496. Squaring the Hyperbola
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496. [C. 2. a, j.] *Squaring the Hyperbola.*

Mr. E. M. Langley's elegant demonstration of the integral of $\sec\theta$, in the *Mathematical Gazette*, May, 1914, p. 335, may be extended so as to "square the hyperbola" at the same time.

In the figure, the line BQU cuts the circle at Q and the straight line OA in U at the same angle; and if displaced into the adjacent position Bqu, crossing NQ in n, $Qq=Qn$,

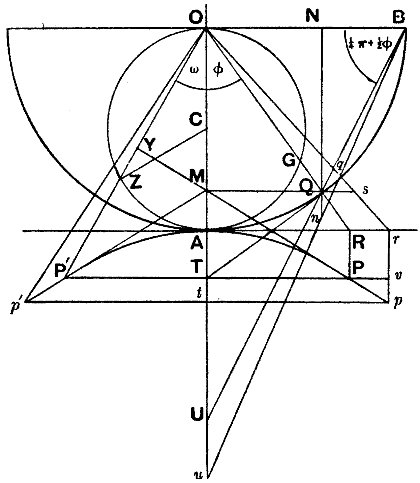
(1) $$\sec\phi\,d\phi=\frac{d\phi}{\cos\phi}=\frac{Qq}{NQ}=\frac{Qn}{NQ}=\frac{Uu}{OU}=\frac{dz}{z},\quad \text{if } OU=z.$$

Then by a definition of the natural logarithm to base e,

(2) $$\int_0 \sec\phi\,d\phi=\int_a \frac{dz}{z}=\log_e\frac{z}{a}=\log\frac{OU}{OA},$$

and $OU=OT+TU=a(\sec\phi+\tan\phi)$, or $OU=OB\tan OBU=a\tan(\frac{1}{4}\pi+\frac{1}{2}\phi)$.

This is Mr. Langley's proof.



In the hyperbola AP, the horizontal ordinate $y=TP=TQ=TU=AR$ to abscissa $OT=x$; and while TQ is the tangent of the circle at Q, the tangent of the hyperbola at P is MP, and at right angles to OP'. Then

(3) $$\frac{\text{hyperbolic sector element } OPp}{\text{circular sector element } OQq}=\frac{OY\,.\,Pp}{OQ\,.\,Qq}=\frac{OM\,.\,Pv}{OM\,.\,Qs}=\frac{Rr}{Qs}=\frac{OA}{OM}=\frac{1}{\cos\phi};$$

(4) $$\text{the sector element } OPp=\tfrac{1}{2}OT\,.\,Qq=\tfrac{1}{2}a^2\sec\phi\,d\phi;$$

and then, by integration, the hyperbolic sector OPP' is a^2 times the integral of $\sec\phi$ in (2), and denoting it by u,

(5) $$u=\int_0 \sec\phi\,d\phi=\log(\sec\phi+\tan\phi)=\log\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)$$
$$=\log\frac{x+y}{a}=\log\frac{a}{x-y},$$

and introducing the notation of the hyperbolic function,

(6) $$\operatorname{sh} u=\tan\phi=\frac{y}{a},\quad \operatorname{ch} u=\sec\phi=\frac{x}{a},\quad e^u=\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi).$$

The line potential at O of AR, or MQ, or any horizontal ordinate from OA to the line OR is given also by u; while the attraction at O by the line AR or the arc AQ is the same.

Brewster quotes from Newton's commonplace book, July 4, 1699: "At such time I found the method of Infinite Series; and in summer 1665, forced from Cambridge by the plague, I computed the area of the Hyperbola at Boothby Lincolnshire to two and fifty figures by the same method."

This refers presumably to a calculation of e, given after to 10 places in the *Principia*, as Squaring the Circle meant the calculation of π; and Newton's attention about the same time must have been first directed to the idea of Gravitation, according to the legend of the apple.

When u is converted from radians to minutes of angle by multiplying by $180\times 60\div\pi=3438$, the result is called the meridional part (M.P.) of ϕ the latitude, so that

$$(7)\quad \text{M.P.}=3438\log_e\tan(45^\circ+\tfrac{1}{2}\text{ latitude})=7916\log_{10}\tan\tfrac{1}{2}(90^\circ+\text{latitude}),$$

and the M.P. is tabulated to every minute of the latitude for use in Navigation and Geography, on the chart of Mercator projection.

The table of M.P. was first calculated by Edward Wright about 1594, and published in his *Errors of Navigation*, 1599. Wright gave here the exact theory of the construction of the Mercator projection, whereas before in Mercator's map of the world, 1569, the degrees of latitude had been laid off empirically.

Later, in 1616, Wright was the translator into English of the Latin of Napier's Canon, and so associated with the early history of the logarithm. Wright's table of the M.P. was obtained by the continued summation of the secant of the angles, proceeding by increments of one minute, as no table of logarithms was then in existence; so that actually he had calculated a table of the logarithm of the tangent, without being aware of it at the time; and the subsequent identification is related in Benjamin Martin's *Mariner's Mirror*, 1772, where they are called nautical logarithms, and discussed by Florian Cajori in the *Napier Tercentenary Memorial Volume*, 1915.

The conformal representation of the (x, y) stereographic projection, with Ox to the pole, and the (u, v) Mercator chart, is discussed in my *Calculus*, Chapter VI, by means of the relation

$$(8)\qquad x+iy=c\operatorname{th}\tfrac{1}{2}(u+iv),\text{ or } c\operatorname{th}\tfrac{1}{2}(u+a+iv)$$

in a linear transformation, v denoting in radians the longitude, and u the M.P. of the latitude.

The transverse axis of the hyperbola has been drawn vertical in the figure, to make it serve in the calculation of the motion of a pendulum, represented by its centre of oscillation G, projected from A along the circle on the diameter AO, with velocity to carry it up to O, and no further.

Then if π/p denotes the beat in seconds of an invisible oscillation at A, and we put $u=pt$ at a subsequent time of t seconds, when the pendulum has advanced through an angle θ, we find

$$(9)\qquad \phi=\tfrac{1}{2}\theta,\quad \tan\tfrac{1}{2}\theta=\operatorname{sh}pt,\quad \sec\tfrac{1}{2}\theta=\operatorname{ch}pt,\quad \frac{d\theta}{dt}=2p\operatorname{sech}pt,$$

so that the motion is expressed by the hyperbolic function and the M.P.

If the half beat of the pendulum in invisible oscillation is taken as the unit of time interval, and the corresponding M.P. in minutes, the angle ϕ or $\tfrac{1}{2}\theta$ is read off in degrees and minutes in the table of M.P.

pt	$\tfrac{1}{2}\pi$	π	$\tfrac{3}{2}\pi$	2π	$\dfrac{\frac{1}{2}\pi}{2}$	$\dfrac{\frac{1}{2}\pi}{10}$
M.P.	5400	10800	16200	21600	2700	540
ϕ	66° 30′	85° 3′	89°	89° 47′	40° 59′	8° 57′
θ	123°	170° 6′	178°	179° 34′	81° 58′	17° 54′

Thus in two beats the pendulum will have reached within half a degree of the highest point O.

The line CZ oscillates about C like a magnet, disturbed from the magnetic meridian OA with velocity just sufficient to carry it away through a right angle into the position of unstable equilibrium, and then

(10) $\tan\omega=\frac{TP}{OT}=\frac{TQ}{OT}=\sin\phi=\operatorname{th} pt,\quad \tan 2\omega=\operatorname{sh} 2pt,\quad \frac{d\omega}{dt}=p\cos 2\omega.$

Sept. 14, 1916. G. GREENHILL.

497. [R. 8. d.] *On the Compound Pendulum.*

The pendulum should be a rod (as uniform as possible) and one sliding piece, carrying a knife-edge, which can be fixed in any position by a screw. The sliding piece is moved down the rod to a position such that the periodic time about the knife-edge which it carries is a minimum. The sliding piece oscillates with the rod.

If m be the mass of the rod,
m_1 the mass of the sliding piece and knife-edge,
h the distance of the C.G. of the *rod* from the knife-edge,
h_1 the distance of the C.G. of the sliding piece from the knife-edge,
k the radius of gyration of the rod about an axis through *its* C.G. parallel to the knife-edge,
k_1 the radius of gyration of the sliding piece about the knife-edge,

then the periodic time $t=2\pi\sqrt{\frac{m(k^2+h^2)+m_1k_1^2}{g(mh+m_1h_1)}}.$

h is the only variable in this formula for t, and t is a minimum when

$$m(k^2+h^2)+m_1k_1^2=(mh+m_1h_1)2h.$$

Hence for the minimum periodic time, $t=2\pi\sqrt{\frac{2h}{g}}.$

This formula is exactly the same as that which is obtained when the knife-edge carrier does not oscillate, but it should be noticed that h is not equal to k in the above. The difficulty of finding the C.G. of the rod can be overcome by inverting the rod and using the same knife-edge carrier to find the minimum periodic time for the inverted rod.

If T and H are the new corresponding values of t and h,

$$T=2\pi\sqrt{\frac{2H}{g}};$$

$$\therefore\ t^2+T^2=\frac{8\pi^2}{g}(h+H).$$

$h+H$, being the distance between the two positions of the knife-edge, can be easily determined.

Thus the difficulty of finding the C.G. is overcome, as in Kater's principle, while the method is much simpler in application.

Also, since the knife-edge carrier itself oscillates, the rod does not need to be specially constructed.

The advantage of the minimum-period method is that g is independent of a small error in h.

For $$t=2\pi\sqrt{\frac{m(k^2+h^2)+m_1k_1^2}{g(mh+m_1h_1)}}=2\pi\sqrt{x},\text{ say};$$

$$\therefore\ g=\frac{4\pi^2.x}{t^2};$$

$$\therefore\ \delta g=\frac{\partial g}{\partial t}\delta t+\frac{\partial g}{\partial h}\delta h=-\frac{8\pi^2}{t^3}x\,\delta t+\frac{4\pi^2}{t^2}\frac{\partial x}{\partial h}\delta h.$$

But $\frac{\partial x}{\partial h}=0$, being the condition that t is a minimum;

$\therefore\ \delta g$ is independent of a small error in h. A. W. LUCY.