addition of fractions, with the most gratifying results.

Superintendent Charles W. Kline writes of this work in Waterloo: "I have observed the work of pupils in the fourth and fifth grades in addition and subtraction of fractions where they followed the usual method of reducing to a common denominator and where they used the objective method. Pupils work much more rapidly and secure better results by the new method. They seem to understand the whole subject of fractions much better than those taught by the old method. On a test given on the same problems in fractions the results were as follows:

New Method:

4 A No. of minutes 10 Median No. problems right 27.

5 B No. of minutes 12 Median No. problems right 28.

Old Method:

4 A No. of minutes 30 Median No. problems right 5.

5 B No. of minutes 10 Median No. problems right 14.

The results of this test would seem to indicate that much better results can be secured in the teaching of fractions by the objective method than by the usual method of reducing to a common denominator."

ABSOLUTE TEMPERATURE.

BY KIRSTINE MEYER, NEE BJERRUM¹*

*Translated by Willard J. Fisher, 28 Baker St.,
Worcester, Mass.*

To fix the conception of temperature as measure of thermal condition, we must make a series of assumptions and base ourselves upon a series of experiences. First, we must define what is meant by

1. Two bodies have equal temperatures;
2. One body has a higher temperature than another.

Then we must proceed from the following experiences:

A. If of two bodies each has the same temperature as one and the same third body, then they have mutually the same temperature.

B. If we choose a particular body A, of which certain properties change under the influence of heat, and bring this body A in contact with a series of other bodies which exert an influence on these properties, then these bodies can be divided

¹*The following pages were published under the title, "From what hypotheses must we start, in order to be able to define the concept temperature," in *Zeitschrift für physikalischen und chemischen Unterricht*, May, 1910.

into two groups; those in one group have higher temperatures than A, the others lower. This grouping remains unchanged, if the body A be replaced by another which has a temperature equal to its temperature.

The body A is called a thermometer, i.e., a contrivance with which one recognizes temperature differences.

Further there come into account

3. The body used as the thermometer;
4. The property of this body, whose change measures the thermal condition.
5. The notation used in the measurement, i.e., the temperature scale.

From the preceding, it appears how hard it was to get clearness on these points. Aristotle could not overcome the difficulties appearing in 1 and 2; Boyle first essentially attained this, when he used the thermometer with inclosed liquid.

In the course of the eighteenth century, facts were learned by which all these points could be cleared up. Hereupon there arose the new problem, to find a scale whose reading measures the "quantity" or "power" of applied heat, independent of the thermometer used.

In the middle nineteenth century the so-called absolute temperature scale was proposed, i. e., a scale whose indications are independent of the thermometer used, and also independent of the property, whose change is used in the measurement. We wish here to show what further points are to be considered with the introduction of this scale.

The newly discovered facts and the assumptions founded on them are:

C. Heat is a form of energy; if heat is produced by work, there is a constant ratio between the heat produced and the mechanical work used.

By this we can define unit heat independently of the temperature scale. If, for example, we imagine in Joule's experiments on the production of heat from work an ice calorimeter substituted for the ordinary calorimeter, then 3 *cg.* ice are melted for every meterkilogram of work that disappears. As a unit for measuring quantity of heat we could then introduce the amount capable of melting 3 *cg.* ice at the freezing point under the pressure of 1 atmosphere.

D. Principle of S. Carnot. If a reversible cyclic process goes on between two bodies with different but constant tempera-

tures, then the ratio between the heat quantities Q_1 and Q_2 given off by the first body and taken up by the second depends only on their temperatures and is entirely independent of the body in which the cyclic process goes on, and also of its properties, which perhaps change during the process. According to the proposition of W. Thomson (later Lord Kelvin), we measure the temperature of the changing body during the intake and outgo of heat by the numbers θ_1 and θ_2 satisfying the equation $Q_1/Q_2 = \theta_1/\theta_2$.

If we fix the number measuring one of the two temperatures, then the other is determined independently of the body which goes through the cyclic process. If we consider this body as a thermometer, we get in absolute measure the temperature of one of the two heat sources, as during the process these were in temperature equilibrium with the thermometer.

The meaning of the zero point of this absolute scale we find thus:

$$Q_1/Q_2 = \theta_1/\theta_2, \text{ or } Q_1 - Q_2/Q_1 = \theta_1 - \theta_2/\theta_1.$$

If we assume that θ_1 is the temperature of that heat source from which the work (or heat quantity) Q_1 is received, while θ_2 is the temperature of that which received the work Q_2 , then we see that the useful effect (efficiency)

$$Q_1 - Q_2/Q_1 = \theta_1 - \theta_2/\theta_1$$

approaches 1 as θ_2 approaches zero. Consequently the absolute zero is the temperature of the colder body approached as the efficiency approaches 1:

It is now to be studied, how the temperature number can be expressed on the scale of any thermometer whatever. For any reversible cyclic process² the equation holds, $dQ/\theta = 0$; i. e., for any arbitrary portion of the process the value of this integral depends only on the initial and final conditions, or dQ/θ is a perfect differential.^{3*} If we assume for the two

²*For elementary discussions of reversible and other cycles, see the textbooks of physics or heat, as Duff's, pp. 281-295, Watson's, pp. 299-335, Reed and Guthe, pp. 216-229, Anderson, pp. 323-326, Carhart, 409-448, Spinney, 230-244

³*For discussions of the meaning and properties of perfect differentials, see textbooks on calculus, as

Murray, *Differential Equations*, pp. 17-22.

Johnson, *Differential Equations*, pp. 22-29.

Murray, *Infinitesimal Calculus*, p. 141.

That dQ is not a perfect differential, or that the amount of heat taken in or given out by a substance during a thermal change depends not only on the initial and final states of the substance but also on the way in which the change from initial to final state is made is clear from the following:

Imagine a piston and cylinder of heat-insulating material containing a hot compressed gas; allow the gas to expand and push the piston before it (which happens when we pull the trigger of an airgun); the gas will cool down as it overcomes the pressure of the external atmosphere, until its own pressure is equal to atmospheric. But, as the walls are insulating, no heat has entered or escaped. The pressure and temperature, however, have fallen, and the volume increased to perfectly definite and fixed values. Line AB represents the change.

variables which determine the state of the body, first the temperature t , measured on any thermometer, second, any y whatever, then $dQ = P dt + R dy$.^{4*}

$$dQ/\theta = (P/\theta)dt + (R/\theta)dy.$$

As dQ/θ is a perfect differential,

$$[\partial(P/\theta)]/\partial y = [\partial(R/\theta)]/\partial t$$

then, since θ is a function of t only^{5*}

$$[(1/\theta)(\partial P/\partial y)] = [(1/\theta)(\partial R/\partial t)] - [(R/\theta^2)(d\theta/dt)], \text{ or} \\ [(1/\theta)(d\theta/dt)] = 1/R[(\partial R/\partial t) - (\partial P/\partial y)].$$

As the left hand side is a function of t only, so is the right

$$\text{hand, hence} \quad \theta = \theta_0 E^{\int_{t_0}^t [1/R(\partial R/\partial t) - (\partial P/\partial y)] dt} \quad (1)$$

So we have the absolute temperature expressed in terms of the temperature t , measured on the scale of any thermometer.

In particular, we can choose as second independent variable the volume v . Then

$$dQ = Pdt + Rdv.$$

If the pressure to which the body is subjected is p , then the increase of internal energy during the reversible process is $dU = dQ - pdv$, which is a perfect differential^{6*}, being dependent only on the initial and final states, according to the first law.^{7*} But if $dV = Pdt + (R - p)dv$ is a perfect differential, then

$$\partial P/\partial v = [(\partial R/\partial t) - (\partial p/\partial t)], \therefore [(\partial R/\partial t) - (\partial P/\partial v)] = \partial p/\partial t$$

and, by substituting in (1)

Suppose again the same gas with the same initial temperature and pressure in a cylinder and piston of the same dimensions but of a conducting material. If we cool the whole down, not by allowing the gas to expand, but by means of cold water poured on, we can bring the pressure down to atmospheric without any change in volume, without allowing the piston to move. Heat has been taken away in definite amount. Line AC represents the change. Now we can release the piston, and, by adding heat, as by pouring on hot water, we can make the gas expand, all the while at atmospheric pressure, until it has the same volume as at the end of the first case. Line CB. The gas is now of a certain volume, pressure and temperature, reached in the first case with no addition or subtraction of heat, in the second case with first an addition and then a subtraction, which, however, do not balance each other. The heat abstracted by the cold water was more than that added by the warm water. So the quantity of heat used in carrying the gas from the initial condition to the final depends greatly on the way the change is made.

^{4*} P and R are coefficients which are known to be in general functions of t and y , but of a form as yet unknown. The equation says that Q depends on t and y . This statement is allowable, since we have tacitly assumed that the substance in the engine—steam, air, or whatever it may be—has an equation of state, so that its condition as to pressure, etc., and the amount of energy change equivalent to any amount of heat introduced or leaving can be completely expressed in terms of the temperature t and one other variable, which latter may be selected from a wide range of physical qualities.

^{5*}The relation between θ and t is purely a numerical one, they being both temperatures, and depends only on the fixed points arbitrarily selected for the two scales, and on the law for the size of a degree, as that the degrees are uniform in size, or some other law.

^{6*}Though dQ and pdv are not dependent solely on the initial and final states, their difference is.

^{7*}The first law is really stated in the equation above, or thus, $dQ = dU + pdv$, or, the heat added to a body is equivalent to the increase of the internal energy of the body plus the external work done by the body during the addition of the heat.

$$\theta = \theta_0 E^{\int_{t_0}^t [(1/R)(\partial p/\partial t)] dt} \quad (2)$$

in which R is the quantity of heat measured in work units, called the heat of expansion along an isotherm,^{8*} or, $R = (\partial Q/\partial v)t = \text{const.}$, whence it follows from (2) that $(1/R)(\partial p/\partial t)$ is a function of t only.

Now consider one or two examples.

(1) Let the body considered be a saturated vapor; then p is a function of t only,^{9*} and (2) can be written

$$(\theta/\theta_0) = E^{\int_{p_0}^p (dp/R)}.$$

(2) Let the body be air, obeying Joule's law;^{10*} then the heat quantity absorbed during expansion along an isotherm is equal to the external work done, or $p dv = R dv$, or $p = R$.

$$\text{Then from (2),} \quad \theta = \theta_0 E^{\int_{t_0}^t [(1/p)(\partial p/\partial t)] dt}$$

To get the relation of the two temperature scales, imagine the integration carried out from a point of the isotherm t_0 to a point on the isotherm t ; choosing these two points on the same ordinate, corresponding to equal volumes, and integrating along

$$\text{this ordinate, then } (\theta/\theta_0) = E^{\int_{p_0}^p (dp/p)} = (p/p_0) \quad (3)$$

in which p and p_0 designate the pressures of the gas at constant volume and at the temperatures considered.

In an air thermometer, in which the inclosed air is kept at constant volume, the temperature is expressed by

$$p = p_0 \propto T.$$

The temperature thus defined agrees with the absolute temperature, if the inclosed gas obeys the law of Joule, of which the law of Mariotte and Gay-Lussac is a special case. This follows from (3), for $p = (p_0/\theta_0)\theta = F(v) \cdot \theta$

since p_0/θ_0 is some function or other of the volume, though not necessarily that demanded by the law of Mariotte and Gay-Lussac.

Finally the question arises, can the absolute scale be prac-

^{8*}i.e., the heat required to do work of expansion without change of temperature.

^{9*}So that dp/dt , partial derivative, becomes dp/dt , ordinary derivative.

^{10*}Joule's law is based on the experiment described in college texts as follows: Duff, p. 235; Watson, p. 305; Reed and Guthe, p. 220; Kimball, p. 273.

tically completely realized? Is there a gas which obeys Joule's law exactly? Direct experiments for answering this question were made, as is well known, by Joule and W. Thomson; of later such experiments nothing is known.^{11*} The result was that none of the gases investigated followed the law exactly; for hydrogen the deviations were least. Consequently, in the year 1888 the hydrogen constant volume thermometer was by international agreement made standard for all temperature measurements, and the temperature scale was defined by $\theta/\theta_0 = p/p_0$, wherein the pressure p_0 at the freezing point of water is made 1000mm. mercury. The temperature difference between the freezing point of water and its boiling point under normal pressure was made 100; so that θ_0 , the temperature of melting ice, is determined by

$$(p_{100} - p_0)/p_0 = 100/\theta_0; \theta_0 = (1)/(p_{100} - p_0)/100p_0 = 1/\beta$$

in which β is the pressure coefficient at constant volume. According to P. Chappuis' latest investigations¹², for hydrogen = 0.00366256, so that $\theta_0 = 273.033$. If we use as thermometric substances other gases than hydrogen, we get somewhat different values for β and consequently for θ_0 . Chappuis gets for

$$\text{Nitrogen} \quad \beta = 0.0036617, \quad \theta_0 = 273.097$$

$$\text{Carbon dioxide} \quad \beta = 0.00367, \quad \theta_0 = 273.48$$

The value of θ_0 obtained with one of the real gases varies little from that which would be got with an ideal gas; the same holds for other numbers on the scale. A series of experiments has been made to determine these variations, i. e., the temperature of a body on the absolute scale has been compared with that according to the international standard scale just mentioned. For this purpose (2) can be applied if the quantities under the integral sign are known, i. e., if, for example, p and R are known for the gases as functions of the temperature within a large interval; at present sufficient data for the purpose have not been obtained. Further, such a comparison can be made on the basis of the Joule-Thomson experiment^{13*} with gases which expand adiabatically without complete expenditure of work through a porous diaphragm. The increase of the internal energy of such a gas, going without access of heat from the condition p, v, θ to the condition $p+dp, v+dv, \theta+d\theta$, is equal to the applied external work; i. e.,

^{11*}The papers of Joule and Joule and Thomson are reprinted in Harper's Scientific Memoirs, I. The Free Expansion of Gases; J. S. Ames, ed., 1898.

¹²Trav. et Mém. du Bureau International des Poids et Mesures, T. 13, 1907.

^{13*}The Joule-Thomson experiment is described in Duff, p. 238; Anderson, p. 277; Reed and Guthe, p. 225; Watson, p. 305-307.

$$-d(pv) = -(p(dv) + (v)dp).$$

The same change of internal energy could be brought about by a reversible process; in this the change of internal energy is

$$dU = dQ - (p)dv,$$

in which $dQ = C_p(d\theta) + (h)dp$, $(h)dp$ meaning the heat quantity necessary for an isothermal increase of pressure dp , and C_p the specific heat at constant pressure.

Consequently $-v(dp) - p(dv) = C_p(d\theta) + h(dp) - (p)dv$.

In this $h = -\theta(\partial v / \partial \theta)_p$

and we have $d\theta = (\theta(\partial v / \partial \theta) - v) / (C_p) dp$ (4)

Introducing now the temperature measured on any thermometer as new independent variable, and denoting $d\theta$ and dp in (4) by

$$\Delta\theta \text{ and } \Delta_p, \partial v / \partial \theta = \partial v / \partial t \cdot dt / d\theta$$

$$C_p = \partial q / \partial \theta = (\partial q / \partial t)(dt / d\theta) = C'_p(dt / d\theta)$$

$$\Delta\theta = (d\theta / dt) \Delta t$$

$$\text{Substituting in (4), } (\theta / \theta_0) = E \int_0^t [(\partial v / \partial t)_p dt] / [v + C'_p(\Delta t / \Delta p)]$$

in which the quantities under the integral sign can be determined by the $\Delta t / \Delta p$ of Joule and Thomson. Here again the determinations, particularly of $\Delta t / \Delta p$, are too few, and also too uncertain. From their experiments Joule and Thomson thought they could deduce $\Delta\theta = (\alpha / \theta_2) \Delta p$ in which α is a constant.

Introducing this expression for $\Delta\theta$ into (4), we get an equation of state for the gas under experiment, from which we can calculate θ and compare this with the temperature got with the normal thermometer. But the equation $\Delta\theta = (\alpha / \theta_2) \Delta p$ has no general validity; it is not grounded theoretically, but set up from the results of experiments over only a small range of temperature.

In the comparison of the two scales we go on in general as just shown; we try to find an equation of state for the real gases containing pressure, volume and absolute temperature; with its help we then find in the given case the absolute temperature expressed in terms of the known values of pressure and volume. This value we compare with the temperature of the gas, got by measurement with the normal thermometer, and so find the discrepancy of the two scales at the point in question. In this way Rose-Innes¹⁴ and D. Berthelot¹⁵ in recent years have tried

¹⁴Phil. Mag., (6) 2, p. 130, 1903.

¹⁵Trav. et Mem., T. 13, 1907.

to solve the problem. Rose-Innes has on the whole followed the method sketched in the example above. The equation of state is in general got by a combination of theoretical considerations and empirical laws. Berthelot, for example, starts out from the equation of Van der Waals, derived theoretically. But since it agrees badly with determined values, certain alterations are made in it. Considering the range of their validity he extrapolates rather far.

In these various ways the final conclusion has been reached that there is a close agreement between the two scales, at least between zero and 100° .¹⁶

At higher temperatures we may expect a better agreement, inasmuch as the isotherms of all gases with rising temperatures approach more nearly equilateral hyperbolas; now, however, a new difficulty is encountered, that of finding a suitable receptacle for the thermometric substance. In fact, it is found that at high temperatures hydrogen reduces glass, porcelain and even quartz; as it further diffuses through a wall of platinum, it cannot be used as thermometric substance in the measurement of high temperatures. Also, at very high temperatures, there is possible a dissociation of the hydrogen molecule into two atoms. Consequently it has been proposed to use nitrogen as the thermometric substance for the normal scale at temperatures above 100° . Experiments on this basis were carried out by P. Chappuis.¹⁷ A comparison between a constant volume nitrogen and a similar hydrogen thermometer showed that the indications of the former varied from the normal scale between 0° and 40° , i. e., about $0^{\circ}.01$, and then grew smaller. The quantity changed, diminishing up to 80° , then it was constant or increased slightly, but only within the limits of observational error. At 100° nitrogen behaves like hydrogen at ordinary temperatures, for its compressibility is smaller than according to the law of Boyle and Mariotte. Hence Chappuis concludes, that from 80° upward nitrogen can be used instead of hydrogen as thermometric substance. To reduce the readings of the nitrogen thermometer to the normal scale, Chappuis proceeds as follows:

$$\alpha = (1/P)(dP/dt) = 0.0036738.$$

Were nitrogen a perfect gas, this would be constant to 0°

¹⁶According to Berthelot's calculations, the discrepancy in this interval between the thermodynamic and normal temperature scales should amount at most to $0^{\circ}.00055$.

¹⁷Phil. Mag., (5), 50, p. 433, 1900.

and during cooling from 100° to 0° , the pressure would diminish by 100×0.0036738 . Had the nitrogen thermometer been so filled that $P_0 = 1.000$ m., then at 100° the observed pressure was $P_{100} = 1.367466$ m., and the corresponding pressure of an ideal gas at 0° would be

$$P''_0 = 1.367466 - 0.0036738 \times 100 = 1.000086 \text{ m.}$$

With this the temperatures are computed corresponding to the measured pressure P for temperatures over 100° , by the formula $P/P'_0 = \theta/\theta_0$.

Helium would be the best adapted thermometric substance; the objections to the use of hydrogen are here of no account;¹⁸ helium approximates to an ideal gas in high degree, and can therefore be used for the measurement of both high and low temperatures. "It is therefore clear, that the normal scale must be that of the helium thermometer."¹⁹ Travers says that for measurement of great temperature intervals constant pressure thermometers would be more practical than those of constant volume, as the long mercury column makes trouble in many ways, partly because the high pressure deforms the receptacle, partly because it is hard to find accurately the temperature of the mercury column.

Low temperatures, producible in the most recent times, are measured with the constant volume helium or hydrogen thermometer. Even in the eighties of the last century Olzewski, in his work on the condensation of gases, measured temperatures as low as -220° with the hydrogen thermometer.²⁰ He compared the indications of gas thermometers filled with hydrogen, nitrogen and oxygen, to -150° , and found that to this limit they agree; at -150° his oxygen thermometer was about 2° too low as compared with the hydrogen thermometer; at the critical temperature of nitrogen his nitrogen thermometer was about 1.7° too low. On this Olzewski remarks, "The good results, which the gas thermometers give in measurements of low temperatures, are due to the fact, that while on cooling the gases approach their condensation temperatures, yet on account of the simultaneous diminution of pressure they also

¹⁸According to investigations of the Physikalisch-Technische Reichsanstalt, helium diffuses through quartz even at ordinary temperatures. *Zeitschr. f. Instrumentenkunde*, 32, p. 122 1912.

¹⁹Travers, *Experimental Study of Gases*.

²⁰*Annalen der Physik*, 31, p. 68, 1887.

recede from it." Later he applied the helium thermometer,² and gives a comparison of its indications with those of the hydrogen thermometer for the boiling points of liquid oxygen at different pressures.

Such measurements were lately carried out in a more accurate manner by Travers and Jacquerod. Temperatures were measured to -260° and Travers proposes to use the helium thermometer for lower temperatures. It seems odd at first that the boiling point of liquid hydrogen can be measured with a hydrogen thermometer, or that of boiling oxygen with an oxygen thermometer; the reason is, that a mass of gas, which is under a pressure of 1000 mm. at 0° , as for the thermometer of Travers and Jacquerod, at -200° shows such a small pressure as to correspond to a vapor so far removed from its saturation point that it still obeys Mariotte's law. In measuring the boiling point of oxygen with a hydrogen, a helium and an oxygen thermometer, we get

H-thermometer	$-182^{\circ}.93$
He-thermometer	$-182^{\circ}.83$
O-thermometer (less complete form)	$-182^{\circ}.4$

The agreement of the values shows that the gases follow nearly the same laws. If helium at this temperature is to be regarded as a perfect gas, so also very nearly are the others.

At very low temperatures there were observed small, but very regularly increasing, discrepancies among measurements of equal temperatures with hydrogen and helium thermometers.

Travers gives the following temperatures, reckoned from -273° as zero:

	He thermometer	H thermometer
Boiling oxygen.....	$90^{\circ}.2$	$90^{\circ}.10$
Boiling hydrogen.....	20.41	20.22
Liquid hydrogen at 100 mm. pressure	15.14	14.93

Such discrepancies must occur, since helium, even at the lowest of these temperatures, has not reached its critical temperature, while that of hydrogen is about 34° . So the helium thermometer gives excellent service at the lowest temperatures, though the discrepancies between the two gases are not large.

After a temperature scale has been found, in which the numbers have a rational meaning, there arises another problem, the same, apparently, which occupied Dalton—"Is there a simple

²Annalen der Physik, 59, p. 184, 1896.

relation between the internal structure of all substances and their temperature numbers?"

Here we meet the theory of corresponding states, whose meaning is in this connection best summarized thus: If we know for a substance at any given temperature the series of states through which it can pass, then we can name for any other substance a temperature at which it can pass through the same series of states.^{22*} An example of such a temperature, at which substances clearly pass through the same series of states, is the critical temperature. Van der Waals, who first set up this theory, asserted, as is well known, that those temperatures correspond which are equal multiples of the critical temperature, and that the corresponding states at those temperatures are those whose pressures and volumes are equal multiples of the critical multiples and volumes.

Some years ago I published a series of researches²³ which showed that Van der Waal's theory of corresponding states did not agree with the available observations, but that a better agreement was reached by computing volume and temperature from zero values characteristic for each substance, and by using units of volume, pressure and temperature also characteristic for each substance.

For the twenty-five substances I plotted a common vapor pressure curve with the abscissa $x = (T_c - T)/K$, and ordinate $y = (P_c - P)/-$, in which P_c and T_c are critical pressure and temperature for a given material and K is its temperature constant.

Investigations on the basis of later obtained experimental material have further substantiated²⁴ these assumptions. For example, with the help of this theory we can calculate in advance the critical temperature of hydrogen.

It would surely be of great importance to our views on the internal structure of matter, if we could find a special "temperature origin" for every substance; recent atomic-kinetic investigations concern themselves with this problem.²⁵

^{22*}i.e., isothermals have the same form.

²³Kgl. danske Videnskab. Selskab. Skrifter, VI, 3 Heft; Ztschr. f. phys. Chemie, 32, p. 1, 1900.

²⁴Ztschr. f. phys. Chemie, 3, p. 325, 1910, and Temperaturbegrebets Udvikling, p. 164 ff.

²⁵W. Nernst, on a general law about the behavior of solid substances at very low temperatures. Phys. Ztschr., 1911, p. 976.

Footnotes marked ^{1*} have been added by the translator; those with numerals are in the original.