622. Geometrical View of Morley's Theorem

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621. [K¹. 1. c.] *Morley's Theorem.*

ABC being any triangle with all its angles trisected: if the two trisectors of angle BAC intersect the adjacent trisectors of angles ABC , BCA in R and Q respectively, and if BR , CQ be produced to intersect in L , then

$$RL = QL.$$

For $RL = BL - BR$

$$\begin{aligned} &= BC \left[\frac{\sin \frac{2}{3}C}{\sin \frac{2}{3}(B+C)} - \frac{\sin C \cdot \sin \frac{1}{3}A}{\sin A \cdot \sin(60^\circ - \frac{1}{3}C)} \right] \\ &= BC \left[\frac{\sin \frac{2}{3}C}{\sin \frac{2}{3}(B+C)} - \frac{\sin \frac{1}{3}C \cdot \sin(60^\circ + \frac{1}{3}C)}{\sin(60^\circ - \frac{1}{3}A) \cdot \sin(60^\circ + \frac{1}{3}A)} \right] \\ &= \frac{BC \cdot \sin \frac{1}{3}C}{\sin \frac{1}{3}(B+C)} \left[\frac{\cos \frac{1}{3}C}{\cos \frac{1}{3}(B+C)} - \frac{\sin(60^\circ + \frac{1}{3}C)}{\sin(60^\circ + \frac{1}{3}(B+C))} \right] \\ &= \frac{BC \cdot \sin \frac{1}{3}C \cdot \sin \frac{1}{3}B}{\sin \frac{2}{3}(B+C) \cdot \sin(60^\circ + \frac{1}{3}(B+C))} \\ &= QL, \text{ by symmetry.} \end{aligned}$$

This equality ($RL = QL$) affords immediate proof of Prof. John Morley's theorem that "the 3 points of intersection of the adjacent trisectors of the angles of any triangle form an Equilateral Triangle."

For, if—in accompanying figure— P be the third point of intersection, obviously, in the triangle BLC , PL will bisect the angle BLC , and the triangles

RLP , QLP will thus be equal in all respects, so that $PR = PQ$; and similarly it can be shown that $RQ = PR$ or PQ . C. H. CHEPMELL.

85 Wilbury Crescent, Hove, Sussex, 13th February, 1922.

622. [K¹. 1. c.] *Geometrical View of Morley's Theorem.*

Let $\alpha + \beta + \gamma = 60$. Along the circumference of any arbitrarily drawn circle set off an arc ZY whose chord ZY subtends an acute angle of α degrees at the circumference.

This arc may be briefly designated by α .

Draw FE , WU , QP parallel chords to ZY , so that the intercepted arcs YE , EU , UP are designated by α , $\beta + \gamma - \alpha$, α respectively; and divide the arc $PQ(\beta + \gamma)$ in A so that $AP = \beta$, $AQ = \gamma$.

Then the angles WUY , UWZ standing on arcs of $\alpha + \beta + \gamma$ are each 60 ; and if UY , WZ are produced to meet in X , then XYZ and XUW are equilateral triangles.

Moreover, since arc $WQAU = 2\alpha + \beta + \gamma = WFZY$, therefore

$$WE = WU = UF.$$

Produce AF , UZ to meet in B ; and AE , WY to meet in C . Then the angle $ABZ = AZU - FAZ = (\alpha + \beta) - \alpha = \beta$. Similarly $ACY = \gamma$.

Since BFA , BZU are secants through B to the circle,

$$AB : AZ = BU : UF = BU : UX, \text{ and angle } BAZ = BUX.$$

Hence the triangles BAZ , BUX are similar, and $UBX = ABZ = \beta$. Thus the triangles BFU , BXU are congruent and $BX = BF$.

Similarly $WCX = \gamma$ and $CX = CE$.

The angle $BXC = BAC + ABX + ACX = 3\alpha + 2\beta + 2\gamma = 120 + \alpha = QAP$.

Since $FZB = \beta = PZA$ and $FZB = FAU = 2\alpha + \beta + \gamma = PAZ$, the triangles FZB , PAZ are similar and $BF : ZF = ZP : AP$.

In the same way the triangles EYC , QAY are similar and $EY : CE = QA : YQ$.

Hence $BF(BX) : CE(CX) = QA : AP$.

Thus the triangle XBC is similar to AQP , and the angle $XBC = AQP = \beta$;

$$XCB = APQ = \gamma.$$

We have thus a triangle ABC whose angles measure in degrees 3α , 3β , 3γ , and whose trisectors form by their intersections an equilateral triangle XYZ .

The same property must be true for any similar triangle by increasing or diminishing the scale of representation.

R. F. DAVIS.

623. [I. 27.] *Note 587. Gazette*, p. 328, Oct. 1921.

A proof of the theorem stated by Mr. Richmond can be derived very simply from a principle enunciated by Dirichlet :

"If $n+1$ particles are placed in n sockets, then there must be at least one socket which contains more than one particle."

Let θ denote any real number, and let

$$\theta_1, \theta_2, \dots, \theta_n$$

be the fractional parts of

$$\theta, 2\theta, \dots, n\theta,$$

so that $\theta_r = r\theta - m_r$, with m_r integral and $0 \leq \theta_r < 1$. The interval $(0, 1)$ can be divided into the n equal sub-intervals

$$\left(0, \frac{1}{n}\right), \left(\frac{1}{n}, \frac{2}{n}\right), \dots, \left(\frac{n-1}{n}, 1\right),$$

it being understood that k/n belongs to the interval $(k/n, (k+1)/n)$ and $(k+1)/n$ to the next one. From the principle just enunciated it follows that at least one of these intervals must contain two (or more) of the numbers

$$\theta_0 (=0), \theta_1, \theta_2, \dots, \theta_n.$$

If θ_r and θ_s belong to the $(k+1)$ th interval,

$$\frac{k}{n} \leq \theta_r = r\theta - m_r < \frac{k+1}{n} \quad \text{and} \quad \frac{k}{n} \leq \theta_s = s\theta - m_s < \frac{k+1}{n}.$$

Hence, taking $r < s$ and subtracting these inequalities, the fractional part of $(s-r)\theta$ lies between $-1/n$ and $+1/n$. Now $s-r$ is one of the integers $1, 2, \dots, n$, so, taking $n=10$, the decimal part of $(s-r)\theta$ must begin either with 0 or 9. Again, taking $n=10^t$, the decimal part of $(s-r)\theta$, with $0 < s-r \leq 10^t$, must begin with either t zeros or t nines. Decimal points can be inserted after any vertical row of figures in the illustration given by Mr. Richmond, so the theorem he stated is proved.

The University, Leeds.

W. E. H. BERWICK.

624. [S. 1. a.] *To find the depth of the centre of pressure of a triangle (1) with one vertex in the surface, (2) completely immersed.*

Let DAB be a triangle with one side DA in the surface of the liquid. Then the depth of the C.P. is half that of B . Let AC be a line through A meeting DB in C , and let the depths of B and C be q and r .