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ARITHMETIC AND REASONING IN CHILDREN

By S. CAROLYN FISHER, Clark University

What is the capacity of the normal child for the higher thought processes,—abstraction, generalization, reasoning? Do these processes exist in children at all, or is the child mind too much absorbed in the acquisition of a mass of experience to admit of the existence of other activity? And if they do exist, can we devise experiments which will adequately reveal them? Are they sufficiently fundamental to justify an endeavor to formulate tests which will serve to measure them in children of widely different environments, and to establish age norms? Closely bound up with reasoning is arithmetical activity. What is the product of the schools in arithmetic? Is it uniform, or variable? If the latter, what is the relation between school conditions,—supervision, courses of study, expenditure of time,—and the proficiency of the pupils in arithmetic? Is arithmetical ability a single ability, or a complex of many different ones? If the latter, what relations obtain between the constituent capacities?

These questions, along with a host of others of educational import, have loomed large in pedagogical controversy since the days of Comenius; it is only within the last decade or so that their serious experimental investigation, in any modern sense of the words, has been attempted. Previous to this period, opinions regarding reasoning agreed in the fundamental assumption that the child's body of knowledge is exceedingly limited; they differed in the matter of emphasis, some stressing the child's efforts to adjust sense data, and finding therein all the essentials of reasoning, others maintaining that the extreme poverty of the child's mind renders impossible, until a comparatively late stage, anything like logical formation of conclusions. Some hold that the capacity for purposive thinking is educable, and deplore the failure of the teaching in the schools to stimulate original observation and sound thought based on that observation (C. A. Scott, 31 and 32; Hancock, 14; Bolton and Haskell, *Ed. Rev.* 15, pp. 473-499); while F. C. Sharp (*Ed. Rev.* 1893) goes so far as to recommend special courses in reasoning, to extend throughout the schools, and to be given by teachers trained for that purpose. Such courses would be based on

the facts learned during the progress of other studies of the curriculum. Regarding the value of arithmetic there has existed a similar conflict of views, as Dr. Burnham points out (6). Opposed to the exaggerated reverence for mathematics, due to its alleged value for formal discipline, were Schopenhauer (1788-1860), who called mathematics the lowest form of mental activity; Holmes, who regarded mathematical thought as a mere calculating grind; Huxley; and most vehemently of all, Sir William Hamilton (1836), with his sweeping denunciations,—“Mathematical study is inferior as a builder of mind; devotion to it is fatal to the development of sensibilities and imagination; continued pursuit of it leaves the mind narrow and dry, meagre and lean.” He admits, however, that study of mathematics may check the habit of mental distraction, and form that of continuous attention. Bledsoe (1877), editor of the *Southern Educational Review*, and later, Pringsheim, accuse Hamilton of deliberately misrepresenting the authorities whom he cites in support of his opinions. Peterson (26) also takes issue with Hamilton, maintaining that arithmetic, with its combinations of numerical properties and its demand for reasoning, is highly beneficial to the child, for the very reasons that Hamilton names. Keyser (17), himself a professor of mathematics, dwells enthusiastically on the value of mathematics, the beauty of its deduction from a score of premises,—a few logical constants,—its appeal to the whole mind; he points out that both Schopenhauer and Hamilton were ignorant of the subject they condemned, the latter even proudly so. Something of Hamilton’s attitude is adopted, on the other hand, by General Francis A. Walker (36) in an address before the Boston School Committee in 1887. It is well, he says, to decrease the time spent in mathematics in the public schools if this can be done without injury; number-work is important, but let us restrict arithmetic to it, and devote the time now spent in the more logical aspects of arithmetic to certain other studies, equally fitted for the cultivation of powers of observation and generalization. The whole conflict seems to hinge on the question of whether arithmetic consists merely in numerical operations, or whether it includes the application of logical principles. Schopenhauer, Huxley, Hamilton and Knilling, taking the first-named position, find arithmetic mechanical and narrowing; Keyser, with the latter viewpoint, exclaims over its breadth. Walker (36) and Padoa (25) also distinguish between number and logic, and confine arithmetic strictly to the former.

A long line of German investigators, taking their cue from

Pestalozzi,—*Anschauungslehre der Zählenverhältnisse*, 1804,—have studied the genesis of the number concept with a view toward devising improved methods of teaching the simple arithmetical operations. The result has been a perennial crop of number-picture charts, and various intricate pedagogical devices, based very largely on the adult consciousness of number, but very slightly on the results of study of the effects of different methods on the learning time of the child. Regarding these it is sufficient to mention two main points of view, that represented by Lay, Messenger, Nanu, Göbelbecker, Walsemann, and others, that the idea of number is a product of direct perception; and that of Dietze, Knilling, Boehme, Kehr, Hartmann, and others, who maintain that the number concept is a product of the act of counting. Good reviews and bibliographies of this literature are given by Meumann (23) and Schanoff (30).

We come now to the experimental studies which bear upon the present topic. Brown (3), Ellison (11), and Eckhardt (9) have made investigations of the thoughts and number ideas of young children. Brown examined the collection of children's sayings gathered by the Worcester State Normal School, and selected and classified 375 of what he deemed the most significant ones. Some of his rubrics are 'misunderstandings of words,' 'misapplications of sayings,' 'explanations of things,' 'false reasonings.' He was much impressed by the children's lack of understanding, due to the "difficulty under which the child labors in trying to deal with the overwhelming multitude of objects and events in the midst of which he finds himself." Yet the child often shows surprising aptness in theorizing; the inadequacy of his theories are due to their being based on false or insufficient data.

Ellison, on the basis of 12,000 definitions of common abstract terms obtained from children of St. Louis schools who ranged in age from eight to fifteen, arrived at a similar conclusion. She found that such mastery of common abstract ideas and of language as would make possible reasonably perfect definitions is lacking in children of thirteen and under, though for several years before this time the abstractions have been clearly enough grasped and the vocabularies of the children large enough to admit of definitions by synonyms. Still earlier, children hold the idea in concrete form, and express it by examples.

Eckhardt gives the results of a statistical study of the number concepts of children upon entering school, and appends figures showing later stages of the development of such

concepts. The method of his investigation consisted in asking the children (45 in number) to count as far as they could; to count by twos, fives, sixes, sevens, and eights; to begin with a certain number and subtract by ones, twos, etc.; to read numbers; and to tell which of two numbers is the higher. His results are given in a single table; no attempt is made to interpret these data by the ordinary methods, and the labor of deciphering them is thus considerable. In so far as we understand his findings they seem to justify the following general statements: Children possess fairly well-developed ideas of number at the age when they ordinarily enter school. His subjects were able to count up to from 3 to 1000; the median number counted was 100 (24 cases). Those who could count to 1000 could also count by twos, threes, etc., and subtract by ones, twos, etc. For most, adding and subtracting by higher numbers was difficult; 4 failed in adding (counting) by twos, and 31 in adding by fives, sixes, sevens, or eights, while the corresponding numbers for subtraction were 10 and 35. In subtracting by ones, some found it necessary to count from the beginning of the series to the number just before that from which 1 was to be subtracted. In the tests for ability to indicate the greater of two numbers, 2 failed; all of those who could count up to 1000, and 10 of those who could count to 100, succeeded. Some could make the distinction within tens,—thus, 47 was known to be greater than 42, but there was doubt about 47 and 22, or 42 and 27. Twelve were unable to read numbers; 6 could read them up to 1000. The children who could count up to the greatest number almost invariably showed the highest school grades.

The remainder of the experimental investigations considered here may be arranged for convenience in three divisions: (1) Those in which the main emphasis is placed on the reasoning, or purposive association, of children; (2) those which have as their object the measuring of the arithmetical ability of children, in its relation to the courses of study and methods of teaching in the schools; and (3) those which have to do chiefly with the relations between different arithmetical operations. There is inevitably more or less overlapping between the different groups, as each contains studies which treat in a minor way of matters properly belonging to another group; but the main distinctions are evident.

1. *Studies of Reasoning in Children.*—Hancock, in 1896, perceiving a general dissatisfaction with the results of arithmetic teaching, due to the fact that children who do well one year often fail the next, undertook to throw light on the actual development of the power of reasoning in children,

knowledge of which is essential, he thinks, before improved methods of teaching can be devised. His method consisted in the presentation of syllabi to the child study section of the Colorado State Teachers' Association, asking them to submit certain tests to their pupils. Four series of tests were used, the first series being presented to the first three grades, and the other three series to the fourth and following grades. The tests consisted in simple arithmetical problems, dealing with small numbers so as to minimize the chances of mechanical errors; each was short enough to obviate any fatigue element. The third series was designed particularly to test pupils' ability to deal with conditions somewhat algebraic; the fourth to test their power to visualize conditions. A sample problem from the last is the following: 'Two boats, starting from the same place, sail up river, one at the rate of two, and the other of three, miles per hour; how far apart will they be at the end of four hours?' The ages of the children tested ranged from seven to fifteen years inclusive.

The results are given in terms of the percentage of pupils who succeeded in solving each problem. The main findings are as follows: for the first two series, confusion in analysis of problems is common; children guess at methods. There is loss of power by girls of ten and boys of eight and fourteen, which loss is compensated for later. The percentage of error decreases most rapidly at nine, thirteen, and fifteen years. As regards the fourth and fifth series, the boys excel in the former, and the girls in the latter. "This agrees with Galton's finding, that women visualize better than men." Hancock's conclusions are as follows: if there are periods when a power begins to develop and function, and forcing it to function before its time is provocative of later weakness, the question of what modifications are needed in the arithmetic course becomes an important one. The large percentage of error found in earlier years suggests that the time for such problems as are contained in the tests is later. Early introduction of algebra and geometry is desirable; more time should be given to arithmetic at the ages of nine and thirteen. It appears that children do not reason far beyond their environment and experience; early training should utilize and extend these two factors.

King (18) was the next investigator along this line. His problem was to throw light on the development of thinking power in children. His procedure consisted in sending tests to teachers in the public schools of various parts of England, to be given to their pupils. The tests were divided into three series, the first to be given to children of from seven to ten,

the second to those of from ten to twelve, and the third to those of from twelve to thirteen. Each series contained two questions, of which the following is typical: "There are several clever boys in a class, and all are careless; therefore a clever boy must be a careless boy. Do you agree?" The last question of each test was identical with the first of the succeeding test, this to permit comparison of stages of growth. Returns were received from 2000 children. The results were as follows: The percentages of pupils answering each question correctly were, for test 1, question 1, 13.1, question 2, 9.4; for test 2, question 1, 20.7, question 2, 30.0; for test 3, question 1, 44.4, question 2, 56.4. Thus there was progress with age. Many glimmerings of solutions,—non-agreement without reasons,—appeared. The girls were more verbose than the boys, but there was little difference in efficiency. Many pupils of fifteen who had received good education showed only the merest germs of logical thinking. Education improves the reasoning capacity. King believes that a more satisfactory, though severer, test would consist in questions further removed from the pupils' experience.

In 1903 J. H. Bair (1), of the Department of Psychology and Education of the University of Colorado, published the results of a study in which an attempt was made to measure the reasoning power of children of different school grades, so as to determine the rate of its development. Two questions were given the children of the public schools of Boulder, Colo. They were (1) "A boy said, 'I know ten doctors who were all good, and ten policemen who were all bad. Therefore doctors are better men than policemen.' Was he right, or wrong? Give reasons for your answer;" and (2) "Mr. Roosevelt said, 'There were thirty boys who were all hard students and were promoted every term. Therefore if you wish to become rich, study hard.' Was he right, or wrong? Give reasons." Bair's results showed a fairly uniform progress throughout the grades. Wide individual variations appeared. Many of the poorer replies showed bias, *argumentum ad hominum*, and lack of ability to understand what was wanted or to use facts. Bair concludes that school curricula do not sufficiently take into account differences in children's power of thinking; moreover, the material given pupils often involves factors beyond the scope of their experience. The reason that children fail in constructive power is not that their capacity for logical thought is immature, but that they lack the information without which it cannot function.

The results of these three studies agree in showing a general progress in ability with age, and striking weaknesses,

especially in younger children. Like most pioneer studies, their value lies in suggestion rather than accomplishment; for certain inadequacies in the work are very clear. We are uninformed regarding many important details of procedure; the possibility of determining grade differences by comparison of the returns of different tests is very doubtful; and treatment of results is incomplete.

Bonser (2), the next and by far the most thorough investigator in this field,¹ removes, in his study, many of the sources of error in previous work. His problem is "to secure concrete basis for some insight into the capacity of children of the fourth, fifth, and sixth school grades for reasoning. His method was essentially the test method of previous investigators. The tests, seven in number, were divided into four series, entitled respectively *mathematical judgment*, *controlled associations*, *selective judgment*, and *literary interpretation*. The tests in mathematical judgment consisted in simple arithmetical problems, such as are found in text-books of arithmetic, part of them, however, being stated in a less conventional way. Those in controlled associations consisted of two sets of sentences, in the first of which a significant word was omitted in each case, to be filled in by the pupil (*i. e.*, A ——— is one who plays a musical instrument), and in the second of which a choice of two suggested significant words, one suitable and the other not, was given (*i. e.*, Glass breaks more (less) easily than tin); also in a list of sixty words, of each of which the pupil was asked to write the opposite. The tests in selective judgment contained two series, the first consisting in a number of statements, with ten reasons for the truth of each statement, some of the reasons correct, others incorrect or irrelevant, the pupil's task being to select the former; and the second consisting in a number of terms, each of which was defined in five different ways, some correct, others incorrect, the pupils being asked to indicate the correct definitions. The tests in literary interpretation contained stanzas of poetry, from the second and third readers, the pupil being asked to write the essential content of each stanza in his own words. All tests were printed on sheets of paper, spaces being left for the pupil's work where necessary.

Bonser discusses the question of what ability it is that his tests really measure. Reasoning, he says, is purposive thinking. It involves: the recognition, in any situation, of previously unnoticed resemblances or common elements, implying an analysis of that situation; the 'guess' that the observed relationship is universal under similar conditions; and

¹ Consideration of the Binet tests is omitted in the present paper.

the verification of this 'guess,' or hypothesis, by testing it under such conditions. Up to and including the formation of the hypothesis, the reasoning is inductive; afterwards it is deductive. Capacity for deductive reasoning, Bonser thinks, is what is measured by the tests; the pupil must select and apply an appropriate principle or hypothesis from those which suggest themselves from his previous experience. Thus, in the tests of mathematical judgment, he must analyze out the essential features of the problem, and from among the principles which he can recall he must choose one that is identical with that of the problem; similarly, the tests in controlled association and selective judgment demand accuracy in recognition of resemblances between the known of experience and the unknown of new situations; and in the test of literary interpretation the child is asked to abstract, to evaluate, and finally to synthesize, the essential thought units.

The subjects in the experiment were 757 children, 385 boys and 372 girls, pupils of the grades named in five schools of Passaic, N. J. The sex distribution throughout the grades was remarkably even; the numbers of children in each grade varied from 109 to 181; ages varied from eleven years seven months to twelve years three months.

Bonser himself gave all the tests, being assisted on two occasions by two graduate students of Columbia University. A time limit of two minutes was set for the opposites test; in all others, the work was collected upon the completion of the first paper. The scoring was done according to rigid objective standards in all cases except that of the tests of literary interpretation, where the reconstructed poems were graded on a basis of their estimated merit.

The main results are given in the following tables:

1. Median ability and variability for each grade, in terms of actual scores made. (Highest possible score, 265.)

Grade	Medians		Higher		Quartiles of Variation	
	boys	girls	boys	girls	boys	girls
4A.....	117.50	130.00		12.50	34.01	22.08
5B.....	155.62	146.25	9.37		29.84	23.31
5A.....	160.62	165.00		4.38	32.81	32.50
6B.....	187.00	188.50		1.50	24.62	22.25
6A.....	205.00	204.50	.50		27.50	21.14
Total.....	165.83	159.25	6.58		37.03	34.08

2. Per cent. of each grade reaching the 75 per centile of all.

Grade	4A	5B	5A	6B	6A
Boys.....	9.67	18.97	19.95	42.85	62.00
Girls.....	2.27	10.45	24.13	45.65	64.37

3. Per cent. of boys reaching the 50 per centile of the girls.

Grade.....	4A	5B	5A	6B	6A
Per cent.....	45.1	58.2	44.6	50.0	50.0

4. Per cent. of gain from grade to grade.

Grades	boys	girls
4A to 5B.....	32.44	12.50
5B to 5A.....	3.21	12.82
5A to 6B.....	16.43	14.24
6B to 6A.....	9.62	8.48

For both boys and girls there is marked progress from grade to grade, the total percentage of gain in ability being 74.4 for boys and 57.3 for girls. Variability for both sexes is slightly lower in the upper grades.

Age differences.—There is, on the whole, a loss in ability with the progress of age for boys, and a gain for girls. The former reach their maximum at twelve years, the latter at thirteen. The youngest 25 per cent. are superior to the oldest 25 per cent. in every grade, with the exception of the boys of grade 6. Both older and younger pupils fall below the median for the grade in certain of the tests. The variability of the younger boys and of the older girls is slightly greater.

Sex differences.—In all, the boys reach a higher median than the girls. If a grouping were made of the best 25 per cent. of all the pupils it would contain a larger per cent. of boys from grades 4A and 5B, and of girls from the other three grades. The boys are generally superior up to twelve, and the girls afterwards. The boys, on the whole, are more variable in every grade. When the tests are considered separately, however, the girls in some are more variable. Where one sex has a higher median, the other shows greater variability. There is, for boys, a marked difference in ability between the A of one grade and the B of the next, but little between the B and the A of the same grade. This Bonser explains on the ground that more care is taken in promoting from one grade to the next, than in permitting continuance in the same grade. The girls apparently show a greater tendency toward regularity in their work.

Sex differences among the tests.—The boys are decidedly superior in the tests of mathematical judgment; they excel also in the tests of selective judgment among reasons for the truth of a statement, and are slightly ahead in the tests in controlled association in which the significant word is supplied by the pupil. The girls excel in the opposites test, and in the test for selective judgment among definitions; they are especially superior in the interpretation of literature. These latter tests, Bonser points out, deal with words in a special way.

The superiority of the girls in them may be due, he thinks, to attention to detail, to more patience in analysis, to keener sense of the meaning of words and phrases, to a more accurate form of expressing meaning,—or to a combination of some or all of these factors. Moreover, boys of this age do not as a rule like poetry; their attitude would be to rid themselves of the matter as soon as possible. On the whole, the resemblances between the sexes are far greater than the differences.

Relative difficulty of the tests.—The most difficult of the tests, as shown by the ratio of the median grades to the highest possible grade, is that for literary interpretation; then comes selective judgment; then mathematical judgment; and finally, easiest of all, comes controlled association.

Correlations of individual abilities.—The coefficients of correlation for the individual abilities in each of the tests with those in each of the others were worked out. Of these the highest are (in order from highest to lowest) .53 for the tests in which sentences are to be completed by the pupil and the opposites test (controlled associations); .49 for the opposites test and the test in selective judgment among reasons for the truth of a statement; .47 for the latter and the test in selective judgment among definitions; and .45 for the test in supplying a significant word in a sentence (controlled association) and that in literary interpretation. The coefficients of correlation for the ability in each test with that in all the tests taken together are (in descending order) .85 for the opposites test (controlled association); .73 for the test in selective judgment among reasons for the truth of a statement; .59 for the tests in mathematical judgment; .58 for the test in selective judgment among definitions; .55 for the tests in supplying significant words in sentences; .37 for the test in literary interpretation. The opposites test thus seems to be one of superior merit as a single test of the several abilities here measured. Correlations on the basis of age are much higher than those on the basis of grade, indicating greater homogeneity within the age groups; there is also a tendency toward higher correlation among the boys than among the girls.

From the progressive development in median ability, and from the high correlations found to exist among the several varieties of ability tested, Bonser concludes that the tests measure one general species of ability, complex though it is. The fact that native capacity rather than the result of training is measured, the author believes to be indicated by the contrasts between grade progress and progress with age, by the

superiority of younger over older children of each grade, by the fairly substantial percentage of pupils of the lower grades in the highest quartile of ability for all, and by the evident sex differences. "The point of greatest pragmatic significance for the school lies in the implications from the two facts that there are substantial percentages from both lower and higher grade groups in the highest quartile (25 per cent.) of ability for all, and that the youngest pupils of each grade show higher ability than the oldest, and sometimes than the grade as a whole. It is that perhaps the worst type of retardation in the schools is the withholding appropriate promotion from those pupils who are the most gifted and therefore of the most significance as social capital."

Bonser points out the fact that mental abilities are too complex to be rated extensively in the mass, as smaller groups within the larger often show opposite tendencies. In an exceedingly thoroughgoing fashion, he gives detailed results for each test, grouping the pupils by age, sex and grade; consideration of these results must be omitted here. The limitation of the tests to deductive reasoning seems unfortunate, as it puts a premium upon memory of certain methods of procedure already learned by the pupil, perhaps in the most routine way; this is particularly true of the tests of mathematical judgment. The questions and statements of the tests of controlled association and selective judgment are not well calculated to arouse any immediate interest in the child; fulfilment of the requirements depends upon information he may chance to possess, rather than upon any observation of his own. The tests have been devised from the point of view of an adult; they present no situations which are vital to the child, and which would be liable to stimulate such powers of thought as he may possess. Again, as regards method, it is a well-known fact that class-room experiments are frequently vitiated by such distractions as are incidental to the intrusion of a stranger into the school-room, *i. e.*, uneasiness and self-consciousness on the part of the pupils; there is no indication that precautions were taken to avoid this very real difficulty. Moreover, the collection of the papers upon the completion of the first one might easily place the best reasoners at a considerable disadvantage, for the reason that the best reasoner is not necessarily the most rapid one. Such a system of timing increases the liability of the use of ready memory of mere routine methods of work, or of careless or slipshod answers.

The preceding studies represent efforts to analyze rather superficially the reasoning processes of a large number of

individuals. A different type of experiment is that seen in Terman's study (35), which aimed at a minute qualitative analysis of the logical processes of a few individuals. This author used as subjects seven of the brightest, and a like number of the most stupid, boys of the schools of Worcester, as selected at his request by the teachers. The ages of the subjects ranged from 10 to 15. The experimentation was carried on in the psychological laboratory of Clark University. Terman tested the boys on puzzles, designed to bring out such logical processes as took place, on mathematical ability, mastery of language, interpretation of fables, learning of chess, memory, and motor ability, as shown by the practice curve in learning such coördinations as those involved in catching a ball in a cup, carrying a book on the head. Only a few of the general results can be given here. The bright group is superior in the mental, and inferior in the motor, tests. The standing of individuals through the tests is strikingly uniform. The duller group discloses no inferiority in persistency and concentration; on the contrary, four of the brighter ones are much more fitful and spurty in their efforts. The methods of work indicate on the part of the brighter subjects associations that are more volatile, spontaneous, and based on more subtle resemblances, and on the part of the duller ones associations that are close, matter-of-fact, and labored. When the boys were asked whether they preferred to read or to play games, all but one of the brighter group preferred reading, while all of the duller preferred games. Endowment, Terman thinks, has a greater importance than training in determining the intellectual rank of an individual.

2. *Arithmetical Ability of Children. Its Nature and Relation to the Course of Study and Method of Teaching.*—Mercante (22), in his investigation of the psychology and cultivation of mathematical efficiency, assumes a neurological viewpoint throughout. He stresses the integrative activity of the nervous system; any mathematical process is an integration, beginning with an auditory or visual perception, and ending with a motor act, after a more or less complex subjective manipulation. Upon the readiness of the associative processes depends the exactitude and speed of the work. Mercante lays down the fundamental postulate that the degree of mathematical efficiency is inversely proportional to the time required for the working of problems in the four fundamental operations of arithmetic, addition, subtraction, multiplication, and division, and for the reading of numbers. Twenty different experiments were made; among the abilities tested were counting lines, reading and copying numbers, writing num-

bers from dictation, reproducing numbers visually (setting them down on paper immediately after their exposure), calculating mentally (simple examples in the fundamentals), and solving problems. The children tested were 134 girls and 93 boys of a school in Argentina, their ages ranging from seven to fifteen. From the vast body of data thus obtained the author constructs numerous tables and curves. Among the important findings are the following: There is rapid progress from the first to the fifth or sixth grades; afterward improvement is slower. In counting lines, the relation between accuracy and time shows a constant positive correlation; boys are more accurate, showing more rapid accommodation and convergence of the eyes; the 'integration' of the girls wavers more. There is greater difficulty and less certainty in reading numbers containing zeros (101,010) than those free from zeros; the girls are more accurate in the latter, and the boys in the former. In writing numbers from dictation, the first place at the left is filled with more accuracy, the place at the right coming next; this Mercante explains on the ground that the first and last figures heard tend to persist most strongly. The writing of numbers is ordinarily unconscious; too close attention to the process results in error. Boys 'integrate' in auditory-motor fashion, girls in auditory-visual-motor; the former are more rapid and accurate. From the experiments in the visual reproduction of numbers Mercante concludes that the visual-writing association is the most rapid one which the brain is able to make. The central digits are more accurately reproduced than initial or terminal digits; boys err more often in permutations of digits, and girls in substitutions. Girls show a higher degree of attention in this test. In mental arithmetic, accuracy and speed show a positive correlation. Multiplication is hardest, then subtraction, and addition. Efficiency improves greatly with practice; there is necessity for much drill in the lower grades, if accuracy is to be secured in the higher ones. In problems (complicated integrations), the boys excel after the ninth year. Attempts are made to analyze the mental concomitants (imagery) of the solutions. Conclusions regarding the imagery employed by the pupils are often rather more decisive than the data warrant; there is throughout an unfortunate tendency to adopt the exceedingly treacherous procedure of theorizing as to the processes which *must* be employed by the pupil in performing a certain task, and interpreting the results accordingly. Little evidence appears that the author is familiar with the work of other investigators in the field.

Ranschburg (28) aimed to test the efficiency of 153 pupils

of the first to the fourth classes in performing simple examples in addition, subtraction, multiplication, and division. He introduced a variation which consisted in notifying of pupils when their answers were incorrect, and allowing ten seconds for correction; the number of corrections necessary served as a measure of objective certainty. The results from 9945 individual records showed a progressive improvement from the lowest to the highest classes. Speed increased with age and practice. The order of difficulty for the fundamentals, beginning with the easiest, was found to be the following: addition, multiplication, subtraction, division. Multiplication was most rapidly performed; next in point of speed came addition, then division, and finally subtraction. The differences in speed Ranschburg accounts for on the basis of the greater mechanization by drill of certain of the fundamentals. Speed he regards as the most significant and satisfactory criterion of mental stage and ability; as a simple mathematical formula for the measurement of efficiency and its expression in a single figure he proposes the following: amount of work done (*Leistung*) equals accuracy divided by time. Ranschburg, in a later study, compares the arithmetical ability of normal and of feeble-minded children. He comes to the conclusion that the time necessary for subtraction furnishes the most important clue to the mental condition.

Rice (29) deplors the fact that "the ship of pedagogy has become waterlogged in a sea of opinions;" he would fain have "efforts made to point the ship in a different direction to find whether we can get out of the trough." Let us take the attitude of scientific inquiry! The method he proposes for establishing facts regarding the much agitated question of the products of different school systems is that of subjecting the children trained under them to one and the same test,—which must be fair and practical,—and comparing the results. He himself gives the data gathered from such a test in arithmetic. His specific problems were to determine in so far as possible what results should be accomplished by the school program; how much time should be devoted to the study, and why some schools secure satisfactory results with a reasonable time allowance, while others fail, despite an inordinate proportion of time. The tests were five in number, one for each grade tested (fourth to eighth, inclusive). Each test contained eight arithmetical problems, selected to suit the grade to be investigated. Some of the problems were carried through several grades; thus, five of the fourth grade problems were repeated in the fifth grade, three in the sixth, etc. Basis for observing the growth from year to year, and for comparison

of different grades in the same and different schools, was thus afforded.

The subjects were 6000 pupils of 18 schools, representing 7 cities. The author was present during the giving of the tests, in all cases. In scoring the papers, only those examples were considered which had been correctly worked in principle; of these, the percentage containing mechanical errors was recorded.

Rice found enormous variations among grades and localities. The grades showing the greatest differences in different localities were the seventh and eighth, class averages ranging from 8.9 to 81.1 per cent. in the former, and from 11.3 to 91.7 per cent. in the latter. The averages for schools as wholes ranged from 25 to 80 per cent. Some showed fair uniformity throughout the grades; in others, marked variations in certain grades were offset by variations in the opposite direction in other grades. In nearly all cases, different schools of the same city showed similar degrees of ability.

Mechanical errors were few. The curve falls from the fourth to the sixth grades, and rises in the seventh and eighth. These fluctuations Rice accounts for on the grounds of effect of drill, and different bases of comparison,—the non-identity of the tests. The schools showing the best thinking (greatest proportion of correctness in the method of solution) made fewest mechanical errors.

No direct relation was discovered between the amount of time devoted to arithmetic and the degree of ability attained. The facts indicate that 45 minutes per day will suffice, though there is no reason to conclude that that amount of time is actually required to obtain satisfactory results.

How shall we explain the variations among schools and grades? Evidently, says Rice, they are due either to the influence of the teaching, to resistances to that influence over which the teacher has no control, or to misdirected effort on the pedagogical side. As regards the first possibility, it appears that the teachers in the poorer schools were as conscientious and as well-trained as those of the better ones. Resistance to the influence of the teacher may consist of three elements,—the home environment of the pupils, the size of the classes, or the ages of the children. Now in the investigation under consideration, there was no relation between the neighborhood (aristocratic or the reverse) of the school and the standard of work attained; the classes were actually larger in the six best schools, and smaller in the poorer ones; and the ages of the pupils of the higher and lower schools were approximately the same, as shown by the grade averages. It

would seem, then, that the principal cause of failure consists in misdirected effort on the pedagogical side.

The pedagogical side of the problem is considered by the author in two divisions, the factors brought into play by the teacher, and by those appointed to direct and supervise the work of the teacher. Under the first division Rice discusses methods of instruction and teaching ability, as represented by a combination of education, training, and personality of the teacher. Methods of teaching, he says, cannot be looked upon as the controlling element, as in the better schools no methods other than the ones commonly used were employed; nor did thoroughness in reviewing invariably make for success, as the review might be merely formal and lacking in qualities that stimulate thought. As to the matter of the qualification of teachers, we should expect variations here to be indicated by marked differences among different class-rooms of the same locality, a condition which does not appear in the results. So the source of the difficulty is left at the door of the supervisor. Now the leading functions of the superintendent are the preparation of the course of study, the suggesting to teachers of improvements, and the establishment of standards and testing for results. The causes for variations could hardly be found in the first two of these, as the tests questions were general in nature and confessedly fair to all, and superintendents' meetings and visits had been in vogue in all the localities tested. Therefore, the controlling elements are contained in the last,—the establishment of standards and testing for results. There are four prevalent modes of standardization. They are: (1) Tests made by the teachers themselves; (2) Tests made by the teachers, supplemented from time to time by others devised by the superintendent; (3) Tests made by the principal, for his own school; (4) Tests made by the principal, supplemented by others of the superintendent. The poorest schools were those in which testing had been done by the teacher alone. This is due, Rice thinks, to a tendency on the part of teachers toward routine work, resulting in the failure of their pupils in problems unusually formulated. Where the tests of teacher or principal are supplemented by those of the superintendent, results are much better; certainty that pupils will be examined on problems formulated in different ways leads the teacher to inculcate knowledge of the underlying principles. Much depends on the superintendent's questions; they must be suggestive and stimulating of independent thought, and based on the normal mental capacity of the children. Knowledge of what the average child who has been well taught is capable of doing is to be gained only by extended study.

There appear, in Rice's investigation, certain obvious defects in method and in treatment of results. The factors influencing the development of arithmetical ability are far too numerous and complex to admit of the separate treatment, and simple logical analysis accorded. The work proved very suggestive in the new lines of educational research indicated, however. Stone (34) was the next to open extended investigations in the field. This author points out the following defects in Rice's study. As to method, he says, Rice makes no statement indicating whether or not he used a time limit; neither have we any means of knowing how the tests were placed before the pupils. Rice's tests, moreover, were inadequate in that they attempted to measure both reasoning capacity and ability in fundamentals by means of the same problems. These are different abilities, and should be separately measured. The tests, likewise, call occasionally for the employment of operations with which the pupils have not yet become familiar. The method of scoring is unsatisfactory; there is no evidence that the relative difficulty of the problems was determined experimentally; and identical scoring units were assigned to problems of different degrees of difficulty.

As to the treatment of data, Stone points out that it is desirable to print more than mere averages; far too few of the data are presented in Rice's results. In his own study he proposes to introduce the following improvements: to take account of the kind and arrangement of the subject-matter in the different courses of study, and also of the place given to drill; and to gather and handle data in a more refined way. He states his problem as follows: "What is the nature of the product of the first six years in arithmetic work? And what is the relation between distinctive procedures in arithmetic work and the resulting abilities?" Again, the test method is used. The tests were two in number, one in fundamentals, containing fourteen examples in addition, multiplication, and division, and having a time limit of twelve minutes; the other in reasoning, containing twelve problems, similar to those found in any text-book of arithmetic, and having a time limit of fifteen minutes. These tests, Stone believes, represent phases of arithmetic work in which no system can afford to fail in producing ability. All tests were personally conducted by the author, directions being given orally by him, except as printed on the test slips. No announcement of the tests was made in advance to the pupils; and teachers were asked to have as little as possible to do with the pupils during the work. The subjects for the investigation were pupils of the 6A grade in each of 26 school systems of New England,

the middle East, and the middle West; 6000 papers were obtained.

The main results were as follows:

Arithmetical abilities; variability and relationship.—There is decided lack of uniformity among systems, both in amount of work done and in accuracy. The variation among individuals within a system is even greater. There are no appreciable sex differences in variation. Extensive tables of coefficients of correlation are given, to show in how far the possession of one ability implies the possession of others. Among different systems, the fundamentals are closely correlated with each other (.92 to .95), the lowest coefficient being that between addition and division; subtraction is most highly correlated with reasoning (.50), and addition least so (.32). The coefficient of correlation of reasoning with all fundamentals is .40 for all the systems, and .73 for all systems except those in which there is a special emphasis upon the fundamentals, a finding which Stone interprets to mean that these coefficients measure the influence of teaching rather than individual capacity of the pupil. Among individuals, the relationship obtaining between different abilities is decidedly less close than among systems. On the whole, then, reasoning ability implies ability in fundamentals no more than mathematical ability implies ability in English. Like Rice, Stone found that systems in which there was supervision by superintendent or supervisor did slightly better than those without such supervision, and that the best work was done under the supervision of both superintendent and principal; he takes pains to state, however, that his results do not show this at all conclusively. He does not attribute so great a potency to the tests as does Rice, for the reason that in those systems in which the supervision was in the form of tests the pupils did better in fundamentals, but their work in reasoning was distinctly inferior.

Arithmetical abilities in relation to time expenditure.—A blank containing careful inquiries as to the amount of time and energy devoted to the different studies of the school curriculum by teachers and pupils was submitted to each principal. From the data thus secured Stone finds no direct ratio between time expenditure and ability. In all schools where home study is the practice, the ability in relation to time spent is somewhat greater. The relation between time spent and ability in fundamentals is closer than that between time spent and ability in reasoning.

Arithmetical abilities in relation to the courses of study.—Careful statements of the courses of study followed were

secured from the various teachers. Then ratings of the different courses of study as to degree of excellence were made by twenty-one professors and graduate students in education, each of whom had had practical experience with and made studies of elementary school curricula. It was found that the systems constituting the inferior half as regards course of study showed a higher degree of ability than those of the superior half. The courses of study were next rated by nine competent judges according to emphasis on concreteness ("extent to which the problems offered are such that their solution will put the pupils in possession of the quantitative aspects of the lives they are experiencing") or emphasis on drill. Here again results were negative, the relation between emphasis on drill and ability being somewhat closer.

Stone concludes from his study that the greatest need of our school systems is standards of achievement. There is apparently no one factor that produces ability,—no single *summum bonum* in arithmetic teaching.

The problem of Rice and Stone is next taken up by Curtis (7), who aimed to ascertain the facts regarding the development of arithmetical ability throughout the grades of a private school for girls in Detroit of which he is principal, and to measure the change produced by a year's regular work under the existing conditions. These two divisions of the investigation will be considered separately.

The development of arithmetical ability was studied by means of tests, two in number, one on fundamentals, containing fourteen examples in addition, subtraction, multiplication, and division, and one on reasoning, consisting of twelve arithmetical problems. The subjects were 317 girls, pupils of eleven grades (third to thirteenth), the average number per grade being 23. The method of instruction was individualistic; the classes were sectioned as soon as they reached 12 in number and frequent interviews were held between teachers and pupils. At the time of the investigation the departments of science and mathematics had for two years been combined, and an objective laboratory system had been in effect. The procedure of the experiment was as follows: The tests were distributed among the girls in the assembly room of the school on successive days, the time being in all cases the middle of the morning session. The pupils were not informed in advance that tests were to be given; the instructions were for each to do her best for the honor of the class. The time limit was twelve minutes for the test on fundamentals, and fifteen for that on reasoning. A record was kept of attempts, as well as of correct solutions; this difference furnished an index of accuracy.

RESULTS OF COURTIS

Curve showing achievements in fundamentals and reasoning. Full lines show fundamentals; dotted lines show reasoning.

From the results it appears that the product of the present educational system is very uncertain. The distribution of ability in all grades after the fifth is very irregular. The curves for fundamentals and reasoning do not follow each other very closely, nor is their relation uniform throughout the grades. Skill in one branch is not necessarily accompanied by skill in other branches; from this it may be inferred that special training has but little value for the development of general ability. Among fundamentals, addition is hardest and multiplication easiest. This is due, Curtis thinks, to the fact that more emphasis is placed on the multiplication and division tables than on the addition; it shows that methods of training are haphazard, not based on insight into the real difficulty. There is great need of all-round early training. The test in fundamentals is more difficult than that in reasoning, as the latter rises to a maximum earlier. The fact that the sixth grade was inferior to the fifth shows the need of special and vigorous training for the former,—although, indeed, the very factors that lead to its inferiority may later bring about increased ability. The inaccuracy of grades five and six was due to the fact that these grades, working with fractions, attempted problems involving fractions, but could not solve problems involving partition, as some of those attempted did.

In the second part of the study,—the investigation of the change produced by a year's regular work,—the tests were somewhat altered. Besides the two previously used, others, especially calculated to test speed in fundamentals and reasoning, and in correct copying of figures, were employed. The reason for the introduction of these extra tests lies in the fact revealed by the previous experiment, that arithmetical ability is really a complex of a large number of separate abilities, among which there is little evidence of transfer. The speed test in fundamentals was composed of very simple examples in the four fundamentals; the time limit was 90 seconds. The speed test in reasoning consisted in a collection of simple one-step problems, after each of which was left a blank space, in which the pupil was asked to write the name of the operation called for; the time allowance was 2 minutes. Particular attention was directed to the operations of borrowing and carrying in the original test on fundamentals, as these in themselves represent particular abilities. This measurement was effected by means of the comparison of examples which did not require borrowing and carrying with others in which these operations were necessary. The tests measuring the rate of motor activity in copying figures consisted in 21 five-

place numbers, to be copied by the pupils. The time limit was one minute. The procedure consisted in giving all five of the tests during the second week of school, and again in the final month; the speed and copying tests were given at intervals during the year,—at the end of six weeks, before and after the Christmas and Easter vacations, and weekly during the last ten weeks.

The results of the previous test were confirmed. In addition to them, it was found that mistakes in copying constituted from 8 to 40 per cent. of the total number of errors made. The introduction of carrying doubles the difficulty of examples; that of borrowing, curiously enough, does not increase it. For grades studying arithmetic, the ability to borrow and carry was greater in June; for those not studying arithmetic, the reverse was the case. There is a close correlation between speed in copying figures and speed in single examples up to a certain point (115 figures per minute); beyond this, there is no indication that increased speed in copying figures is accompanied by more rapid performance of examples. It appears, also, that there is a critical degree of knowledge of the tables, beyond which an increase in score does not carry with it an increase in ability to work examples.

By far the most striking fact brought out in Curtis' work is that of the immense variability of individuals within a grade. This is clearly illustrated by superimposing on one chart the zones between the upper and lower mean variations of the fifth, sixth, seventh, and eighth grades. The remarkable fact appears that for two-thirds of the year a considerable part of the area covered is common to all four of the grades, and for the whole time a large part is common to three grades. In other words, the eighth grade contains a great number of pupils of fifth, sixth, or seventh grade ability only. There is, apparently, no such thing as a grade standard. This, Curtis believes, is largely due to the failure of many pupils to secure all-round development in fundamentals in the lower grades. The present system does not minister adequately to the needs of different individuals. For overcoming these difficulties, without sacrificing the advantages of group work, Curtis recommends the establishment of standard scores, stages of progress, for each grade, to be measured by standard tests; that would make possible individual prescriptions. That such tests are reliable measures of ability is proven, he thinks, by the consistency of grades and individuals in his own results. He is at present engaged in an effort to establish clearly defined grade standards, on the basis of the abilities of a large number of children of different localities.

3. *The Relations Between the Different Arithmetical Operations.*—Lobsien (21) undertook to determine the relation between memory for numbers and ability in arithmetical work. He gave memory and arithmetic tests to 50 pupils, of ages from nine to ten years, in six classes of a boys' *Mittelschule*, fourth year. In the memory tests, two-place numbers were exposed for definite periods of time, the percentage remembered furnishing a measure of memory. For testing visual memory, the numbers were exposed visually, simultaneously; for testing auditory memory, they were read to the pupils. The arithmetic tests were both written and oral; they consisted of examples in the four fundamental operations. The following are the coefficients of correlation obtained: Auditory memory with oral and written arithmetic, .24 and .18 respectively; visual memory and oral and written arithmetic, —.32 and —.06 respectively; oral and written arithmetic, .23; auditory and visual memory, .17. Visual memory for figures has little to do with either oral or written arithmetic; the efficient worker in arithmetic has a good auditory memory. These results appear somewhat unreliable for the reason that visual or auditory presentation furnishes no guarantee that the pupil will employ a visual or auditory method in his retention. Huther's paper (15) is non-experimental; his main contention is contrary to Lobsien's findings. Erler (12) takes up the same problem, using as subjects pupils of the third year classes of a *Bürgerschule*. His memory tests were three in number, each consisting in the recall of as many as possible of eight two-place numbers, read at five-second intervals. In the first test, the instructions were to refrain absolutely from moving vocal organs or hands (mouth to be kept tightly closed, and hands doubled up); in the second, there was no restriction of movement, but no audible speech was permitted; in the third, the figures were repeated aloud by the pupils. The arithmetic tests consisted in examples in the four fundamentals. Erler obtained slightly higher correlations between both kinds of arithmetic and "pure auditory memory" (first memory test) than did Lobsien; the correlations between the results of the second memory test and arithmetical ability were higher, and those between the third memory test and arithmetical ability still higher. The weaknesses of Lobsien's work,—assumption that auditory memory is necessitated by the presentation of the auditory stimuli,—appear again in this investigation.

Brown (5) emphasizes the need of a more satisfactory account of the psychology of reasoning than that furnished by the introspective method. His problem was to collect data

on the various mathematical abilities of children. He gave tests in arithmetic, geometry, and algebra to 83 boys of an English public school, their ages ranging from thirteen to seventeen. The results show a close relation between memory for propositions in geometry and power of recognizing general relations; the most important ability in geometry is that of remembering preceding propositions. The coefficients of correlation between the three branches of mathematics tested were as follows: geometry and algebra, — .05; geometry and arithmetic, .23; algebra and arithmetic, .75. The negative correlation between geometry and algebra indicates that geometrical ability is related to algebraic only through the mediation of arithmetic; this is in harmony with the view that mathematical reasoners fall into two types, geometrical, or intuitional, and analytical, or logical. The close relation between algebra and arithmetic is due mainly to habits of accuracy common to both. The ability to do percentage and proportion in arithmetic was found to be more closely related to geometrical than to algebraic ability. The precise aim of this study is not clear to the writer; Brown apparently intends to throw some light on reasoning ability of children, though just what he means by reasoning is not very evident. He seems at times to be combating something akin to the faculty doctrine of reasoning. To his tests the objection may be made from this viewpoint that they measure little besides memory for foregoing methods of work. A number of his conclusions seem hardly warranted by his data, notably that of the mediation of arithmetic between geometrical and algebraic ability.

Fox and Thorndike (13), investigating the relationships between the different abilities involved in the study of arithmetic, gave tests in addition, multiplication, fractions, rational computation, and simple arithmetical problems to 77 high school students of ages ranging from fourteen to twenty, 28 boys and 49 girls. All tests were given by the same individual; each was timed, the time being uniform and short enough to prevent any from finishing. The individual records of ability thus obtained were correlated; the coefficients are as follows: addition with multiplication, and fractions, .75 and .19 respectively; addition and multiplication with fractions, with problems, and with both fractions and problems, .46, .55, and .54 respectively; fractions with problems, and rational computation, .44 and .58 respectively. Thus, different relationships and mutual independence exist between the different forms of ability tested. Skill in the solution of problems is no more closely allied to skill in working with

fractions than to efficiency in the mere associative processes involved in adding and multiplying, a finding which the authors attribute to the alleged fact that the vast majority of pupils use neither arithmetical analysis nor reasoning. The girls do about 5 per cent. better, on the whole, than the boys; their variation is 92 per cent. as great. The data show no reason for sex differentiation in arithmetical training. The authors point out, however, that in the town where the experiment was conducted many of the brightest boys go to work immediately upon graduation from the grammar school, and that consequently the high school has an unfairly large proportion of mediocre boys and of bright girls.

Starch's (33) purpose was to obtain data on the transfer of training in arithmetical operations. His subjects, 15 in number, he divided into two groups of 7 and 8 each; his method consisted in giving to all the subjects two sets of tests, one before and one after an interim of two weeks, during which the group of 8 was drilled in mental multiplication of three-place numbers by one-place multipliers. The tests were eight in number, six dealing with the four fundamentals and two with memory span for numbers and words. Comparison of the efficiency of the drilled and non-drilled groups after the final tests revealed an improvement of from 20 to 40 per cent. for the former, and of 20 per cent. for the latter. The memory span was not affected. Introspective data showed that the improvement was due to the acquisition of an increased ability to keep numbers in mind, and to the application to other operations of elements acquired during the training. Visualization, supplemented and prevented from confusion by inner speech, was common. Winch's (37) problem was this: Is there such a thing as a habit of accuracy? Will accuracy be transferred from one subject-matter to another? If so, is the relation a causal one, or is it due to a factor which is common to both? The specific object was to test the transfer from practice in numerical accuracy to arithmetical reasoning. The method consisted in giving to all his subjects,—pupils of four schools, ages for the most part averaging ten years,—three tests each, alternately, in arithmetical problems and 'rule sums;' then, on the basis of the ability shown, in dividing them into two approximately equal groups, one of which was practiced for a time (4, 6, and 10 days in different schools) in 'rule sums;' and finally, in giving to all another test in problems. The first tests showed rather high positive correlations between ability in reasoning and accuracy in numerical computation (coefficients, .69 to .85). The drill sections showed decided improvement in accuracy during

the course of the drill. As regards the existence of transfer, however, the results are negative, or practically so. In the two schools which show a slightly greater improvement in the final test on the part of the drilled than of the non-drilled,—for both sections show improvement,—there are large sub-groups in both sections which show superiority in the non-drilled pupils. There is some superiority for the drilled sections in the mechanical work involved in solving the problems; this may account for the slightly higher scores, where they occur. In a second test, the following changes in method were made: The first tests were given just before the summer vacation; the drill took place during the first two weeks of the following fall semester; and the instructions were for the pupils to indicate how they would solve the problems. Again, there was absolutely no evidence of transfer.

J. C. Brown (4), the subject of whose investigation was the value of drill work, examined 51 pupils, their average age being thirteen years six months; a test in fundamental operations and arithmetical reasoning (problems) was given; the pupils were divided into two equal groups on the basis of the marks, and one of these groups was drilled, orally and by written work, during a specified part of the next thirty recitations. A final test in reasoning and fundamentals showed improvement in both, especially the latter, the improvement being most marked in the drilled section. The benefit of the drill was greatest in the sixth, and least in the eighth, grade; it persisted during the summer vacation, as was shown by tests in the fall.

A general survey of the field covered in this paper reveals some fairly well-established facts, many possibilities, and a large number of problems yet to be solved. As regards the arithmetical ability of children, perhaps the most striking fact is that of the immense variability among different individuals who have been subjected to the same school training. The reason for this disparity remains to be determined. Certainly none of the investigators mentioned have succeeded in ascribing it to any definite cause, or set of causes. Stone's results are confessedly largely negative; Rice reached his conclusion,—that inequalities must be ascribed to poor supervision,—through a process of logic rather than from his experimental data. Courtis and others have as yet done little more than to state the problem in a slightly different form,—that the relative amount of training given to different branches of arithmetic is based on conjecture, rather than on experimental determination of where training is needed. Courtis proposes, as a means for overcoming this lack of uniformity, the use of

careful diagnostic tests, with individual or small group remedial measures.

It appears also that the abilities implied in the arithmetic work of the school are many and varied; that they are largely independent of one another; that there is little evidence of transfer in any such sense as that a brief drill in one operation is attended by an improvement in other operations. This whole general problem and its numerous attempted solutions raises a series of interesting questions. Is the non-existence of transfer due to the method of drill which has been employed? Would not a training which consists in a conscious and deliberate attempt to create in the child's mind the attitude of analysis and synthesis in conformity with certain well-understood principles, in the presence of any problematic situation which might be presented to him, with the habits of concentration involved,—would not such a training lead to an improvement in any field of endeavor where such an attitude is useful or advantageous? Investigators of transfer have drilled their subjects in an exceedingly blind fashion; and it seems safe to say that the acquisition of skill and efficiency during the period of practice has consisted essentially in the unreflective mechanization of a circumscribed group of motor acts or in the acquisition and facilitation of a circumscribed group of associations. There has been little or no attempt to promote the acquisition of principles which possess any degree of generality; and there has been no attempt to inculcate an understanding of the purpose of it all.

The reasoning capacity, if it be conceded that such tests as have been employed really measure it, makes its appearance rather late in the grades. Children frequently fail through employment of imperfect or false concepts, lack of ability to express themselves, or tendency to apply principles in a hit or miss fashion, showing that they have failed to comprehend general laws, or have learned to apply them only in a routine way. Children learn to perform specifically formulated problems according to a given method, and any change in the formulation of the problems leaves them wholly at sea.

But there is, perhaps, one main objection to all the tests which have been used. Before considering it, it would be well to determine, tentatively at least, what is the nature of the series of purposive associations called reasoning. Reasoning has been defined as an attempt to meet adequately any situation, by recognizing certain elements in that situation which are identical with known situations, and reacting to them accordingly. Acuteness of reasoning depends on ability to analyze situations carefully, and to detect minute resem-

blances. The formation and testing of hypotheses consists ultimately in grouping together a number of resemblances which experience has shown to be valid, and in determining whether they also hold in further experience, *i. e.*, the present instance. Reasoning is implicit if, in proceeding from the known to the unknown by way of detected resemblances, no effort is made to discover in just what that resemblance consists; it is explicit if the relation of identity is analyzed. If such a definition of reasoning be accepted, few, if any, of the tests which have been employed really measure its presence in children. The situations which have been presented do not constitute problems which appeal to the child; they are selected from the point of view of the adult. Success in meeting them depends largely on the child's memory of certain methods of past procedure, not on his capacity to reason either in an implicit or explicit fashion. The 'native capacity' which Bonser claims to test is, from this point of view, little else than ability that a given method of procedure already learned is now to be applied. Some of the tests, to be sure, involve analysis, but it is analysis of an artificial sort.

If this criticism is valid, the investigations so far must leave most of the questions regarding the reasoning of children. The vast differences in the range of experience, environment, and interests of different children make it an extremely difficult task to devise tests which would determine the extent to which the child mind possesses the attitude of the reasoner.

In conclusion, it seems wise to call attention to an antithesis which has been pointed out between two different modes of mental activity. Meumann speaks of an objective and a subjective type of thinker. Poincaré distinguishes two types of mathematicians, an intuitionist, or geometrical, and an analytic, or algebraic. Lessing (19) undoubtedly has much the same distinction in mind when he speaks of an extrospective and an introspective type of scientist. The former type includes the natural scientists, those whose main interest lies in the observation of objective facts; to the latter type belong the metaphysicians and the pure mathematicians, whose chief interest is to analyze and to systematize the logical processes by means of which their conclusions are reached. This antithesis is to be found in children, as well as in adults, according to Lessing. Certain children take pleasure in numerical combinations and in mathematical speculations, while others prefer the observation of nature, the collecting of plants and animals, and the like. Race plays a part in this differentiation; Jewish children, for instance, are extremely concrete and practical in thought as well as in action. Lessing's sug-

gestion offers an interesting subject for further investigation; such differences as he indicates, if they exist in children, must be taken into account in any attempt to devise tests which will serve for the establishment of reasoning norms. The problem, as it stands, is one of the most important ones for future investigation.

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