Review

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REVIEW.

Gyrostatics and Rotational Motion. By A. GRAY.

Pp. 530. 42s. 1918. (Macmillan.)

In opening the Section on Didactics at the Cambridge International Mathematical Congress 1912, the chairman, Mr. C. Godfrey, had occasion to mention "the Exhibition." But turning to the index of the volumes of the Proceedings, the existence cannot be traced of any such Exhibition.

It was a display of scientific books of all the chief publishers of the world, a rival to a similar display at the preceding Congress in Heidelberg, and at the same time a collection of simple mechanical apparatus suitable for mathematical instruction; also of calculating machines, very complete of every sort. A catalogue of the apparatus had been printed, and would have added to the interest, if given a place in the volumes of the Proceedings.

The chief attraction to the foreign visitor was the show of Kelvin's gyrostatics and their improvements, in action by Dr. James Gray, son of our author. Much of the obscurity was cleared up at a glance, due to the confused diagrams as given in the *Natural Philosophy* of Thomson and Tait.

The preface of our author explains the scope of his treatment of Gyroscopic Theory—"to supply a systematic discussion of gyrostatic action and rotational motion, which may be of use to students of dynamics and practical men.

To give a training in the analytical and numerical discussion of problems of actual apparatus, accompanied by the practical study of gyrostatic devices. As a rule the discussion of dynamical and especially of gyrostatic problems in the lecture room and text book has either been restricted to one or two simple cases, or when more comprehensive, been far too much an affair of Euler's or Lagrange's equations; which, by providing a walled-in path along which the mind can travel, withdraw its attention from the incidents, often exceedingly instructive, which attend its progress."

Lagrange's Equations have become a stock piece of cram, confidently expected to be set in every examination. But they are far too delicate a tool to be put into the hand of a novice; and Lagrange himself cannot be said to have understood his own Equations thoroughly, unaware as he was of the distinction between coordinates holonomic and anholonomic, to use the cacophonous jargon. As for Euler's Equations, who is there among us can say he thoroughly understands and can interpret every term of them, and point out where the coordinates are inapplicable?

The author follows Thomson and Tait in a free use of the fluxional notation, of a dot above a letter. The idea of the Fluxion is preferable compared with the Differential Coefficient, dot-age versus d-ism; but it lost the victory from the difficulty of printing, though so easy to write.

It is related that Lagrange wrote the fluxion in his manuscript of the *Mécanique analytique*, and being a careful methodical man, he sent the whole manuscript complete to the printer. But when the mass of proofs arrived, he found the printer to suit his convenience had substituted the dash for the dot, replacing $\dot{x}$ by x' ; and it was then too much expense and trouble to make the correction.

Professor Bryan's suggestion removes the whole difficulty, if we practise writing and then printing the dot, not above, but after the letter, like the dash, and then x' , x'' give the printer no more trouble than x' , x'' . Because $\dot{x}$ and $\ddot{x}$ require a separate line above for the dots; and as several letters of the alphabet are already in their dotage, they cannot be made more dotty. The saving of expense would have been considerable on Bryan's principle, if adopted by Gray in this book.

The author is not of the Lagrange-Laplace school, too much in favour to-day, in suppressing all figures and diagrams, trusting to intuition, or drawing them with a finger in the air as a visualisation. On the contrary the work is profusely illustrated with careful drawings.

An apparatus is shown in Fig. 30 (b), p. 128, suitable for the lecture table, as visible to a large class, for showing off gyroscopic motion. Instead of a

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spinning top, requiring string, or a gyrostat actuated by an electric motor, an ordinary bicycle wheel is taken, moving freely on ball bearings, with the hub screwed to the end of the suspending stalk. The wheel can be spun by hand, and the axle projected in any manner, to show off the motion, undulating, looped, or dropping from a cusp.

Kelvin's rule "hurry the precession and the axle rises against gravity" can be illustrated, the instinctive action in riding a bicycle; as well as the reverse effect of retarding the precession; and if the precession is stopped the axle falls down vertical. This is the great principle for the inventor to grasp, in guiding a new design of harnessing a flywheel.

The suspension of the axle is of the astronomical Alt-Azimuth nature, and the three angles employed in the analytical discussion, θ , ϕ , ψ , introduced by Euler in 1760, are still in use, in which θ may be called the altitude, ψ the azimuth, and ϕ the hour-angle, angular displacement of the wheel over the axle.

There is a difference of opinion among writers as to the measurement of altitude θ . Darboux and Klein reckon θ from the upper vertical, the zenith as in Fig. 30 (a); but in the apparatus of Fig. 30 (b) it is more convenient to change to the nadir, downward vertical of the position of rest. The change from one reckoning to the other must be practised (p. 329); as in the ordinary spinning top asleep, and in the centrifuge, the motion is in the neighbourhood of the zenith upward vertical, realised by the large wheel of Fig. 30 (a).

But starting from the position of rest, give the axle the displacement θ . Then holding the axle at this angle θ , give it the displacement ψ in azimuth.

With vector representation of rotation on a right-handed screw, positive rotation of ψ will be with the sun, deasil. But with a change to the zenith, ψ would be measured against the sun, widdershins, and with the rotation of the Earth. These differences in space relation are difficult to indicate on the plane of a diagram, and here the Maxwell top comes in useful on the table, to make them clear.

Generally in any of these conventional representations of a sphere on a plane, as in Fig. 1, p. 39, the figure should be looked at as a star chart, on the concave side of the celestial sphere; not as a sea chart, on the convex surface of the terrestrial globe.

Euler's angle ϕ may be called hour-angle, by analogy with astronomical use, where the rotating gyroscopic wheel is the Earth itself, and the vertical is the equivalent of the pole of the ecliptic, in the theory of Precession. The physical interpretation of ϕ is rather delicate, as requiring angular displacement about a moving axis.

In making the ψ displacement in azimuth, with θ fixed in altitude, the wheel of Fig. 30 must be held fixed on the axle by the thumb; otherwise, if free to turn on the axle considered smooth, it will be found to have turned round over the axle, and ϕ will have come into existence.

But after the step ψ has been made, the thumb can release the wheel, and the displacement ϕ can follow, as shown in Fig. 12, p. 71; where the zenith measurement is employed; but as a star chart, the vectors are left-handed screws.

If the thumb was not applied to keep the wheel from moving round, then in any conical movement of the axle, the wheel would rub over the axle. This is easily shown with a pencil or penholder held lightly between finger and thumb, and revolved conically by a finger applied at the point. In a complete circuit, the body will be seen to have turned round through an angle proportional to the conical solid angle swept out.

In this experiment Aristotle's dictum can be disproved, that it is impossible to turn a smooth sphere. But spitted on a smooth diametral axle stalk, the sphere can be revolved to any extent by moving the axle round in a conical movement; the sphere always coming to rest with the axle.

These are the preliminary questions discussed in Chapter IV, after which in Chapter V the application is made to the gyroscopic motion, tractable by the Elliptic Function; and here the reader must not be deterred, if he is ambitious to look across the frontier of the Science.

This function seems created to speak the language of gyroscopic motion, long before the name was invented. Euler was using it unconsciously in his

Dynamics, 1760, for expressing the motion of θ in altitude, by comparing it with a corresponding beat of a plane pendulum.

But a corresponding interpretation of azimuth ψ was beyond his powers, as requiring the Third Elliptic Integral, and the world had to wait another sixty years for the developments of Legendre and Jacobi.

The invention of the theta-function made the expression analytically complete of the Third Elliptic Integral. But as the integrals which occur in gyroscopic motion are of Legendre's circular kind, with imaginary parameter, the argument of the theta-function is complex, and so of no practical value for numerical computation.

Here Abel's theory of the pseudo-elliptic integral can be utilised for giving an expression in a finite algebraical form, by choosing the elliptic parameter a rational aliquot part fraction of a period; and the construction of a series of such cases of soluble top motion is in interesting analytical diversion. A few examples are given here in the final chapter, and the results can be shown on a stereoscopic card, as drawn by the late Mr. T. I. Dewar. Professor Gray has checked some of the results by a direct calculation of the integrals by quadrature in Chapter XII, p. 259, the only practical procedure in the intermediate state of a general case of motion.

In the symmetrical top, of uniaxial symmetry, the vector of angular momentum (A.M.) describes a curve in a horizontal plane, which can be identified with a Poincot herpolhode, and the projection on this plane of a point on the axle will describe its hodograph.

In a complete treatment, by the consideration of hour angle ϕ as well as azimuth ψ , a similar statement can be made, where the fixed horizontal plane of the azimuth ψ is replaced for ϕ by a plane fixed at right angles to the axis of the top.

Thus ϕ and ψ may be interchanged to give a new state of top motion; and Jacobi's statement is explained, that the most general state of top motion can be resolved into two Poincot movements.

This Poincot state is the representation of the motion of a body of any shape about a fixed point, say about the c.g., when tossed in the air.

The c.g. proceeds to describe a parabola, and taking a stand at the c.g., the body wobbles about it in a very complicated manner, which can be studied at ease in the Maxwell top on the table, twirled by a finger and thumb, when the screws are adjusted in any manner so as to bring the c.g. into coincidence with the point in the supporting cup.

Poincot reduced this complication to order by taking the A.M. vector, now fixed in direction and magnitude, and drawing a plane through its end at right angles. Then if the momental ellipsoid is rolled on this plane, it will accompany the body as if fixed in it, the angular velocity about the instantaneous axis being made proportional to the radius vector of the ellipsoid.

It is conducive to simplicity if Prof. Gray would make the equations of his Chapter XXI homogeneous, by the introduction of a quantity, which may be denoted by D , such that

$$Ap^2 + Bq^2 + Cr^2 = 2T = Dh^2, \quad A^2p^2 + B^2q^2 + C^2r^2 = H^2 = D^2h^2,$$

and then (A, B, C, D) are of the same dimensions, and also (p, q, r, h) .

And Poincot's polhode is given by the intersection of the two ellipsoids

$$Ax^2 + By^2 + Cz^2 = D\delta^2, \quad A^2x^2 + B^2y^2 + C^2z^2 = D^2\delta^2,$$

the first being the momental ellipsoid; and then δ is the distance from O of the fixed plane on which it rolls, to trace out the herpolhode. The identification with the geometry of Darboux's representation by the deformable hyperboloid is facilitated thereby.

The simplicity is evident then in Dynamics of abstracting the gaze from the body and its visible motion about a changing axis of rotation, and in fixing the attention first on the invisible axis of angular momentum (A.M.), which in this case of no external applied force is a fixed vector. Here Maxwell's top comes in useful, as the motion of the instantaneous axis is revealed on it by a coloured cardboard disc, placed on the axle. But apparently there is no demand to-day for this top in Dynamical instruction, and it is no longer manufactured for sale.

Kirchoff's Kinetic Analogue of Chapter XXII is useful not merely in its application to the problem of the stability and springing of a shaft, but is of analytical interest in showing a double interpretation of the equations of gyroscopic motion. The Analogue states that if a shaft is sprung into a general helix under the action of a wrench at each end, composed of equal opposing forces and couples, the central line is a curve such that if a point travels along it with a certain constant velocity, the tangent of the line will follow the direction of the axle of a certain top in associated motion. The velocity must be chosen so that a wave of the helix is passed over in the period of the motion of the top. In this way the slight tremor of nutation of a top asleep upright is associated with the stability of a screw shaft between bearings, transmitting a thrust and a couple, and sprung slightly.

Problems on the vibration of a system of bodies, flexible, elastic, or jointed, may be treated in the manner called elementary, without the use of the differential equation, when the associated steady motion is considered of bodily rotation about an axis, performed in the same period as the vibration; and when the displacement is small, the shadow on a wall will show the vibration. In this way we compare the simple and conical pendulum, and discuss the problem of the bell and clapper, a loaded string, a vertical rope, or Kelvin's chain of gyrostats, model of an electromagnetic wave, with circular polarisation, and the treatment is carried out on elementary principles, as here in Chapter XXII. The Hindu rope trick, on these dynamical considerations, is no longer a physical impossibility.

The greatest spinning top we know is the Earth itself, revolving once in twenty-four hours, and precessing at a rate of $50''$ per annum, thus making the year 20 minutes shorter, the pole completing a circuit of the sky in 26,000 years. The problem is described here in Chapter I, and resumed in Chapters X, XI, with full development, in a manner that should satisfy the physical astronomer.

The baffling question of the "Variation of Latitude" is exciting the interest of astronomers. According to Chandler's observations the period of a variation is fourteen months, but in Euler's theory a rigid Earth, with $\frac{A, C}{C - A} = 305, 306$, should complete a wobble in about ten months; the discrepancy is laid to the elastic yielding of the body of the Earth.

This wobble could be increased, and swing got up in the Earth, by well timed salvoes of artillery, such as we have been hearing in the years past. The idea was anticipated by Maxwell, who imagined the effect of a man going to bed when his latitude is a minimum and getting up when at a maximum; the statement will be found given as an example in Routh's *Advanced Rigid Dynamics*.
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UNIVERSITY OF OTAGO, NEW ZEALAND.

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