of v and T when $vT < 2a$.

If we restrict ourselves to the case where $T = 2\pi/n_1 = T_1$, then $n_1T_1 = 2\pi$, and we get for the case where $vT_1 < 2a$ [*i.e.* $v < 26{\cdot}9$ (mile/hour)]

$$\Theta = 2\left\{\left(\frac{\pi}{\theta}\right)^2 - \frac{\pi}{\theta}\right\};$$
$$\Phi = -\frac{2\pi}{\theta^2}.$$

In the case under consideration, therefore, the functions Θ and Φ are determined as follows:—

$$\left.\begin{aligned}\Theta &= \frac{1-\cos 2\theta}{\theta^2} - \frac{\sin 2\theta}{\theta} \text{ when } 0 \leqslant \theta \leqslant \pi; \\ &= 2\left\{\left(\frac{\pi}{\theta}\right)^2 - \frac{\pi}{\theta}\right\} \text{ when } \theta \geqslant \pi. \\ \Phi &= \frac{\sin 2\theta}{\theta^2} - \frac{1+\cos 2\theta}{\theta} \text{ when } 0 \leqslant \theta \leqslant \pi; \\ &= -\frac{2\pi}{\theta^2} \text{ when } \theta \geqslant \pi\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (38).$$

The graphs of Θ and Φ are given in figs. 21 and 22.

The greatest possible increase of amplitude is given by

$$\partial k_1 = \tfrac{3}{2}\partial p(\Theta^2 + \Phi^2)^{\frac{1}{2}}.$$

If $vT_1 > 2a$, this leads to

$$\partial k_1 = 3\partial p\left(\frac{\sin\theta}{\theta^2} - \frac{\cos\theta}{\theta}\right) \quad (\theta < \pi) \qquad . \quad . \quad . \quad . \quad (39).$$

The graph of the function

$$\Psi = \frac{\sin\theta}{\theta^2} - \frac{\cos\theta}{\theta}$$

is shown in fig. 23.

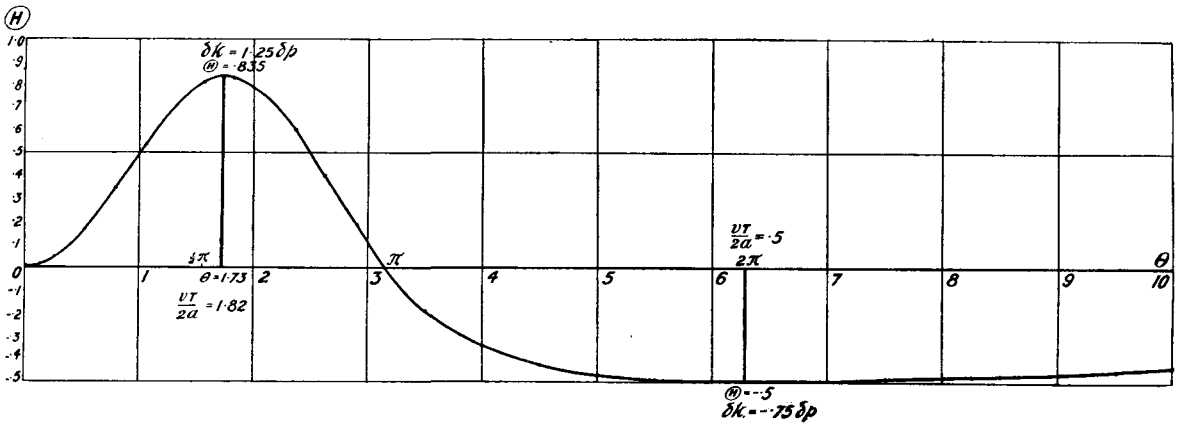


FIG. 21.

The maximum value of ∂k_1 is $1{\cdot}31\partial p$, corresponding to $vT_1/2a = 1{\cdot}51$. Hence, since $T_1 = 14{\cdot}5^{\mathrm{m}}$ and $2a = 6$ miles, the velocity of propagation of the pressure disturbance which has most effect on the uninodal seiche is about 37 (mile/hour).

If $vT_1 < 2a$, then

$$\partial k_1 = 3\pi\partial p\left(\frac{\pi^2+1}{\theta^4} - \frac{2\pi}{\theta^3} + \frac{1}{\theta^2}\right)^{\frac{1}{2}} \quad (\theta > \pi) \qquad . \quad . \quad . \quad . \quad (40).$$

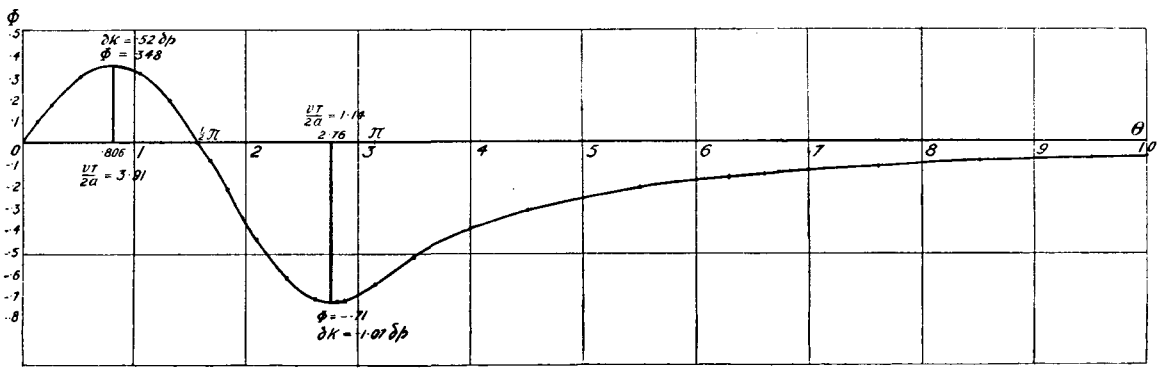


FIG. 22.

This has a minimum value when $\theta = 4{\cdot}0287$, and a maximum when $\theta = 5{\cdot}4061$.

Hence the maximum value of ∂k_1 is given by

$$\partial k_1 = 3\pi \times {\cdot}08234\partial p = {\cdot}776\partial p.$$

The corresponding velocity for the pressure disturbance is about 14 (mile/hour).

12. As a further example, we may take the case where $T = T_1$ as before, and we

suppose the pressure disturbance to reach the negative end of the lake just after the extreme amplitude there has reached a maximum; so that $\tau_1 = \pi/n_1$. In this case we have

$$\partial k_1 = -A_1' = \tfrac{3}{2}\Theta\partial p\,;$$
$$\partial\tau_1 = -B_1'/k_1n_1 = -\tfrac{3}{2}\Phi\partial p/k_1n_1\,.$$

In particular, the maximum increase of the extreme amplitude is $1{\cdot}253\partial p$, corresponding to $vT_1/2a = 1{\cdot}82$. Since $T_1 = 14{\cdot}5^{\mathrm{m}}$, this would give for the velocity of

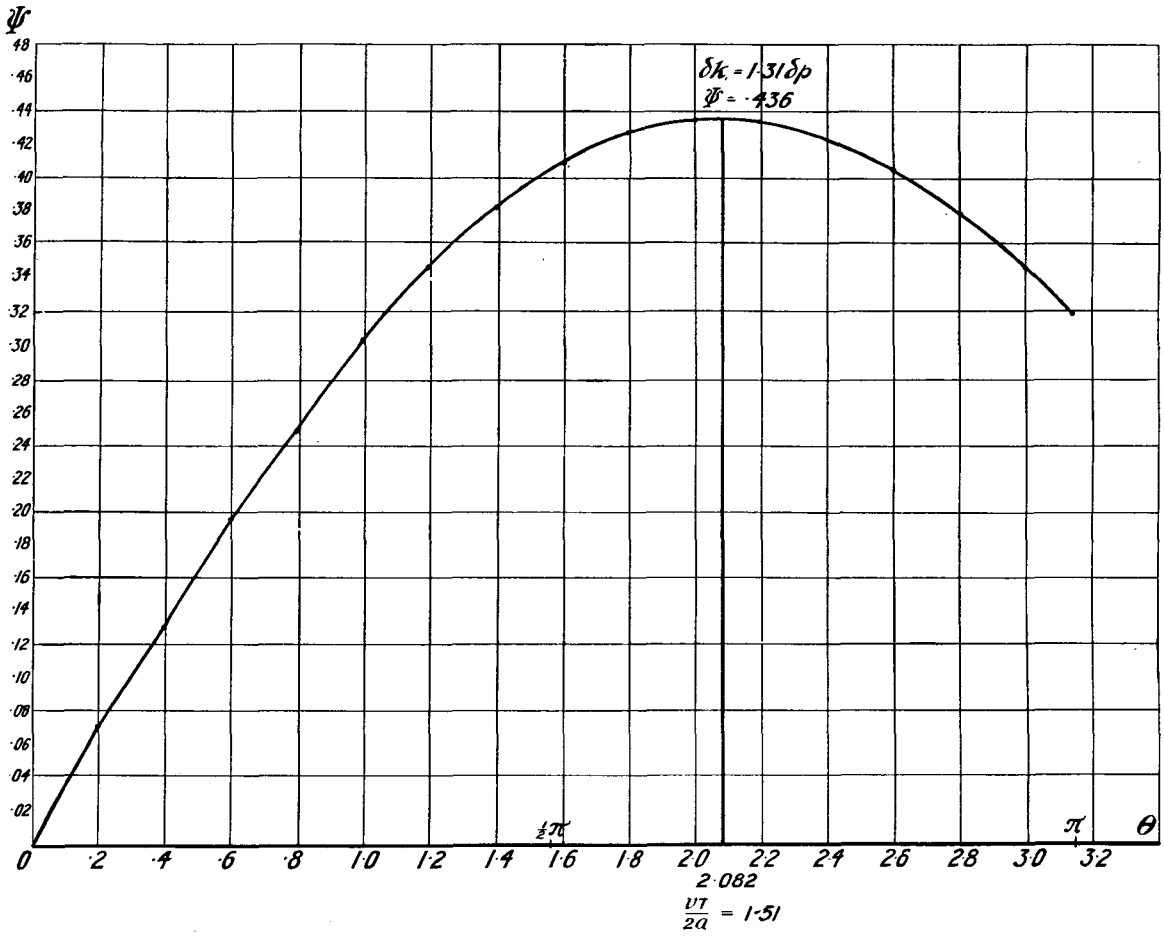


FIG. 23.

propagation of the pressure disturbance which produces the greatest effect about 45 (mile/hour).

If we put $\tau_1 = 0$, and suppose $vT_1 < 2a$, we get

$$\partial k_1 = A_1' = -\tfrac{3}{2}\Theta\partial p\,;$$
$$\partial\tau_1 = B_1'/k_1n_1 = \tfrac{3}{2}\Phi\partial p/k_1n_1\,;$$

and the maximum value of ∂k_1 is $\cdot 75\partial p$, the corresponding value of v being about 12 (mile/hour).

Again, if we take $T = T_1$ as before, but $\tau_1 = \frac{\pi}{2}$,

$$\partial k_1 = B_1' = \tfrac{3}{2}\Phi\partial p\,;$$
$$\partial\tau_1 = -A_1'/k_1n_1 = \tfrac{3}{2}\Theta\partial p/k_1n_1\,;$$

and the maximum value of ∂k_1 is $\cdot 52\partial p$, corresponding to $v = 97$ (mile/hour).

Lastly, for $T = T_1$, $\tau_1 = -\frac{\pi}{2}$,

$$\partial k_1 = -B_1' = -\tfrac{3}{2}\Phi\partial p\,;$$
$$\partial\tau_1 = A_1'/k_1n_1 = -\tfrac{3}{2}\Theta\partial p/k_1n_1\,;$$

and the maximum value of ∂k_1 is $\cdot 71\partial p$, corresponding to $v = 28$ (mile/hour).

Alternative Method.

13. If the disturbance of phase is not required, the following method, by means of which I originally obtained some of the results given above, will furnish the disturbance of the extreme amplitudes of the various seiche components due to a given disturbance of pressure, to the same degree of approximation as Rayleigh's method.

If $K = \mathfrak{T} + \mathfrak{V}$ denote the whole energy of the seiche motion, p the pressure at any point of the water surface, and v_ν the velocity of the water at that point in the direction of the normal to the surface drawn towards the water, then the following equation holds: *

$$\frac{DK}{Dt} = \int_{-a}^{a} dxpv_\nu = a\int_{-1}^{+1} dwpv_\nu\,.$$

It is easy to show that, for our purposes, the above equation may be written

$$\frac{DK}{Dt} = -a\int_{-1}^{+1} dwp\dot{\zeta} \qquad . \quad . \quad . \quad . \quad . \quad (41);$$

for in so doing we neglect only quantities of the orders of k_ν^2/ha ($<1/10^8$) or k_ν^2/a^2 ($<1/10^{10}$), already negligible if we are to apply the theory of long waves.

Suppose now the seiche motion to be analysed into uni-, bi-, tri-, ... nodal components whose amplitudes at the ends of the lake are k_1, k_2, k_3, ...

Since these components are normal modes of motion for the parabolic lake, we may calculate the total energies for each of these seiches separately and independently; and the sum of these energies will be K. Let these partial energies be K_1, K_2, K_3, ...

Taking the ν-nodal seiche by itself, we have the equation

$$\frac{DK_\nu}{Dt} = -a\int_{-1}^{+1} dwp\dot{\zeta}_\nu \qquad . \quad . \quad . \quad . \quad . \quad (42).$$

In the integral on the left-hand side of (42) we need pay no attention to any constant

* See Lamb's *Hydrodynamics*, 3rd ed. (1906), p. 9.

added to p, as the integral of a constant pressure all over the surface of the lake must obviously be zero. Hence we need only consider the disturbing pressure, which may be expressed in centimetres of water as heretofore by $\partial p = f(w, t)$.

If, therefore, ∂K_ν denote the total increment of the energy of the ν-nodal seiche by the disturbing pressure $\partial p = f(w, t)$ acting from $t = 0$ to $t = T$, we have

$$\partial K_\nu = ag \int_{-1}^{+1} dw Q'_\nu(w) \int_0^T dt\, k_\nu n_\nu \sin n_\nu(t - \tau_\nu) f(w, t) \quad . \quad . \quad . \quad (43).$$

The energy of the ν-nodal seiche is equal to its potential energy in the configuration of maximum potential and zero kinetic energy. Hence we have by (11)

$$K_\nu = \frac{ga}{2\nu + 1} k_\nu^2 \quad . \quad . \quad . \quad . \quad . \quad (44);$$

and therefore

$$\partial K_\nu = \frac{2ga}{2\nu + 1} k_\nu \partial k_\nu \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Strictly regarded, k_ν is a function of the time; for the energy of the seiche is being continually altered by the disturbing surface pressure, so that the extreme amplitude k_ν of the seiche at each moment, which would be left if the disturbing pressure were suddenly to cease, varies with the time. Inasmuch, however, as the variation of k_ν is small, and $f(w, t)$ is also small, if we neglect quantities of the order $\partial k_\nu \partial p$, we may regard k_ν as constant in the integral on the right-hand side of the equation (43).

We thus get from (43) and (45)

$$\partial k_\nu = \tfrac{1}{2}(2\nu + 1) \int_{-1}^{+1} dw Q'_\nu(w) \int_0^T dt\, n_\nu \sin n_\nu(t - \tau_\nu) f(w, t) \quad . \quad . \quad . \quad (46),$$

a formula which summarises our whole theory so far as disturbance of the extreme amplitudes of the various seiches is concerned. It follows that

$$\left.\begin{aligned} \partial k_\nu &= A'_\nu \cos n_\nu \tau_\nu + B'_\nu \sin n_\nu \tau_\nu ; \\ A'_\nu &= \tfrac{1}{2}(2\nu + 1) \int_{-1}^{+1} dw Q'_\nu(w) \int_0^T dt\, n_\nu \sin n_\nu t \,.\, f(w, t) , \\ B'_\nu &= -\tfrac{1}{2}(2\nu + 1) \int_{-1}^{+1} dw Q'_\nu(w) \int_0^T dt\, n_\nu \cos n_\nu t \,.\, f(w, t) \end{aligned}\right\} \quad . \quad . \quad . \quad (47).$$

where

It will be found that the formulæ (47) lead to the same results, so far as amplitude is concerned, as we have already found in the special cases discussed above. We add some important examples of its application.

14. *Example* 1.—Let us consider the effect of a uniform time-change in a pressure gradient which has a uniform space variation along the lake. This will be represented by taking

$$f(w, t) = \tfrac{1}{2} a t w ,$$

where α is the difference of pressure between the ends of the lake generated in unit of time.

Then we get from (46)

$$\partial k_\nu = \tfrac{1}{4}\alpha(2\nu+1)\int_{-1}^{+1} dw Q'_\nu(w) w \int_0^T dt\, n_\nu t \sin n_\nu(t-\tau_\nu)\,.$$

Since $w = Q_1'(w)$, and $\int_{-1}^{+1} dw Q'_\nu(w) Q_1'(w) = 0$, unless $\nu = 1$, in which case the value is 2/3, it follows that a pressure disturbance of the kind under consideration can only generate a uninodal seiche in a symmetric parabolic lake; and we have, putting $n_1\tau_1 = \phi$ for shortness,

$$\partial k_1 = \tfrac{1}{2}\alpha(\mathrm{U}\cos\phi - \sin\phi)\,,$$

where

$$\mathrm{U} = \int_0^T dt\, n_1 t \sin n_1 t\,,$$
$$= \frac{1}{n_1}\{\sin\theta - \theta\cos\theta\}\,,$$

if $\theta = n_1 \mathrm{T}$; and

$$\mathrm{V} = \int_0^T dt\, n_1 t \cos n_1 t\,,$$
$$= \frac{1}{n_1}\{\theta\sin\theta - (1-\cos\theta)\}\,.$$

If we take the special case where the pressure disturbance begins when the uninodal seiche is at its culmination, $\phi = 0$; and we have

$$\partial k_1 = \frac{\alpha}{2n_1}(\sin\theta - \theta\cos\theta)\,.$$

It is easy to see that $\sin\theta - \theta\cos\theta$ has a maximum value when $\theta = \pi$, *i.e.* when $\mathrm{T} = \pi/n_1 = \tfrac{1}{2}\mathrm{T}_1$, as might be expected. The greatest possible disturbance under the present supposition regarding the phase would therefore be given by

$$\partial k_1 = \pi\alpha/2n_1 = \tfrac{1}{4}\alpha\mathrm{T}_1\,.$$

In other words, the alteration in the range of the seiche ($2\partial k_1$) would be equal to the number of millimetres (Aq.) of difference in pressure between the two ends of the lake generated in half the uninodal period.

If the initial phase be not given, but so chosen as to give the maximum effect to the disturbance, then

$$\partial k_1 = \frac{\alpha}{2n_1}\sqrt{(\mathrm{U}^2+\mathrm{V}^2)}\,,$$
$$= \frac{\alpha}{2n_1}\sqrt{(\theta^2 - 2\theta\sin\theta - \cos\theta + 2)}\,.$$

This has a maximum value when $\theta = 2\pi$, viz. :—

$$\partial k_1 = 2\pi\alpha/2n_1 = \tfrac{1}{2}\alpha\mathrm{T}_1\,.$$

15. *Example* 2.—Consider the effect of a steady (*i.e.* non-progressive) harmonic disturbance of pressure on the ν-nodal seiche during a single period of that seiche.

If a be the range of the pressure disturbance, we may put in (46)

$$f(w, t) = \tfrac{1}{2}a \sin(mt - \phi) f(w) \quad . \quad . \quad . \quad . \quad . \quad (48),$$

where $2\pi/m$ is the period of the pressure disturbance, and ϕ/m its phase. We then get, putting $\tau_\nu = 0$ for convenience,

$$\partial k_\nu = \tfrac{1}{4}(2\nu + 1)a \int_{-1}^{+1} dw Q'_\nu(w) \int_0^T dt\, n_\nu \sin n_\nu t \sin(mt - \phi) f(w) \quad . \quad . \quad . \quad (49),$$

where $T = 2\pi/n_\nu$.

Hence, if

$$P = \tfrac{1}{4}(2\nu + 1)a \int_{-1}^{+1} dw Q'_\nu(w) f(w), \quad . \quad . \quad . \quad . \quad (50),$$

a quantity independent of t or T, we have

$$\left.\begin{aligned} \partial k_\nu / P &= U \cos\phi - V \sin\phi\,; \\ U &= \int_0^T dt\, n_\nu \sin n_\nu t \sin mt\,, \\ V &= \int_0^T dt\, n_\nu \sin n_\nu t \cos mt \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (51).$$

where

If $\theta = m/n_\nu$, we find

$$\left.\begin{aligned} U &= -2 \sin\pi\theta \cos\pi\theta/(1-\theta^2)\,, \\ V &= 2 \sin\pi\theta \sin\pi\theta/(1-\theta^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (52).$$

Whence

$$\partial k_\nu = -P\left(\frac{2\sin\pi\theta}{1-\theta^2}\right)\cos(\pi\theta - \phi) \quad . \quad . \quad . \quad . \quad (53).$$

So far as ϕ is concerned, the numerical value of ∂k_ν is a maximum when $\phi = \pi\theta$, or $\phi = \pi(1-\theta)$.

The maximum disturbance possible is therefore produced when θ is so chosen that $f(\theta) \equiv 2 \sin\pi\theta/(1-\theta^2)$ is a maximum, *i.e.* when $\theta = \cdot 838$ approximately, which gives $f(\theta) = 3\cdot 273$. It will be seen, however, from the graph of $f(\theta)$ (fig. 24) that this function varies very slowly indeed near its maximum. We have in fact $f(\cdot 7) = 3\cdot 173$, $f(1) = \pi = 3\cdot 142$. Hence, between $\theta = \cdot 7$ and $\theta = 1$ the divergence from the maximum value of $f(\theta)$ is only about 4 per cent.*

If we take the special case of the uninodal seiche, and suppose $f(w) = w$, we find $P = \tfrac{1}{2}a$; and for the maximum possible value of ∂k_1

$$\partial k_1 = 1\cdot 18a\,,$$

* This result may seem at first sight to be in contradiction with the ordinary theory of forced vibration; but it is not really so. In the ordinary theory we consider a practically infinite number of oscillations, and take into account the viscosity of the system. In the present case we consider only one oscillation, and neglect the viscosity. It is obvious that this latter supposition is nearer the truth in the case of lake oscillations, because the disturbances of pressure are always transient, and usually periodic only for a very few oscillations.

while the effect will not vary from this by more than 4 per cent., if the period of the pressure disturbance lies between $14{\cdot}5^{m}$ and $20{\cdot}6^{m}$.

Of course, in practice the maximum result is not usually attained, even if only one oscillation is considered, owing to divergence from the most effective phase of disturbance: also, if there are several periods of the disturbance, and θ is not unity, the effect produced during the first oscillation may not be equalled in those that follow, or may be partially destroyed.

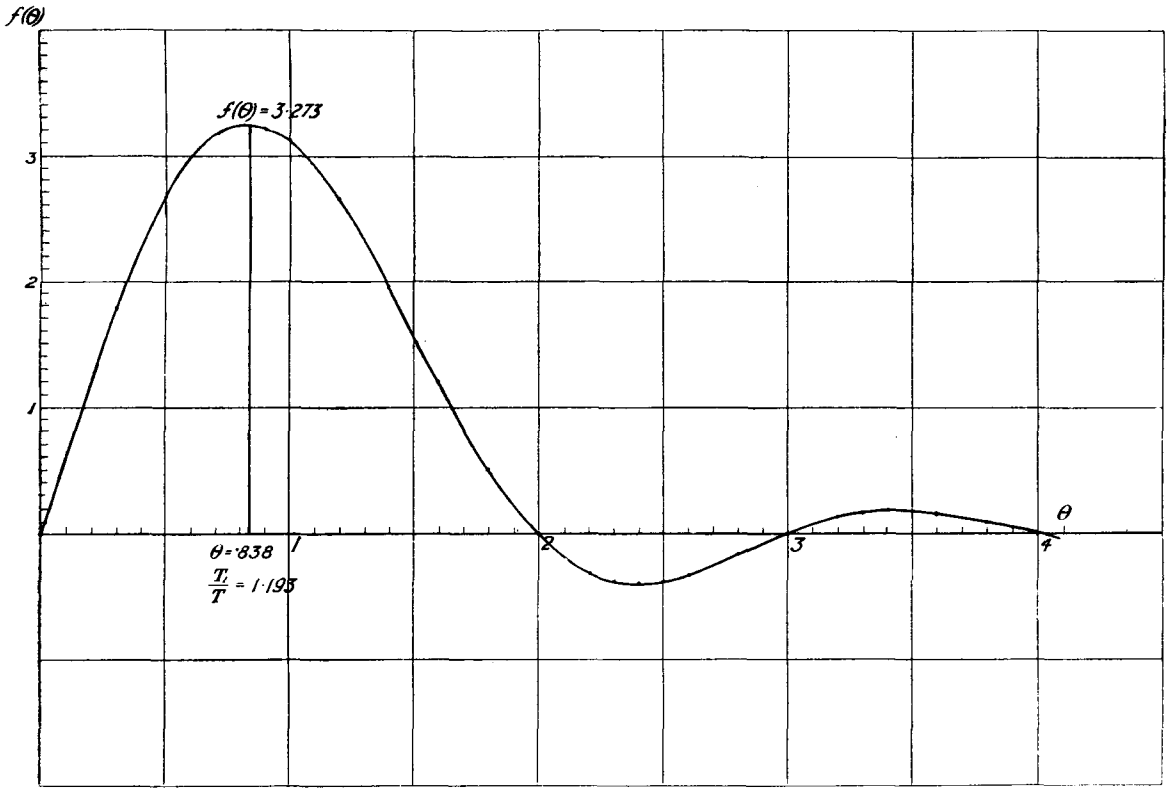


FIG. 24.

16. *Example* 3.—Consider the effect upon the uninodal seiche during one of its periods of a train of progressive harmonic waves of pressure disturbance given by

$$f(w, t) = \tfrac{1}{2}a \sin\left\{\frac{m}{v}(vt - aw) - \phi\right\} \quad . \quad . \quad . \quad . \quad (54),$$

so that $2\pi/m$ is the period, and v the velocity of propagation. Then we have

$$\partial k_1 = \tfrac{3}{4}a \int_{-1}^{+1} dw w \int_0^{T_1} dt n_1 \sin n_1 t \sin\{mt - \lambda w - \phi\},$$

where $\lambda = ma/v = 2\pi a/vT$, T being the period of the disturbance.

Then

$$4\partial k_1/3a = \mathrm{U}\cos\phi - \mathrm{V}\sin\phi\,,$$

where

$$\left.\begin{aligned}\mathrm{U} &= \int_{-1}^{+1} dw\, w \int_0^{\mathrm{T}_1} dt\, n_1 \sin n_1 t \sin(mt - \lambda w),\\ \mathrm{V} &= \int_{-1}^{+1} dw\, w \int_0^{\mathrm{T}_1} dt\, n_1 \sin n_1 t \cos(mt - \lambda w)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (55).$$

We find

$$\left.\begin{aligned}\mathrm{U} &= \left(-\frac{\sin\lambda}{\lambda^2} + \frac{\cos\lambda}{\lambda}\right)\frac{4\sin^2\pi\theta}{1-\theta^2},\\ \mathrm{V} &= \left(-\frac{\sin\lambda}{\lambda^2} + \frac{\cos\lambda}{\lambda}\right)\frac{4\sin\pi\theta\cos\pi\theta}{1-\theta^2}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (56),$$

where $\theta = m/n_1 = \mathrm{T}_1/\mathrm{T}$.

Hence

$$\partial k_1 = -\tfrac{5}{2}a\left(\frac{\sin\lambda}{\lambda^2} - \frac{\cos\lambda}{\lambda}\right)\frac{2\sin\pi\theta}{1-\theta^2}\sin(\pi\theta - \phi) \quad . \quad . \quad . \quad (57).$$

The function $\sin\lambda/\lambda^2 - \cos\lambda/\lambda$ has already been considered above. Its maximum value (for $\theta < \pi$) is ·436. Also, as we have seen, the maximum value of $2\sin\pi\theta/(1-\theta^2)$ is 3·273.

Hence the maximum possible value of ∂k_1 is given approximately by

$$\partial k_1 = 2{\cdot}14a \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

The methods of calculation which we have used for a symmetric parabolic lake are, of course, applicable to any lake for which the normal modes of motion can be found. All we have to do is to use, instead of the Legendrian functions, the general Seiche functions, Bessel's functions, or other functions appropriate to the special form of lake-basin in question.

CORRIGENDA IN PREVIOUS MEMOIRS.

"HYDRODYNAMICAL THEORY OF SEICHES," *Trans. R.S.E.*, vol. xli. p. 599.

PAGE	LINE	
603	23	*Delete from* "which is the period" *to* "depth of water."
616	19	*For* $(+\partial\xi/\partial x)dy1$ *read* $(1+\partial\xi/\partial x)dy$.
622	8	*For* n_2 *read* n_1.
634	9	*For* $\frac{A}{a}$ *read* $\frac{A}{2a}$.
636	9 & 12	*For* $\frac{2a}{h}$ *read* $\frac{h}{2a}$.
637	1	*For* $\xi=\xi$ *read* $\xi=\xi'$.
,,	4 & 14	*For* $\frac{2a^2}{h}$ *read* $\frac{h}{2}$.
,,	6 & 16	*For* $\frac{2a'^2}{h}$ *read* $\frac{h}{2}$.
,,	8 & 9	*Delete* a^2 *and* a'^2
,,	9	*For* $Y_1(na)$ *read* $Y_1(na)\}$.
,,	18	*Delete* a^2 *and* a'^2.
638	$\left\{\begin{matrix}20, 22\\31, 32\end{matrix}\right\}$	*For* $\frac{2a}{h}$ *read* $\frac{h}{2a}$.

"CALCULATION OF THE PERIODS AND NODES OF LOCH EARN," ETC., *Trans. R.S.E.*, vol. xli. p. 823.

831	20	*For* $c(1-1/1.2)\ .\ .\ .\ .\ (1-1/(n-1)n)$ *read* $c(1-c/1.2)\ .\ .\ .\ .\ (1-c/(n-1)n)$.
833	14	*For* $<\cdot000008$ *read* $<\cdot0000008$.

"AN INVESTIGATION OF THE SEICHES," ETC., *Trans. R.S.E.*, vol. xlv. p. 361.

385	25	*For* $\left(t-a_1+\frac{T}{2}\right)$ *read* $\sin\frac{2\pi}{T_2}\left(t-a_2+\frac{T'_1}{4}\right)$.