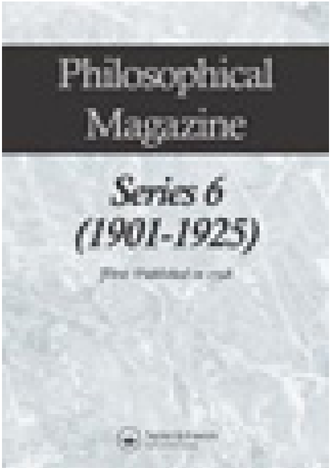

Philosophical Magazine Series 6
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tphm17

LX. On the decrease of velocity of swiftly moving electrified particles in passing through matter

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Published online: 08 Apr 2009.

shown by noting that if the homogeneous equation admitted a solution the boundary relation (23) would read

$$-\lambda^{-}\frac{dU}{-dn}(s^{-})-\lambda^{+}\frac{dU}{dn}(s^{+})=0.$$

Multiplying by U(s) and integrating over Σ we have

$$\lambda^{-}\int_{\Sigma}-U\frac{dU}{-dn}(s^{-})ds+\lambda^{+}\int_{\Sigma}-U\frac{dU}{dn}(s^{+})ds=0.$$

In this equation the integration may be extended over Θ also, seeing that on the conductor side of those surfaces the normal derivative is zero. But the potential U and its derivative are continuous in the regions bounded by Σ and Θ, so that the last equation makes the sum of two essentially positive expressions equal to zero. U(p) must therefore vanish identically and hence $\mu(s)$ also. Thus since the homogeneous equation has no solution except zero (24) admits a unique finite and continuous solution $\mu(s)$. From this the potentials $v(p)$ and $v_1(p)$ are determined, and therefore their sum U(p). The final solution to our problem is then

$$V(p)=U(p)+u(p).$$

December 24, 1914.

LX. *On the Decrease of Velocity of Swiftly Moving Electrified Particles in passing through Matter.* *By* N. Bohr, *Dr. Phil. Copenhagen; p. t. Reader in Mathematical Physics, University of Manchester**.

THE object of the present paper is to continue some calculations on the decrease of velocity of α and β rays published by the writer in a previous paper in this magazine†. This paper was concerned only with the mean value of the rate of decrease of velocity of the swiftly moving particles, but from a closer comparison with the measurements it appears necessary, especially for β rays, to consider the probability distribution of the loss of velocity suffered by the single particles. This problem has been discussed briefly by K. Herzfeld‡, but on assumptions as to the mechanism of decrease of velocity essentially different

* Communicated by Sir Ernest Rutherford, F.R.S.

† Phil. Mag. xxv. p. 10 (1913). (This paper will be referred to as I.)

‡ *Phys. Zeitschr.* 1912, p. 547.

from those used in the following*. Another question which will be considered more fully in the present paper is the effect of the velocity of β rays being comparable with the velocity of light. These calculations are contained in the first three sections. In the two next sections the theory is compared with the measurements. It will be shown that the approximate agreement obtained in the former paper is improved by the closer theoretical discussion, as well as by using the recent more accurate measurements. Section 6 contains some considerations on the ionization produced by α and β rays. A theory for this phenomenon has been given by Sir J. J. Thomson†.

§ 1. *The average value of the rate of decrease of velocity.*

For the sake of clearness it is desirable to give a brief summary of the calculations in the former paper. References to the previous literature on the subject will be found in that paper.

Following Sir Ernest Rutherford, we shall assume that the atom consists of a central nucleus carrying a positive charge and surrounded by a cluster of electrons kept together by the attractive forces from the nucleus. The nucleus is the seat of practically the entire mass of the atom and has dimensions exceedingly small compared with the dimensions of the surrounding cluster of electrons. If an α or β particle passes through a sheet of matter it will penetrate through the atoms, and in colliding with the electrons and the nuclei it will suffer deflexions from its original path and lose part of its original kinetic energy. The deflexions will give rise to the scattering of the rays, and the second effect will produce the decrease in their velocity. The relative parts played by the nuclei and the electrons in these two phenomena are very different. On account of the intense field around the nuclei the main part of the scattering will be due to collisions of the α or β particles with them ; but on account of the great mass of the nuclei the total kinetic energy lost in such collisions will be negligibly small compared with that lost in collisions with the electrons. In calculating the decrease of velocity we shall therefore consider only the effect of the latter collisions.

* *Note added in proof.* I have only now had an opportunity of seeing a recent interesting paper by L. Flamm (*Sitzungsber. d. K. Akad. d. Wiss. Wien, Mat.-nat. Kl.* cxxiii. II *a*, 1914), who has discussed the problem of the probability variation in the ranges of α particles in air on assumptions corresponding with those used in the present paper, and has obtained some of the results deduced in section 2 (see the note on page 599).

† Phil. Mag. xxiii. p. 449 (1912).

Consider a collision between an electrified particle moving with a velocity V and an electron initially at rest. Let M, E, m, and e be the mass and the electric charge of the particle and the electron respectively, and let the length of the perpendicular from the electron to the path of the particle before the collision be p. If the electron is free, the kinetic energy Q given to the electron during the collision can simply be shown to be

$$Q = \frac{2E^2e^2}{mV^2}\frac{1}{p^2+a^2}, \quad \ldots \ldots \quad (1)$$

where

$$a = \frac{eE(M+m)}{MmV^2}. \quad \ldots \ldots \quad (2)$$

Consider next an α or β particle penetrating through a sheet of some substance of thickness Δx, and let the number of atoms in unit volume be N, each atom containing n electrons. The mean value of the number of collisions in which p has a value between p and $p+dp$ is given by

$$dA = 2\pi N n \Delta x p dp. \quad \ldots \ldots \quad (3)$$

If we now could neglect the effect of the interatomic forces on the electrons, the average value of the loss of kinetic energy of the swiftly moving particle in penetrating through the sheet of matter would consequently be

$$\Delta T = \frac{4\pi e^2 E^2 N n \Delta x}{mV^2}\int \frac{pdp}{p^2+a^2}, \quad \ldots \ldots \quad (4)$$

where the integration is to be performed over all the values for p, from $p=0$ to $p=\infty$. The value of this integral, however, is infinite. We therefore see that in order to obtain agreement with experiments it is necessary to take the effect of the interatomic forces into consideration.

Let us assume, as in the electron theory of dispersion, that the electrons normally are kept in positions of stable equilibrium and, if slightly displaced, they will execute vibrations around these positions with a frequency ν characteristic for the different electrons. In estimating the effect of the interatomic forces it is convenient to introduce the conception of the "time of collision," *i. e.* a time interval of the same order of magnitude as that which the α or β particle will take in travelling through a distance of length p. If this time interval is very short compared with the time of vibration of the electron, the interatomic forces will not have time to act before the α or β particle has escaped again from the atom, and the energy transferred to the electron will therefore be very nearly the same as if the electron were free. If, on the other hand, the time of collision is long compared with the time of vibration, the electron will behave almost as if it were rigidly bound, and the energy transferred will be exceedingly small. The effect of the interatomic forces is therefore equivalent to the introduction of an upper limit for p in the integral (4), of the same order of magnitude as V/ν. The rigorous consideration of the general case would involve complicated mathematical calculations, and would hardly be adequate in view of our very scanty knowledge as to the mechanism of the forces which keep the electrons in their positions in the atom. However, it is possible over a considerable range of experimental application to introduce great simplifications and to obtain results which to a high degree of approximation are independent of special assumptions as to the action of the interatomic forces.

The calculation of the total loss of energy suffered by the α or β particle is very much simplified if we assume that, for all collisions in which the interatomic forces have an appreciable influence on the transfer of energy, the displacement of the electron during the collision is small compared with p as well as with the maximum displacement from which it will return to its original position. It can be simply shown that the displacement of the electron during the collision if it were free would be of the same order of magnitude as the above quantity a. The first assumption is therefore equivalent to the condition that V/ν is great compared with a. The second assumption is equivalent to the condition that the value for Q which we obtain by putting $p=V/\nu$ in (1) is small compared with the energy W necessary to remove the electron from the atom. Under these conditions we get by a simple calculation, the detail of which was given in the former paper, that the effective upper limit p_ν for p in the integral (4) is equal to

$$p_\nu = \frac{k}{2\pi}\frac{V}{\nu},$$

where $k=1{\cdot}123$. Introducing this, we get for the integral in (4), performing the integration from $p=0$ to $p=p_\nu$ and neglecting a^2 in comparison with p_ν^2,

$$\log\left(\frac{p_\nu}{a}\right) = \log\left(\frac{kV^3Mm}{2\pi\nu Ee(M+m)}\right).$$

From (4) we now get, noticing that ν has different values $\nu_1 \nu_2 \ldots \nu_n$ for the different electrons in the atom,

$$\Delta \mathrm{T} = \frac{4\pi e^2 \mathrm{E}^2 \mathrm{N} \Delta x}{m \mathrm{V}^2} \sum_1^n \log \left(\frac{k \mathrm{V}^3 \mathrm{M} m}{2\pi \nu \mathrm{E} e (\mathrm{M}+m)} \right)^* . \qquad (5)$$

In the above we have assumed, as in the ordinary theory of dispersion, that the electrons in the atoms normally are at rest. On the theory of the nucleus atom it seems, however, necessary to assume that normally the electrons rotate in closed orbits round the central nucleus. In this case it is a further condition for the validity of the above calculations that the velocity of rotation of the electrons in their orbits is small compared with the velocity of the α or β particle and that the dimensions of the orbits are small compared with V/ν. In a previous paper the writer† has attempted to apply the quantum theory of radiation to the theory of the nucleus atom. It was pointed out that there appears to be strong evidence for the assumption that for every electron in the atom the energy W will be of the same order of magnitude as $h\nu$, where h is Planck's constant. On this assumption it was deduced that in an atom containing n electrons the highest characteristic frequency of an electron will be of the same order of magnitude as

$$\nu = \frac{2\pi^2 e^4 m}{h^3} n^2;$$

the corresponding values for the velocity of rotation, for the diameter of the orbit, and for W will be of the same order of magnitude as

$$\mathrm{V} = \frac{2\pi e^2}{h} n, \quad d = \frac{h^2}{2\pi^2 e^2 m} \frac{1}{n}, \quad \text{and} \quad \mathrm{W} = \frac{2\pi^2 e^4 m}{h^2} n^2$$

respectively. From these expressions it will be seen that the conditions underlying the above calculation will be the better satisfied the smaller the number n of the electrons in the atom. Introducing the numerical values for e, m, and h, it can be shown that all the conditions will be fulfilled, in case of α particles ($\mathrm{V}=2.10^9$, $\mathrm{E}=2e$, $\mathrm{M}=10^4 m$) if $n<10$, and in case of β particles ($\mathrm{V}=2.10^{10}$, $\mathrm{E}=e$, $\mathrm{M}=m$) if $n<100$. Now according to Rutherford's theory the number of electrons in the atom is approximately equal to half the atomic weight in terms of the atomic weight of hydrogen as unity. If, therefore, the main assumptions as to the

* I. p. 19.

† Phil. Mag. xxvi. p. 476 (1913).

mechanism of transfer of energy from the α or β particle to the electrons are correct, we should expect that the formula (5) will hold for absorption of α rays in the lightest elements, and for β rays also for the absorption in the heavier elements. In case of β rays it must, however, be remembered that the formula (1) is deduced under the assumption that V is small compared with the velocity of light. We shall return to this question in Section 3, when we have considered the probability variation in the loss of energy suffered by the single particles.

§ 2. *The probability distribution of the losses of energy suffered by the single α or β particles.*

The questions to be discussed in this section are intimately connected with the probability of the presence of a given number of particles at a given moment in a small limited part of a large space, in which a large number of the particles are distributed at random. This problem has been investigated by M. v. Smoluchowski*, who has shown that the probability for the presence of n particles is given by

$$W(n) = \frac{\omega^n}{n!}\epsilon^{-\omega}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where ϵ is the basis for the natural logarithm and ω is the mean value of the number of particles to be expected in the part of the space under consideration. If ω is very large this probability distribution is to a high degree of approximation represented by the formula

$$W(s)ds = \sqrt{\frac{\omega}{2\pi}}\,\epsilon^{-\frac{1}{2}\omega s^2}ds, \quad . \quad . \quad . \quad . \quad (7)$$

where s is defined by $n = \omega(1+s)$, and $W(s)ds$ denotes the probability that s has a value between s and $s+ds$.

In the paper cited K. Herzfeld uses the formula (7) in calculating the probability distribution of the distance R which an α particle of a given initial velocity will penetrate through a gas before it is stopped. Herzfeld makes the simple assumption that a certain number of collisions with the gas molecules is necessary to stop the particle, and he takes this number A to be equal to the total number of ions formed by the particle in the gas. Now the number of collisions suffered by an α particle in penetrating a given distance through the gas is the same as the number of molecules

* Boltzmann-Festschrift, 1904, p. 626; see also H. Bateman, Phil. Mag xxi. p. 746 (1911).

present in a tubular space round the path of the particle. The probability distribution of the number of collisions can therefore be obtained from the above formulæ, if for ω we introduce the mean value of the number of collisions. Since A is supposed to be very great the variation in the ranges R of the single particle will be very small. The probability that R has a value between $R_0(1+s)$ and $R_0(1+s+ds)$, where R_0 is the mean value of the ranges, will therefore, on Herzfeld's assumption, be simply given by (7) if we put $\omega=A$. On the present theory the calculations cannot be performed quite so simply. The total number of collisions is not supposed to be sharply limited, but it is supposed that the amount of energy lost by the α or β particle in collisions with the electrons will depend on the distance of the electron from the path of the particle, and will decrease continuously for an increase of this distance. In order to apply considerations similar to Herzfeld's, it is therefore necessary to divide the collisions up into groups in such a way that the amount of energy lost by the particles will be very nearly equal for all the collisions inside each group.

Consider an α or β particle penetrating through a thin sheet of some substance of thickness Δx, and let us divide the number of collisions of the particle with the electrons into a number of groups in such a way that the distance p has a value between p_r and p_{r+1} for the collision in the rth group.

Let us now for the present assume that it is possible in this way to divide the collisions into groups so that the number in each group is large at the same time as the difference between any two values for the energy Q lost by a collision in the same group is small. Let the value for Q corresponding to the rth group be Q_r and let the mean value of the number of collisions in this group be A_r, and the actual number of collisions in this group suffered by the given α or β particle be $A_r(1+s_r)$. The total energy lost by the particle in passing through the sheet in question is then given by

$$\Delta T = \Sigma\, Q_r A_r (1+s_r).$$

From this we get, denoting the mean value of ΔT by $\Delta_0 T$,

$$\Delta T - \Delta_0 T = \Sigma\, Q_r A_r s_r.$$

Since the A's are large numbers, we get from (7) for the probability that s_r has a value between s_r and s_r+ds_r,

$$W(s_r)ds_r = \sqrt{\frac{A_r}{2\pi}}\,\epsilon^{-\frac{1}{2}A_r s_r^2}\,ds_r.$$

Now similarly denoting the probability that ΔT has a value between ΔT and $\Delta T+dT$ by $W(T)dT$, we get by help of a fundamental theorem in the theory of probability,

$$W(\Delta T)dT=(2\pi P\Delta x)^{-\frac{1}{2}}\epsilon^{-\frac{(\Delta T-\Delta_0 T)^2}{2P\Delta x}}dT, \quad . \quad . \quad (8)$$

where
$$P\Delta x=\Sigma\frac{1}{A_r}(Q_r A_r)^2=\Sigma A_r Q_r^2.$$

On the above assumptions this can simply be written

$$P\Delta x=\int Q^2 dA.$$

Introducing in this expression the values for Q and dA given by (1) and (3), and integrating for every kind of electron from $p=0$ to $p=p_\nu$ we get

$$P=\frac{4\pi e^4 E^4 N}{m^2 V^4}\sum_1^n\left(\frac{1}{a^2}-\frac{1}{p_\nu^2+a^2}\right).$$

Assuming, as in the former section, that p_ν is large compared with a, we get, neglecting the last term under the Σ and introducing in the first the value of a from (2),

$$P=\frac{4\pi e^2 E^2 M^2}{(M+m)^2}Nn \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

It will be noticed that this expression is very simple. It depends only on the total number of electrons in unit volume, but neither on the velocity of the α or β particle nor on the interatomic forces.

From (8) and (9) we can simply deduce the probability distribution of the thickness of the layers of matter through which particles of given initial velocity will penetrate before they have lost all their energy. Putting $\Delta T=\Delta_0 T(1+s)$, we get for the probability that s has a value between s and $(s+ds)$,

$$W(s)ds=\sqrt{\frac{u}{2\pi}}\,\epsilon^{-\frac{1}{2}us^2}ds, \quad . \quad . \quad . \quad . \quad (10)$$

where
$$u=\frac{(\Delta_0 T)^2}{P\Delta x}=\frac{\phi}{P}\Delta_0 T, \quad . \quad . \quad . \quad . \quad (11)$$

ϕ being the mean value of $\frac{\Delta T}{\Delta x}$.

If we now suppose that the straggling of the rays is small —this assumption is already indirectly involved in the assumptions used in the deduction of (8)—the formula (10) will express also the probability that a particle in order to lose the energy $\Delta_0 T$ will penetrate through a layer of thickness between $\Delta x = \Delta_0 x(1+s)$ and $\Delta x + dx = \Delta_0 x(1+s+ds)$, where $\Delta_0 x = \Delta_0 T/\phi$. In order to find the probability $W(R)dR$, that a particle in order to lose all its energy will penetrate through a layer of thickness between R and $R+dR$, let us now divide the interval from 0 to T in a great number of small steps $\Delta_1 T$, $\Delta_2 T$ ... and let us for the rth step denote the quantities corresponding to Δx, u, ϕ, and s by $\Delta_r x$, u_r, ϕ_r, and s_r. The distance through which a given particle will penetrate is equal to

$$R = \Sigma\, \Delta_r x = \Sigma \frac{\Delta_r T}{\phi_r}(1+s_r).$$

From this we get, denoting the mean value of the ranges of the particles by R_0,

$$R - R_0 = \Sigma \frac{\Delta_r T}{\phi_r} s_r.$$

In exactly the same manner as that used in obtaining (8) we now get

$$W(R)dR = (2\pi U)^{-\frac{1}{2}} \epsilon^{-\frac{(R-R_0)^2}{2U}} dR, \quad . \quad . \quad (12)$$

where

$$U = \Sigma \left(\frac{\Delta_r T}{\phi_r}\right)^2 \frac{1}{u_r} = P\,\Sigma \frac{\Delta_r T}{\phi_r^3};$$

or simply

$$U = P \int_0^T \left(\frac{dT}{dx}\right)^{-3} dT, \quad . \quad . \quad . \quad . \quad (13)$$

where the differential coefficient stands for the mean value of $\frac{\Delta T}{\Delta x}$.

The equations (8) and (9) and consequently also (12) and (13) are deduced under the assumption that the collisions suffered by the swiftly moving particle in penetrating a thin sheet can be divided into groups in such a way that the variation of Q for each group is small, while at the same time the number of collisions in the group is large. The condition for this is that the quantity $\lambda = dA \Big/ \frac{dQ}{Q}$ is large compared with unity. Substituting from (1) and (3) we get

$$\lambda = \pi N n \Delta x (p^2 + a^2). \quad . \quad . \quad . \quad . \quad (14)$$

We see that λ is equal to the average number of electrons inside a cylinder of radius $\sqrt{p^2+a^2}$. Since λ decreases for decreasing p, we shall only have to consider its value for $p=0$. Substituting for a we get

$$\lambda_0 = \frac{\pi e^2 E^2 (M+m)^2 N n \Delta x}{M^2 m^2 V^4}.$$

If we consider a gas at ordinary temperature and pressure and introduce the numerical values for e, m, E, M, and N, we obtain both for α and β rays approximately

$$\lambda_0 = 2\cdot 3 \,.\, 10^{37} \,.\, \frac{n\Delta x}{V^4}.$$

This expression varies very rapidly with V, and gives quite different results for α and for β particles.

For α rays from radium C we have $V=1\cdot 9 \,.\, 10^9$, this gives $\lambda_0 = 1\cdot 7 \,.\, n\Delta x$. Now the range of α rays from radium C in hydrogen and helium is about 30 cm., and according to Rutherford's theory, the number n of electrons in a molecule of these gases is equal to 2. We therefore see that λ_0 will be large compared with unity, provided the sheet of matter be not exceedingly thin compared with the range. For other gases λ_0 will be even greater, since the product of the number of electrons in the molecule and the range of rays is greater than for hydrogen and helium. In case of α rays we may therefore expect that the formulæ deduced above should give a close approximation. In order to get an idea of the order of magnitude of the variation to be expected in the loss of energy suffered by an α particle, consider for instance a beam of α rays penetrating a sheet of hydrogen gas 5 cm. thick. Using the experimental values for the constants, we get from (11) $u=3.10^3$ approximately. Introducing this in (10) we see that the probability variation is very small. Thus about half the particles will suffer a loss of energy which differs less than 1 per cent. from the mean value, and less than 1 per cent. of the particles will suffer a loss which differs more than 5 per cent. In section 4 we shall return to this question and compare the formula (12) with the measurements.

For β rays of velocity about $2 \,.\, 10^{10}$, we get for a sheet of aluminium $0\cdot 01$ gr. per cm.^{2}—a thickness corresponding to that used in the experiments discussed in section 5—$\lambda_0 = 1\cdot 6 \,.\, 10^{-2}$. Since this is very small compared with unity, it is clear that the assumptions used in deducing the formulæ (8) and (12) are in no way satisfied. Still, it appears that it is possible from the calculations to draw some conclusions of importance for the comparison of the theory with the measurements.

Consider a β particle passing through a sheet of matter, and let us for a moment assume that no collision occurs for which λ is smaller than a certain value τ. Let the value for p determined from (14) by putting $\lambda=\tau$ be p_τ. If τ is not small compared with unity the probability distribution of the loss of energy will with considerable approximation be given by (8), if in the expression for P the integral is performed from $p=p_\tau$ instead of from $p=0$. According to the above p_τ will be great compared with a, and we get instead of the expression (9) for P

$$P_\tau = \frac{1}{\tau}\frac{4\pi^2 e^4 E^4 N^2 n^2 \Delta x}{m^2 V^4}. \quad . \quad . \quad . \quad . \quad (15)$$

Introducing this in (11) we find for a sheet of aluminium 0·01 gr. per cm.^{2} for u approximately $u_\tau = 250\tau$. If τ is not small compared with unity, we therefore see that we obtain a probability distribution of the loss of energy which is of the same character as that for α rays. The mean value for the loss of energy for the collisions in question is simply obtained from the formula (5) in the former section by replacing a by p_τ. This gives

$$\Delta_\tau T = \frac{4\pi e^2 E^2 N \Delta x}{m V^2}\sum_1^n \log\left(\frac{p_\nu}{p_\tau}\right). \quad . \quad . \quad . \quad (16)$$

In the applications the logarithmic term in this formula will be large and $\Delta_\tau T$ will depend very little upon the exact value of τ. Thus for an aluminium sheet $\Delta_\tau T$ will vary only 4 per cent., if τ varies from 1 to 2.

Let us now consider the probability distribution of the loss of energy due to the collisions for which p is smaller than p_τ. Since p_τ is large compared with a, it follows from (14) that the average number of these collisions is very nearly equal to τ. If now τ is a small number, *e. g.* $\tau=1$, it is evident that the probability distribution of the loss of energy due to the collisions will be of a type quite different from that considered above. In the first place, there is a certain probability that there will be no loss of energy at all; from (6) we get that this probability is equal to $\epsilon^{-\tau}$. Next, if Q_τ is the value given by (1) if we put $p=p_\tau$, no loss of energy greater than zero and smaller than Q_τ is possible. At Q_τ the probability curve suddenly rises and falls off for increasing values of Q approximately as Q^{-2}. For the aluminium sheet considered above we have approximately $\Delta_\tau T/Q_\tau = 16\tau$.

From these considerations it will appear that the probability distribution of the loss of energy suffered by a β particle of given initial velocity in penetrating through a thin sheet of matter will show a sharp maximum at a value very close to $\Delta_\tau T$, if $\tau=1$, and fall rapidly off on both sides. The value for the decrease of energy measured in the experiments is evidently this maximum, and not the mean value for ΔT given by the formula (5), such as was supposed in my former paper. The considerable difference between the two values is due to a very small number of very violent collisions left out in deducing the formula (16) but included in (5). Putting $\tau=1$ and introducing for p_ν and p_T, we get from (16)

$$\Delta_1 T = \frac{2\pi e^2 E^2 N \Delta x}{m V^2} \sum_1^n \log\left(\frac{k^2 V^2 N n \Delta x}{4\pi\nu^2}\right). \quad . \quad (17)$$

In section 5 we shall consider the question of the loss of energy suffered by a beam of β rays when penetrating through a sheet of matter of greater thickness.

§ 3. *Effect of the velocity of β particles being comparable with the velocity of light.*

The calculations in the former sections are based on the formula (1) for the energy transferred to an electron by a collision with an α or β particle. In the deduction of this formula it is assumed that the velocity V is small compared with the velocity of light c. This condition is not fulfilled in case of high speed β particles. If V is of the same order of magnitude as c, the calculation of the amount of energy transferred by a collision involves complicated considerations for the general case. The problem, however, with which we are concerned is very much simplified by the circumstance considered in the former section, that the value for the loss of energy of β particles, measured in the experiments, will depend only on collisions in which the energy transferred is very small compared with the total energy of the β particle, *i. e.* collisions in which a is small compared with p. Considering such collisions and calculating the force exerted on the electron by the β particle, we can neglect the displacement of the electron during the collision as well as its reaction on the β particle. We need, therefore, only consider the way in which this force is influenced by the velocity of the β particle itself.

In the electron theory it is shown that the electric force, exerted on an electron at rest by a particle of charge E and uniform velocity $V=\beta c$, will be directed along the radius vector from the particle to the electron and given by*

$$F=\frac{eE}{r^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\omega)^{\frac{3}{2}}},$$

where r is the distance apart and ω the angle between the radius vector and the path of the particle. Let the shortest distance from the path to the electron be p, and let $\omega=\frac{\pi}{2}$ at the time $t=0$. We have then $\sin\omega=\frac{p}{r}$ and $r^2=(Vt)^2+p^2$. For the components of the force perpendicular and parallel to the path of the swiftly moving particle we now get

$$F_1=\frac{p}{r}F \quad \text{and} \quad F_2=\frac{Vt}{r}F$$

respectively. Introducing for r, and putting $(1-\beta^2)^{-\frac{1}{2}}=\gamma$, we get

$$F_1=\frac{pe\gamma E}{((\gamma Vt)^2+p^2)^{\frac{3}{2}}} \quad \text{and} \quad F_2=\frac{\gamma VteE}{((\gamma Vt)^2+p^2)^{\frac{3}{2}}}.$$

We see from these expressions that the force at any moment is equal to that calculated on simple electrostatics, if we everywhere replace the velocity V of the swiftly moving particle by γV, and, in calculating the component perpendicular to the path, replace the charge E of the particle by γE, while leaving it unaltered in calculating the component parallel to the path. In the calculation of the correction due to the high speed of the β rays we shall, therefore, have to consider the effects of the two components separately.

If the electron is free it will be simply seen that the velocity of the electron, after a collision in which a is small compared with p, will be very nearly perpendicular to the path of the β particle. In calculating the energy transferred in this case we need therefore consider only the component of the force perpendicular to the path. If V is small compared with c we get from (1), neglecting a compared with p,

$$Q=\frac{2e^2E^2}{mV^2p^2}.$$

If in this expression we introduce γV for V and γE for E,

* See, for instance, O. W. Richardson, 'The Electron Theory of Matter,' p. 249, Cambridge 1914.

we see that it is unaltered. If the electrons were free there would thus be no correction to introduce in the calculation due to the effect of the velocity of the β particle being of the same order as c. If, however, we take the effect of the interatomic forces into account, the problem is a little more complicated. In this case it is necessary to introduce a correction in the expression for p_ν. In addition the effect of the interatomic forces will involve a certain transfer of energy due to the component of the force parallel to the path of the β particle; this is due to a sort of resonance effect which comes into play when the "time of collision" is of the same order of magnitude as the time of vibration of the electrons.

In the former paper it was shown that the contribution to ΔT due to the component parallel to the path is given by*

$$Z = \frac{2\pi e^2 E^2 N n \Delta x}{m V^2}.$$

From (17) it therefore follows that the contribution to ΔT_1, due to the component perpendicular to the path of the β particle, is given by

$$Y = \Delta_1 T - Z = \frac{2\pi e^2 E^2 N \Delta x}{m V^2} \sum_1^n \left(\log\left(\frac{k^2 V^2 N \quad n x}{4\pi \nu^2}\right) - 1\right).$$

If we now in the expression for Y replace V and E by γV and γE, and in the expression for Z replace V by γV but leave E unaltered, we get, by adding the two expressions together and substituting for γ, the following corrected formula for $\Delta_1 T$:

$$\Delta_1 T = \frac{2\pi e^2 E^2 N \Delta x}{m V^2} \sum_1^n \left[\log\left(\frac{k^2 V^2 N n \Delta x}{4\pi \nu^2}\right) - \log\left(1 - \frac{V^2}{c^2}\right) - \frac{V^2}{c^2}\right]. \quad (18)$$

* I. p. 17. The expression deduced in this paper was

$$Z = \frac{4\pi e^2 E^2 N n \Delta x}{m V^2} L,$$

where

$$L = \int_0^\infty \frac{1}{x}\left(f'(x)\right)^2 dx \quad \text{and} \quad f(x) = \int_0^\infty \frac{\cos xz}{(1+z^2)^{\frac{3}{2}}} dz.$$

L formed part of a complicated expression, used in determining p_ν and evaluated by numerical calculation. The value of L, however, can be simply obtained by noticing that

$$f''(x) - \frac{1}{x} f'(x) - f(x) = 0.$$

This gives

$$L = \int_0^\infty f'(x)\left(f''(x) - f(x)\right) dx = \frac{1}{2}\Big|_0^\infty \left(f'(x)\right)^2 - \left(f(x)\right)^2 \Big|.$$

Now $f(0) = 1$ and $f'(0) = f(\infty) = f'(\infty) = 0$; consequently $L = \frac{1}{2}$.

It will be seen that the correction is very small unless V is very near to the velocity of light, since in other cases the two last terms will approximately cancel each other out.

§ 4. *Comparison with measurements on α rays.*

In the former paper it was shown that the formula (5) in section 1 gives values which are in close agreement with the measurements on absorption of α rays for the light elements hydrogen and helium, if we assume that the atoms of these elements contain 1 and 2 electrons respectively, and if for the characteristic frequencies we introduce the frequencies determined by experiments on dispersion. It was also shown that an approximate agreement with the measurements of the absorption in heavier elements could be obtained by assuming that these elements, in addition to a few electrons of optical frequencies, contain a number of electrons more rigidly bound and of frequencies of the same order of magnitude as those determined in experiments on characteristic Röntgen rays; the values deduced for the number of electrons were in approximate agreement with those calculated on Sir E. Rutherford's theory of scattering of α rays. In this section we shall therefore only consider the new evidence obtained by later more accurate measurements.

Since the velocity of α particles is small compared with the velocity of light, we have $T=\frac{1}{2}MV^2$. From (5) we therefore get

$$\frac{dV}{dx}=K_1\frac{n}{V^3}\left(\log V^3-\frac{1}{n}\Sigma\log\nu+K_2\right), \quad . \quad (19)$$

where

$$K_1=\frac{4\pi e^2E^2N}{mM} \quad \text{and} \quad K_2=\log\left(\frac{kMm}{2\pi eE(M+m)}\right).$$

This expression depends on two quantities characteristic for the different substances, *i. e* the number of electrons in the molecule n, and the mean value of the logarithm of the characteristic frequencies of the electrons $\frac{1}{n}\Sigma\log\nu$. The latter quantity determines the characteristic differences in the "velocity curve," *i. e.* the curve connecting corresponding points in a (x, V) diagram. In the former paper formula (19) was compared with values for dV/dx deduced from the measûrements. Since the quantity directly observed is the value of V corresponding to different values of x, it is simpler first to integrate formula (14). This gives

$$x=\frac{V_0^4-V^4}{3nK_1}\frac{1}{z_0-z}\int_z^{z_0}\frac{dz}{\log z}, \quad . \quad . \quad . \quad (20)$$

where

$$\log z=\frac{4}{3}\left(\log V^3-\frac{1}{n}\Sigma\log\nu+K_2\right).$$

A table for the logarithm integral in (20) is given by Glaisher *.

Considering a gas at 15° and 760 mm. pressure we have $Ne=1{\cdot}224\,.\,10^{10}$. Putting $e=4{\cdot}78\,.\,10^{-10}$, $E=2e$, $\frac{e}{m}=5{\cdot}31\,.\,10^{17}$, and $E/M=1{\cdot}448\,.\,10^{14}$, we get $K_1=1{\cdot}131\,.\,10^{34}$ and $K_2=-21{\cdot}80$. In most measurements rays from radium C are used. This corresponds to $V_0=1{\cdot}922\,.\,10^9$ †.

Assuming that the hydrogen atom contains one electron, we get for the hydrogen molecule $n=2$. If we further assume that the characteristic frequency of both electrons in the hydrogen molecule is equal to the frequency determined by experiments on dispersion in hydrogen, we get ‡

$$\nu_1=\nu_2=3{\cdot}52\,.\,10^{15} \quad \text{and} \quad \frac{1}{n}\Sigma\log\nu=35{\cdot}78.$$

Using these values and the above values for V_0, K_1 and K_2, we get $\log z_0=8{\cdot}75$. Introducing this in formula (20) we get for the distance travelled in hydrogen gas by α rays from radium C before their velocity has decreased to half of its original value, $x_1=24{\cdot}0$ cm. The first column of the table below contains values for x/x_1 corresponding to different values for V/V_0. No accurate measurements on the velocity curve in hydrogen have been made. Such measurements would form a very desirable test of the theory since the assumptions underlying the calculations may be expected to be closely fulfilled in case of this gas. T. G. Taylor § has recently determined the range of α rays from radium C in hydrogen. He found 30·9 cm. at 15° and 760 cm. Using the theoretical value $x_1=24{\cdot}0$ cm., we should expect from the table that the range would be close to 27 cm. This is not far from the range observed. At present it seems difficult to decide whether the small deviation may be ascribed to experimental errors in the constants involved.

* Phil. Trans. Roy. Soc. clx. p. 367 (1870).
† E. Rutherford and H. Robinson, Phil. Mag. xxviii. p. 552 (1914).
‡ C. & M. Cuthbertson, Proc. Roy. Soc. A. lxxxviii. p. 166 (1909).
§ Phil. Mag. xxvi. p. 402 (1913).

V/V_0.	I.	II.	III.	IV.	V.
1·0	0	0	0	0	0
0·9	0·338	0·315	0·300	0·318	0·289
0·8	0·592	0·561	0·539	0·560	0·520
0·7	0·780	0·751	0·730	0·750	0·729
0·6	0·911	0·894	0·879	0·889	0·882
0·5	1·000	1·000	1·000	1·000	1·000
0·4	1·055	1·080			
0·3	1·087				
0·2	1·104				

According to Rutherford's theory the helium atom contains two electrons. Since helium is a monatomic gas this gives $n=2$ as for hydrogen. Experiments on dispersion in helium give $\nu=5{\cdot}92\,.\,10^{15}$. Introducing these values for n and ν in (20) we get values for x which are a little greater than those for hydrogen. The theoretical ratio between the ranges in helium and in hydrogen is 1·09. The measurements of E. P. Adams' *, discussed in the former paper, were in disagreement with the calculation that the range in helium was shorter than in hydrogen; the ratio between the ranges observed being only 0·87. Taylor's recent measurements, however, give for this ratio 1·05, in close agreement with the theoretical value.

For air Marsden and Taylor † have recently made an accurate determination of the velocity curve. They found that α rays from radium C will travel through 5·95 cm. of air at 15° and 760 mm. pressure before their velocity is reduced to $\frac{1}{2}V_0$. If we assume that the nitrogen atom contains 7 electrons and the oxygen atom 8 electrons, we get for the air molecule in mean $n=14{\cdot}4$. Introducing this in the formula (20) and putting $x_1=5{\cdot}95$ for $V=\frac{1}{2}V_0$, we find $\log z_0=5{\cdot}37$ and $\frac{1}{n}\Sigma\log\nu=38{\cdot}32$. The values for x/x_1 corresponding to this value for $\log z_0$ are given in the column II. of the table. Not so many values are given as for hydrogen, since the fulfilment of the conditions mentioned in section 1, on account of the higher frequencies, claims greater values for V for air than for hydrogen. Column IV. contains the values for x/x_1 observed by Marsden and Taylor. The agreement between the calculated and the observed values is very close. At the same time it will be seen that the values in column IV. differ considerably from those in

* Physical Review, xxiv. p. 113 (1907).
† Proc. Roy. Soc. A. lxxxviii. p. 443 (1913).

columns I. and III. The values in these columns are calculated by putting $\log z_0 = 8{\cdot}75$ (see above) and $\log z_0 = 4{\cdot}44$ (see below) respectively. If instead of $\log z_0 = 5{\cdot}37$ we had used one of the latter values, we should instead of 14·4 have to put $n = 8{\cdot}1$ or $n = 22{\cdot}5$ respectively, in order to obtain the observed value for x_1. It will therefore be seen that the considerable difference between the values in the columns I., II., and III. offers a method of determining n, even in cases where $\frac{1}{n}\Sigma \log \nu$ is not known beforehand.

Marsden and Taylor could not observe any α particle with a velocity smaller than $0{\cdot}42\,V_0$. When the velocity had decreased to this value the particles apparently disappeared suddenly. This peculiar effect is in striking contrast to what should be expected on the theory. It appears, however, that it may possibly be explained by a statistical effect due to a small want of homogeneity in the α-ray pencils used. In the first part of the velocity curve the slope varies gradually, and a possible small want of homogeneity will have only a very small effect on the mean value of the velocity. But near the end of the range the slope of the curve is very steep, and if the pencil for some reason is not quite homogeneous, the effect will be that, as we recede from the source, more and more of the particles will so to speak suddenly fall out of the beam. In this way the velocity will not start to decrease rapidly until almost all the particles are stopped; but then the beam will contain so few particles that the final descent may be very difficult to detect.

The values in column V. correspond to Marsden and Taylor's results for the velocity curve of rays from radium C in aluminium. The value for x_1 corresponding to $V = \frac{1}{2}V_0$ was $9{\cdot}64 \,.\, 10^{-3}$, measured in gr. per cm.^{2} The value for K_1 in aluminium if x is measured in gr. per cm.^{2} is $9{\cdot}81 \,.\, 10^{36}$. If for aluminium we assume $n = 13$, and in (20) introduce $x_1 = 9{\cdot}64 \,.\, 10^{3}$ for $V = \frac{1}{2}V_0$, we get $\log z_0 = 4{\cdot}44$ and $\frac{1}{n}\Sigma \log \nu = 39{\cdot}02$. As mentioned above this corresponds to the values in column III. It will be seen that the values in column V are much closer to those in III. than to those in I. and II., but the agreement is not nearly so good as for air. This may partly be due to the difficulty in obtaining homogeneous aluminium sheets, but it may also be due to the fact that the assumptions underlying the calculations cannot be expected to be strictly fulfilled for all the electrons in the aluminium atom (see page 586). For elements of higher atomic weight, the assumptions used in the calculations are satisfied to a still smaller degree than for aluminium, and accurate agreement with the measurements cannot be obtained, although the theory offers an approximate explanation of the way in which the stopping power of an element and the shape of the velocity curve vary with increasing atomic weight.

In section 2 we considered the probability variation in the ranges of the single particles of an initially homogeneous beam of α rays. Denoting the mean value of the ranges by R_0, we get from (12) and (13) for the probability that the range R has a value between $R_0(1+s)$ and $R_0(1+s+ds)$

$$W(s)ds = \frac{1}{\rho\sqrt{\pi}}\,\epsilon^{-\left(\frac{s}{\rho}\right)^2} ds, \quad . \quad . \quad . \quad . \quad (21)$$

where

$$\rho^2 = \frac{2U}{R_0^2} = \frac{2P}{R_0^2}\int_0^T \left(\frac{dT}{dx}\right)^{-3} dT. \quad . \quad . \quad . \quad (22)$$

This expression is much simplified if we use an approximate formula for dT/dx. Putting $x = CT^r$, we get

$$\int_0^T \left(\frac{dT}{dx}\right)^{-} dT = \frac{r^3}{3r-2}C^3T^{3r-2} = \frac{r^2}{3r-2}\frac{x^2}{T}\left(\frac{dT}{dx}\right)^{-1}.$$

Introducing this in (22) we get *

$$\frac{1}{\rho^2} = \frac{3r-2}{r^2}\,\frac{T}{2P}\,\frac{dT}{dx}. \quad . \quad . \quad . \quad (23)$$

* *Note added in proof.* For $r=\frac{3}{2}$, this expression is equivalent to the expression deduced by L. Flamm (*loc. cit.* formula (25)) for the variation in the ranges of α particles due to collisions with the electrons. This author has considered also the collisions with the central nuclei and concluded that, although the effect of these collisions on the mean value of the rate of decrease of velocity of the α particles is very small compared with that due to the collisions with the electrons, their effect on the variation in the ranges is not negligible but will be given by an expression of the type (21) for a value of ρ of the same order of magnitude as that given by (23). From considerations analogous with those applied in section 2 in the case of β rays it appears, however, that the collisions between the α particles and the nuclei will produce a variation in the ranges of a type different from (21). In these collisions only very few of the particles suffer a considerable diminution of their ranges, while the greater part of the particles suffer diminutions which are very small even compared with the average differences in the ranges produced by the collisions with the electrons. It seems therefore that the effect of the collisions with the nuclei may be neglected in a comparison with the measurements.

Geiger has shown that we obtain a close approximation to the velocity curve in air if we put $r=\frac{3}{2}$. For hydrogen we obtain a similar approximation by putting $r=\frac{5}{3}$. The exact value for r, however, is of only little importance since $\frac{3r-2}{r^2}$ is very nearly constant for values of r between 1 and 2. Putting $T=\frac{1}{2}MV^2$ and introducing the theoretical expressions (5) and (9) for dT/dx and P, we get

$$\frac{1}{\rho^2}\frac{r^2}{3r-2}=\frac{M}{4m}\frac{1}{n}\sum_1^n \log\left(\frac{kV^3Mm}{2\pi veE(M+m)}\right)=\frac{3}{16}\frac{M}{m}\log z_0.$$

For α rays from radium C we now get, using the same values for $\log z_0$ as above, for hydrogen and air $\rho=0{\cdot}86\,.\,10^{-2}$ and $\rho=1{\cdot}16\,.\,10^{-2}$ respectively. For α rays from polonium, assuming the initial velocity of the rays to be equal to 0·82 that for radium C, we get for hydrogen and air $\rho=0{\cdot}91\,.\,10^{-2}$ and $\rho=1{\cdot}20\,.\,10^{-2}$ respectively.

Geiger* and later Taylor† have made experiments in order to measure the distribution of the ranges in hydrogen and air of α rays from polonium and radium C. They counted the number of scintillations on a zinc-sulphide screen kept at a fixed distance from the radioactive source and varied the pressure of the gas between screen and source. The results do not agree with those to be expected from the theory. The straggling observed was several times larger than that to be expected and did not show the symmetry claimed by the formula (21). These results, if correct, would constitute a serious difficulty for the theory; they seem, however, inconsistent with the results of some recent experiments by F. Friedmann‡. The latter experiments were made in order to test Herzfeld's theory, which also gave a straggling much smaller than that observed by Geiger and Taylor. Friedmann found a distribution of the ranges in air of α rays from polonium which coincides approximately with that given by (21), if $\rho=1{\cdot}0\,.\,10^{-2}$. As seen, this value is even a little smaller than that calculated from the theory. Further experiments on this point would be very desirable.

§ 5. *Comparison with the measurements on β rays.*

The experimental evidence as to the rate of loss of energy by β particles in penetrating through matter has until

* Proc. Roy. Soc. lxxxiii. p. 505 (1910).

† Phil. Mag. xxvi. p. 402 (1913).

‡ *Sitzb. d. K. Akad. d. Wiss. Wien, Mat.-nat. Kl.* cxxii. II*a*, p. 1269 (1913).

recently been very limited on account of the great difficulties in the measurements. Much light, however, is brought upon this question by the study of the homogeneous groups of β rays emitted from certain radioactive substances. O. v. Baeyer* observed that the lines in the "β ray spectrum," produced when the rays are bent in a magnetic field, were shifted to the side of smaller velocities when the radioactive source was covered by a thin metal foil. The question has recently been more closely investigated by Danysz†, who extended the investigation to a great number of the groups of homogeneous rays emitted from radium B and C. The first two columns in the table below headed by Hρ and Δ(Hρ) contain the values given by Danysz for the product of the magnetic force H and the radius of curvature ρ for a number of groups of homogeneous β rays, and the corresponding values for the alteration in this product observed when the rays have passed through an aluminium sheet of 0·01 gr. per cm.^{2} The limit of error in the values for Δ(Hρ) is stated to be about 15 per cent.

Hρ.	Δ(Hρ).	β.	$\beta^3\Delta$(Hρ).
1391	124	0·635	31
1681	95	0·704	33
1748	90	0·718	33
1918	66	0·750	28
1983	61	0·760	27
2047	56	0·770	26
2224	57	0·795	28
2275	48	0·802	25
2939	37	0·867	24
3227	48	0·885	33
4789	39	0·942	32
5830	32	0·960	28

The values for Hρ are connected with the velocity of the β particles through the equation

$$\frac{eV}{c}H = \frac{V^2}{\rho} m\left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}},$$

deduced on the expression for the momentum of an electron which follows from the theory of relativity. Denoting V/c by β, we get

$$H\rho = \frac{c^2 m}{e}\beta(1-\beta^2)^{-\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (24)$$

* *Phys. Zeitschr.* xiii. p. 485 (1912).
† *Journ. de Physique,* iii. p. 949 (1913).

This gives

$$\Delta(H\rho) = \frac{c^2 m}{e}(1-\beta^2)^{-\frac{3}{2}}\Delta\beta.$$

On the theory of relativity we have further

$$T = c^2 m((1-\beta^2)^{-\frac{1}{2}} - 1);$$

from this we get

$$\Delta T = c^2 m\beta(1-\beta^2)^{-\frac{3}{2}}\Delta\beta. \quad . \quad . \quad . \quad . \quad (25)$$

We have consequently

$$\Delta T = e\beta\Delta(H\rho). \quad . \quad . \quad . \quad . \quad . \quad (26)$$

From (18) we thus have, putting $E = e$ and $V/c = \beta$,

$$\Delta(H\rho) = \frac{2\pi e^3 N\Delta x}{mc^2\beta^3}\sum_1^n\left[\log\left(\frac{k^2c^2Nn\Delta x}{4\pi\nu^2}\right) - \log\left(\frac{1-\beta^2}{\beta^2}\right) - \beta^2\right]. \quad . \quad . \quad . \quad . \quad (27)$$

Except for very high velocities the variation of the last factor will be very small, and we shall therefore, according to the theory, expect $\Delta(H\rho)$ to be approximately proportional to β^{-3}. The third column of the table contains the values for β, and the fourth column the values for $\beta^3\Delta(H\rho)$. It will be seen that the values in this column are constant within the limit of experimental errors.

Putting $n=13$ and using the value $\frac{1}{n}\Sigma\log\nu = 39{\cdot}0$ calculated from experiments on α rays, we get from (27) for an aluminium sheet 0·01 gr. per cm.²

β =	0·6	0·7	0·8	0·9	0·95
$\beta^3\Delta(H\rho)$ =	40	41	42	44	46

Considering the great difficulty in the experiments and the great difference in mass and velocity for α and β rays, it appears that the approximate agreement may be considered as satisfactory. The mean values for $\Delta(H\rho)$, calculated from the formula (5) in section 1, would be about 1·3 times larger for the slowest velocities and would increase far more rapidly with the velocity of the β rays.

Measurements of the decrease of velocity of β rays in sheets of metals of higher atomic weight are more difficult than with aluminium on account of the greater effect of the scattering of rays. Danysz found that the rate of decrease of velocity was approximately proportional to the weight per cm.² of the absorbing sheet. Since the number of electrons in any substance is approximately proportional to the weight, and since the differences in the characteristic frequencies will have a very much smaller influence for fast β rays than for α rays, results of this kind should be expected on the theory.

If we assume that the formula (18) holds also for the loss of energy suffered by β rays in penetrating a layer of matter of greater thickness, we obtain for the "range" of the β particles

$$R=\int_0^R \Delta x=\int_0^T \frac{mc^2\beta^2\Delta T}{2\pi e^4 N\Sigma},$$

where Σ denotes the last factor in (18) and (27). Considering Σ as constant, and using the above formula for ΔT, we get

$$R=\frac{m^2c^4}{2\pi e^4 N\Sigma}\int_0^\beta \frac{\beta^3 d\beta}{(1-\beta^2)^{\frac{3}{2}}}=\frac{m^2c^4}{2\pi e^4 N\Sigma}\left[(1-\beta^2)^{\frac{1}{2}}+(1-\beta^2)^{-\frac{1}{2}}-2\right] \quad . \quad . \quad . \quad . \quad (28)$$

Recently R. W. Varder* has made some interesting experiments on the absorption of homogeneous β rays. He measured the variation in the ionization produced by the rays in a shallow ionization chamber when sheets of different thicknesses were introduced in the beam before striking the chamber. Using aluminium sheets, he found that the ionization varied very nearly linearly with the thickness of the sheets, and his diagrams give a strong indication of the existence of a "range" of the β particles. Varder compared the ranges observed with the last factor S in the formula (28), and found that the ratio between the ranges and S, though nearly independent of the initial velocity of the rays, decreased slowly with this velocity. This should be expected from the above calculations, as Σ will increase slowly with the velocity. Measuring R in gr. per cm.^{2} Varder found $R/S=0{\cdot}35$ for $\beta=0{\cdot}8$ and $R/S=0{\cdot}30$ for $\beta=0{\cdot}95$. The first factor in the theoretical formula is equal to 0·42 for $\beta=0{\cdot}8$ and 0·38 for $\beta=0{\cdot}95$. We see that the agreement may be considered as very satisfactory.

The distribution of the losses of energy, suffered by the individual particles of a beam of initially homogeneous β rays in penetrating through a sheet of matter of considerable thickness, cannot be represented by the formula (12) used in the former section, since—see section 2—already the distribution of the loss of energy suffered in penetrating through a thin sheet differs essentially from that given by formula (8).

* Phil. Mag. xxix. p. 725 (1915).

In addition, the transverse scattering of the rays due to deflexions suffered in collisions with the electrons as well as with the positive nuclei must be taken into account. This scattering will cause the mean value of the actual distances travelled by the particles in the matter to be greater than the thickness of the sheet. If, however, we for a moment neglect all collisions in which the particles suffer either abnormally big losses in their energy or big deflexions, we may, as in section 2, expect that the rest of the rays will behave in a similar way to a beam of α rays and that they will show a range of a similar degree of sharpness. Therefore the distribution of the energy of a beam of initially homogeneous β rays emerging from a thick layer of matter must, as for a thin sheet, be expected to exhibit a well-defined peak sharply limited on the side of the greater velocities, but falling more slowly off towards the smaller velocities. The further the rays pass through the matter the greater the chance that the particles will suffer a violent collision, and the smaller will be the number of particles present at the peak of the distribution. A simple calculation shows that by far the greater part of this effect is due to the deflexions suffered in collisions with the positive nuclei. An estimate of the effect of these collisions may be obtained in the following way.

The orbit of a high speed β particle colliding with a positive nucleus has been discussed by C. G. Darwin*. From his calculations it follows that the angle of deflexion ϕ of a β particle of velocity $V = \beta c$ is given by

$$\cot\left(\frac{\pi-\phi}{2}(1-\beta^2\psi^2)^{\frac{1}{2}}\right) = \psi(1-\beta^2\psi^2)^{-\frac{1}{2}},$$

where

$$\psi = \frac{ne^2(1-\beta^2)^{\frac{1}{2}}}{p\beta^2c^2m}.$$

ne is the charge on the nucleus and p is the length of the perpendicular from the nucleus to the path of the β particle before the collision. Let p_τ be the value of the p corresponding to $\psi=\tau$. The probability that a β particle will pass through a sheet of matter of thickness Δx without suffering a collision for which $\psi > \tau$ is equal to $1-\omega\Delta x$, where

$$\omega = \pi p_\tau^2 N = \frac{\pi n^2 e^4 (1-\beta^2) N}{\tau^2 \beta^4 c^4 m^2}.$$

Since $\omega\Delta x$ is small, this probability can be written $\epsilon^{-\omega\Delta x}$,

* Phil. Mag. xxv. p. 201 (1913).

and the probability that the β particle will penetrate through a sheet of greater thickness without suffering a single deflexion for which $\psi > \tau$ is consequently given by $W = e^{-\lambda}$, where $\lambda = \int \omega \Delta x$. Substituting for Δx from the formula (18) and using the same notation as above, we get

$$\lambda = \int \frac{n^2(1-\beta^2)\Delta T}{2\tau^2\beta^2c^2m\Sigma}.$$

Considering Σ as a constant we get from this, using the expression (25) for ΔT,

$$-\lambda = \frac{n^2}{2\tau^2\Sigma}\int \frac{d\beta}{\beta(1-\beta^2)^{\frac{1}{2}}} = \frac{n^2}{4\tau^2\Sigma}\log\left(\frac{1-(1-\beta^2)^{\frac{1}{2}}}{1+(1-\beta^2)^{\frac{1}{2}}}\right) = \frac{n^2}{8\tau^2\Sigma}\log\left(\frac{S}{S+4}\right),$$

where S as above denotes the last factor in the expression (28) for the range R. We have consequently

$$W = K\left(\frac{S}{S+4}\right)^{\frac{n^2}{8\tau^2\Sigma}}, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where S is approximately proportional to the range of the emergent rays, and K a constant.

The formula (29) gives an estimate of the number of β particles left in the peak of the velocity distribution of the emergent rays, and may be compared with the ionization measured in Varder's experiments. It will be seen that W depends to a very high degree on n, and therefore on the atomic weight of the absorbing substance. As mentioned above, Σ is for these fast rays approximately proportional to n, and the exponent in (29) is therefore proportional to n. If aluminium was used as absorbing substance Varder found that the ionization was approximately proportional to the range of the emergent rays, while for paper it decreased more slowly, and far more rapidly for silver and platinum. For aluminium we have $n=13$ and $\frac{1}{n}\Sigma = 18$ for $\beta = 0{\cdot}9$. Putting the exponent in the expression for W equal to 1, we get in this case $\tau = 0{\cdot}30$ and ϕ approximately equal to $30°$; this is an angle of the right order of magnitude. For paper the exponent in (29) will be halved and for platinum will be more than five times larger than for aluminium, for the same values of τ and ϕ.

In connexion with the calculations in this section, it may be of interest to remark that the approximate agreement obtained between the theory and the measurements seems to give strong support to the expressions used for the momentum and the energy of a high speed electron. Let us for a moment suppose that the ordinary expressions for the momentum and the energy of slowly moving electrons could be used without alteration. This should not alter the equations (26) and (27), but the values for V deduced from the values for Hρ would be $(1-\beta^2)^{-\frac{1}{2}}$ times greater. Introducing this in the formula (27) we should have found a value for $\Delta(\mathrm{H}\rho)$ which for the swiftest rays would be about 30 times smaller than that observed by Danysz, and the values in the last column of the table on p. 601 would, instead of being nearly constant, be more than 20 times smaller for the slowest rays than for the swiftest. If, on the other hand, we had supposed that the expressions for the momentum were correct, but that the "longitudinal" mass of the electron was equal to the "transversal" mass, we should have obtained the same values for V as in the table, but the equations (26) and (27) would have been altered by a factor $(1-\beta^2)^{-1}$. In this case the value calculated for $\Delta(\mathrm{H}\rho)$ for the swiftest rays would have been about 15 times larger than that observed, and the values in the last column, instead of being nearly constant as observed, should have been expected to be 10 times greater for the fastest than for the slowest rays. It thus appears that measurements on the decrease of velocity of β rays in passing through matter may afford a very effective means of testing the formula for the momentum and the energy of a high speed electron.

§ 6. *The ionization produced by α and β rays.*

A theory of the ionization produced by α and β rays in a gas has been given by Sir J. J. Thomson*. In this theory it is assumed that the swiftly moving particles penetrate through the atoms of the gas and suffer collisions with the electrons contained in them. The number of pairs of ions produced is supposed to be equal to the number of collisions in which the energy transferred from the particle to the electron is greater than a certain energy W necessary to remove the latter from the atom. If we neglect the interatomic forces this number can be simply deduced. By differentiating (1) with regard to p and introducing for pdp in (3) we get

$$d\mathrm{A}=\frac{2\pi e^2\mathrm{E}^2\mathrm{N}n\Delta x}{m\mathrm{V}^2}\frac{d\mathrm{Q}}{\mathrm{Q}^2}. \quad . \quad . \quad . \quad . \quad (30)$$

* Phil. Mag. xxiii. p. 449 (1912).

Denoting by Q_0 the value for Q obtained by putting $p=0$ in (1), we get, integrating (30) from $Q=W$ to $Q=Q_0$,

$$A_W=\frac{2\pi e^2 E^2 N n \Delta x}{m V^2}\left(\frac{1}{W}-\frac{1}{Q_0}\right), \quad . \quad . \quad . \quad (31)$$

where

$$Q_0=\frac{2mM^2V^2}{(m+M)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

If we consider a substance in which the different electrons correspond to different values for W, we get instead of (31) simply

$$A_W=\frac{2\pi e^2 E^2 N \Delta x}{m V^2}\sum_1^n\left(\frac{1}{W}-\frac{1}{Q_0}\right). \quad . \quad . \quad (33)$$

Sir J. J. Thomson showed that the formula (31) with close approximation can explain the relative number of ions produced by α and β rays. If, however, in (31) we introduce the values for W calculated from the observed ionization potentials, and the values for the number of electrons in the atoms which were found to agree with the calculations in section 4, we obtain absolute values for A_W which are several times smaller than the ionization observed. It appears, however, that this disagreement may be explained by considering the secondary ionization produced by the electrons expelled from the atoms in the direct collisions with the α and β particles. In Sir J. J. Thomson's paper it is argued that this secondary ionization seems to be very small compared with the direct ionization, since the tracks of α and β particles on C. T. R. Wilson's photographs show only very few branches. A calculation, however, indicates that the ranges of the great number of the secondary rays able to ionize are so short that they probably would escape observation. The rays in question will be the electrons expelled with an energy greater than W, and will be due to collisions in which the α or β particle loses an amount of energy greater than 2W. The number of such collisions is given by (31) if W is replaced by 2W. Let this number be A_{2W}. The total energy lost by the particle in the collisions in question is equal to

$$\int_{2W}^{Q_0} Q\,dA=\frac{2\pi e^2 E^2 N n \Delta x}{m V^2}\log\frac{Q_0}{2W}=2W\log\frac{Q_0}{2W}A_{2W},$$

approximately. The mean value of the energy of the electrons expelled is therefore $P=W(2\log(Q_0/2W)-1)$. For hydrogen and α rays from radium C this gives approximately

$P=10W$; corresponding to a velocity of $6 \cdot 10^8$ and a range of about 10^{-4} cm. in hydrogen at ordinary pressure.

The number of ions produced by the secondary rays cannot be calculated in the same simple way as the number produced by the direct collisions with the α or β particles, for in the case of the secondary rays we cannot neglect the effect of the interatomic forces. From the considerations in section 1 it is seen that the conditions for the neglect of the interatomic forces is that the value of p corresponding to $Q=W$ is very small compared with V/ν. By help of the expression (1) for Q and the expressions for W and ν on p. 585, it can be simply shown that this condition is equivalent to the condition that the energy $\frac{1}{2}mV^2$ of the rays is very great compared with W. This condition is fulfilled for α and β rays in light gases, but is not fulfilled for rays as slow as the seoondary rays.

Recently J. Franck and G. Hertz* have made some very interesting experiments which throw much light on the question of ionization by slowly moving electrons. Experimenting with mercury vapour and helium gas, they found that an electron will rebound from the atom without loss of energy if its velocity is less than a certain value. As soon, however, as the velocity is greater than this value the electron will be able to ionize the atom, and it was shown that the probability that ionization will occur at the first impact is considerable. For other gases the results were somewhat different, but in all cases a sharply defined limiting value for the velocity of the ionizing electrons was observed. These experiments indicate that slowly moving electrons are very effective ionizers. We may, therefore, obtain an approximate estimate of the number of ions produced by the secondary rays, if we simply assume that each of these rays will produce s ions if their energy has a value between sW and $(s+1)W$. This would give for the total number of ions formed

$$I=A_W+A_{2W}+\ldots=\frac{2\pi e^2E^2Nn\Delta x}{mV^2}\left[\left(\frac{1}{W}-\frac{1}{Q_0}\right)+\left(\frac{1}{2W}-\frac{1}{Q_0}\right)+\ldots\right].$$

If Q_0 is very great compared with W this gives approximately

$$I=\frac{2\pi e^2E^2Nn\Delta x}{mV^2}\frac{1}{W}\log\frac{Q_0}{W}=A_W\log\frac{Q_0}{W}\ldots \quad . \quad (34)$$

This formula applies only to substances for which W has the same value for all the electrons in the atom. For other substances we must take into account that an electron expelled may produce ions, not only in collisions with electrons

* *Verh. d. Deutsch. Phys. Ges.* xvi. p. 457 (1914).

corresponding to the same value for W, but also in collisions with other electrons in the atom. Considering, however, the rapid decrease in the chance of ionization with increasing W, we may in a simple way obtain an approximate estimate, if we assume that all the ionization produced by the secondary rays is due to collisions with electrons corresponding to the smallest value for W. This value will be the one which is determined in experiments on ionization potentials; let it be denoted by W_r. In the same way as above we now get

$$I = \sum_1^n (A_W + A_{W+W_1} + A_{W+2W_1} + \ldots) = \frac{2\pi e^2 E^2 N \Delta x}{m V^2} \sum_1^n \left[\left(\frac{1}{W} - \frac{1}{Q_0} \right) + \left(\frac{1}{W+W_1} - \frac{1}{Q_0} \right) + \ldots \right].$$

If Q_0 is big compared with all the W's, we get approximately

$$I = \frac{2\pi e^2 E^2 N \Delta x}{m V^2} \frac{1}{W_1} \sum_1^n \log\left(\frac{Q_0}{W}\right). \quad . \quad . \quad . \quad (35)$$

On account of the simplifying assumptions used, the formulæ (34) and (35) can only be expected to indicate an upper limit for the ionization.

The minimum fall of potential P necessary to produce ionization in hydrogen, helium, nitrogen, and oxygen was measured by Franck and Hertz*. They found 11, 20·5, 7·5, and 9 volts respectively. By help of the relation $W = \frac{Pe}{300}$, we get from this W equal to $1{\cdot}75 . 10^{-11}$, $3{\cdot}25 . 10^{-11}$, $1{\cdot}20 . 10^{-11}$, and $1{\cdot}45 . 10^{-11}$ respectively.

The absolute number of ions produced by α rays in air is determined by H. Geiger†. He found that every α particle from radium C in passing through 1 cm. of air at ordinary pressure and temperature produced $2{\cdot}25 . 10^4$ pairs of ions From this we get, using T. S. Taylor's‡ measurements of the relative ionizations in air, hydrogen, and helium, that the number of pair of ions produced by an α particle from radium C in passing through 1 cm. of one of the two latter gases is very nearly the same and equal to $4{\cdot}6 . 10^3$.

If now in (31) we introduce the above value for W for hydrogen and use the same values for N, n, e, E, m, and V as in section 4, we get for α rays from radium C

* *Verh. d. Deutsch. Phys. Ges.* xv. p. 34 (1913).
† Proc. Roy. Soc. A. lxxxii. p. 486 (1909).
‡ Phil. Mag. xxvi. p. 402 (1913).

in hydrogen $A_W = 1{\cdot}15 \, . \, 10^3$. The value given by (34) is $5{\cdot}9 \, A_W$. The first value is 4 times smaller than the ionization observed. The latter value is of the right order of magnitude, but is a little larger than the experimental value.

For helium W is nearly twice as great as for hydrogen. From (31) and (34) we should therefore expect a value for the ionization only half of that in hydrogen. Taylor, however, found the same ionization in hydrogen as in helium. Since in this case the value observed is greater than that calculated from (34), the disagreement is difficult to explain, unless the high value observed by Taylor possibly may be due to the presence of a small amount of impurities in the helium used. This seems to be supported by experiments of W. Kossel* on the ionization produced by cathode rays. This author found that the ionization in helium was only half as large as that in hydrogen—in agreement with the theory. The cathode rays used had a velocity of $1{\cdot}88 \, . \, 10^9$, corresponding to a fall of potential of 1000 volts, and the number of ions produced in passing through 1 cm. of hydrogen at a pressure of 1 mm. Hg was equal to $0{\cdot}882$. This corresponds to 670 pairs of ions at atmospheric pressure. Putting $V = 1{\cdot}88 \, . \, 10^9$ and $E = e$, and using the same values for W, e, m, N, and n as above, we get from (31) $A_W = 300$. From (34) we get $T = 4{\cdot}5 \, A_W$.

If we consider a substance such as air, which contains a greater number of electrons in the atoms, we do not know the value of W for the different electrons. A sufficiently close approximation may, however, be obtained, if in the logarithmic term of (35) we put $W = h\nu$, where h is Planck's constant. This gives, if we at the same time introduce the value for Q_0 from (32),

$$I = \frac{2\pi e^2 E^2 N \Delta x}{m V^2 W_1} \sum_1^n \log\left(\frac{2V^2 m M^2}{h\nu (M+m)^2}\right). \quad . \quad . \quad (36)$$

If now in this formula we introduce the values for n and $\frac{1}{n}\Sigma \log \nu$ used in section 4 in calculating the absorption of α rays in air, and put $W_1 = 1{\cdot}25 \, . \, 10^{-11}$, we get $I = 3{\cdot}6 \, . \, 10^4$. This is the same order of magnitude as the value $2{\cdot}25 \, . \, 10^4$ observed by Geiger, but somewhat larger; this would be expected from the nature of the calculation. The value to be expected from the formula (33) cannot be stated accurately on account of the uncertainty as to the magnitude of the W's,

* *Ann. d. Physik*, xxxvii. p. 393 (1912).

but an estimate indicates that it would be less than a fifth of the observed value. While the formulæ (31) and (33) give values which vary simply as the inverse square of the velocity, the formula (36) gives a variation of I with V which is similar to the variation of ΔT given by the formula (5). Using the same value for $\frac{1}{n}\Sigma \log \nu$ as above, we thus get for α rays in air, that the ratio between the values for I, given by (36), for $V=1{\cdot}8\,.\,10^9$ and for $V=1{\cdot}2\,.\,10^9$ is equal to 1·65. The corresponding ratio for ΔT given by (5) is 1·54. This is in agreement with Geiger's* measurements, which gave that the ionization produced by an α particle in air, at any point of the path, was nearly proportional to the loss of energy suffered by the particle; both quantities being approximately proportional to the inverse first power of the velocity.

The number of ions produced by cathode rays in air has been measured by W. Kossel† and J. L. Glasson‡. For a velocity of $1{\cdot}88\,.\,10^9$ Kossel found 3·28 pairs of ions per cm. at 1 mm. pressure. Under the same conditions Glasson found 2·01 and 0·99 pairs of ions for the velocities $4{\cdot}08\,.\,10^9$ and $6{\cdot}12\,.\,10^9$ respectively. At atmospheric pressure this gives for the same three velocities $2{\cdot}49\,.\,10^3$, $1{\cdot}53\,.\,10^3$, and $0{\cdot}75\,.\,10^3$ pairs of ions respectively; or 9·0, 14·7, and 30·0 times smaller than Geiger's value for α rays from radium C. The values calculated from (36) for cathode rays of the three velocities in question are 7·1, 17·4, and 31·2 times smaller than the value calculated for α rays from radium C.

The calculations in this section cannot be immediately compared with measurements of the ionization produced by high speed β rays, since we have made use of the formula (1) which is valid only if V is small compared with the velocity of light. In a manner analogous to the considerations in section 3, it can, however, be simply shown that the correction to be introduced in the formula (36) is very small and will only affect the logarithmic term. For high speed β rays, the variation of this term with the velocity V will further—as in the calculations in section 5—be very small compared with the variation of the first factor. From (36) we shall therefore expect that the ionization produced by these rays will be approximately proportional to the inverse square of the velocity. This is in agreement with W. Wilson's§ measurements.

* Proc. Roy. Soc. A. lxxxiii. p. 505 (1910).

† *Loc. cit.* ‡ Phil. Mag. xxii. p. 647 (1911).

§ Proc. Roy. Soc. A. lxxxv. p. 240 (1911).

Summary.

According to the theory discussed in this paper, the decrease of velocity of α and β rays in passing through matter depends essentially on the characteristic frequencies of the electron in the atoms, in a similar way as the phenomena of refraction and dispersion.

In a previous paper it was shown that the theory leads to results which are in close agreement with experiments on absorption of α rays in hydrogen and helium, if we assume that the atoms of these elements contain 1 and 2 electrons respectively, and if the frequencies of these electrons are put equal to the frequencies calculated from experiments on dispersion. It was also shown that an approximate explanation of the absorption of α rays in heavier substances can be obtained, if we assume that the atoms of such elements, in addition to a few electrons of optical frequencies, contain a number of electrons more rigidly bound and of frequencies of the same order of magnitude as characteristic Röntgen rays. The number of electrons deduced was in approximate agreement with those calculated in Sir E. Rutherford's theory of scattering of α rays. These conclusions have been verified by using the later more accurate measurements.

In my former paper, very few data were available on the decrease of velocity of β rays in traversing matter and the agreement between theory and experiment was not very close. The agreement between theory and experiment is improved materially, partly by using new measurements and partly by taking the probability variations in the loss of energy suffered by the individual β particles into account. In this connexion it is pointed out that it appears that measurements on the decrease of velocity of β rays afford an effective test of the formulæ for the energy and momentum of a high speed electron deduced on the theory of relativity.

In connexion with the calculations of the absorption of α and β rays, the ionization produced by such rays is considered. It is shown that the theory of Sir J. J. Thomson gives results in approximate agreement with the measurements if the secondary ionization produced by the electron expelled by the direct impact of the α and β rays is taken into account.

I wish to express my best thanks to Sir Ernest Rutherford for the kind interest he has taken in this work.

University of Manchester,
July 1915.