CXII. *The Problem of the Whispering Gallery.*

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THE phenomena of the whispering gallery, of which there is a good and accessible example in St. Paul's cathedral, indicate that sonorous vibrations have a tendency to cling to a concave surface. They may be reproduced upon a moderate scale by the use of sounds of very high pitch (wave-length = 2 cm.), such as are excited by a bird-call, the percipient being a high pressure sensitive flame†. Especially remarkable is the narrowness of the obstacle, held close to the concave surface, which is competent to intercept most of the effect.

The explanation is not difficult to understand in a general way, and in 'Theory of Sound,' § 287, I have given a calculation based upon the methods employed in geometrical optics. I have often wished to illustrate the matter further on distinctively wave principles, but only recently have recognized that most of what I sought lay as it were under my nose. The mathematical solution in question is well known and very simple in form, although the reduction to numbers, in the special circumstances, presents certain difficulties.

Consider the expression in plane polar coordinates (r, θ)

$$\psi_n = J_n(kr)\cos(kat - n\theta), \quad . \quad . \quad . \quad (1)$$

applicable to sound in two dimensions, ψ denoting velocity-potential; or again to the transverse vibrations of a stretched membrane, in which case ψ represents the displacement at any point‡. Here a denotes the velocity of propagation, $k = 2\pi/\lambda$, where λ is the wave-length of straight waves of the given frequency, n is any integer, and J_n is the Bessel's function usually so denoted. The waves travel circumferentially, everything being reproduced when θ and t receive suitable proportional increments. For the present purpose we suppose that there are a large number of waves round the circumference, so that n is great.

As a function of r, ψ is proportional to $J_n(kr)$. When z is great enough, $J_n(z)$, as we know, becomes oscillatory and admits of an infinite number of roots. In the case of the membrane held at the boundary any one of these roots might be taken as the value of kR, where R is the radius of the boundary. But for our purpose we suppose that kR is

* Communicated by the Author.

† Proc. Roy. Inst. Jan. 15, 1904.

‡ 'Theory of Sound,' §§ 201, 339.

the *first* or lowest root (after zero) which we may call z_1. In this case $J_n(z)$ remains throughout of one sign. For the aerial vibrations, in which we are especially interested, the boundary condition, representing that $r=R$ behaves as a fixed wall, is that $J_n'(kR)=0$. We will suppose that k and R are so related that kR is equal to the *first* root (z_1') of this equation. The character of the vibrations as a function of r thus depends upon that of $J_n(z)$, where n is very large and z less than z_1 or z_1'. And we know that in general, n being integral,

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\omega - n\omega)\,d\omega. \quad . \quad . \quad . \quad (2)$$

Moreover, the well known series in ascending powers of z shows that in the neighbourhood of the origin $J_n(z)$ is very small, the lowest power occurring being z^n.

The tendency, when n is moderately high, may be recognized in Meissel's tables*, from which the following is extracted:—

z.	$J_{18}(z)$.	$J_{21}(z)$.	z.	$J_{18}(z)$.	$J_{21}(z)$.
24	−0·0931	+0·2264	16	+0·0668	+0·0079
23	+0·0340	0·2381	15	0·0346	0·0031
22	0·1549	0·2105	14	0·0158	0·0010
21	0·2316	0·1621	13	0·0063	0·0003
20	0·2511	0·1106	12	0·0022	0·0001
19	0·2235	0·0675	11	0·0006	0·0000
18	0·1706	0·0369	10	0·0002	
17	0·1138	0·0180	9	0·0000	

From the second column we see that the first root of $J_{18}(z)=0$ occurs when $z=23·3$. The function is a maximum in the neighbourhood of $z=20$, and sinks to insignificance when z is less than 14, being thus in a physical sense limited to a somewhat narrow range within $z=23·3$.

The above applies to the membrane problem. In the case of aerial waves the third column shows that $J_{21}(z)$ is a maximum when $z=23·3$, so that $J_{21}'(23·3)=0$. This then is the value of kR, or z_1'. It appears that the important part of the range is from 23·3 to about 16.

The course of the function $J_n(z)$ when n and z are both large and nearly equal has recently been discussed by Dr. Nicholson†. Under these circumstances the important part

* Gray and Matthews' Bessel's Functions.

† Phil. Mag. xvi. p. 271 (1908); xviii. p. 6 (1909).

of (2) evidently corresponds to small values of ω. If $z=n$ absolutely we may write ultimately

$$J_n(n)=\frac{1}{\pi}\int_0^{\pi}\cos n(\omega-\sin\omega)d\omega=\frac{1}{\pi}\int_0^{\infty}\cos n(\omega-\sin\omega)d\omega$$
$$=\frac{1}{\pi}\int_0^{\infty}\cos\frac{n\omega^3}{6}\,d\omega=\frac{1}{\pi}\left(\frac{6}{n}\right)^{\frac{1}{3}}\int_0^{\infty}\cos\alpha^3\,d\alpha$$
$$=\Gamma\left(\tfrac{1}{3}\right).\,2^{-\frac{2}{3}}\,3^{-\frac{1}{6}}\,\pi^{-1}\,n^{-\frac{1}{3}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

one of Nicholson's results.

In like manner when $n-z$, though not zero, is relatively small, (1) may be made to depend upon Airy's integral. Thus

$$J_n(z)=\frac{1}{\pi}\int_0^{\infty}\cos\left\{(n-z)\,\omega+\tfrac{1}{6}z\omega^3\right\}d\omega. \quad . \quad . \quad (4)$$

In the second of the papers above cited Nicholson tabulates $z^{\frac{1}{3}}J_n(z)$ against $2\cdot1123\,(n-z)/z^{\frac{1}{3}}$. It thence appears that

$$z_1=n+\frac{2\cdot4955}{2\cdot1123}n^{\frac{1}{3}}=n+1\cdot1814\,n^{\frac{1}{3}}. \quad . \quad . \quad . \quad (5)$$

The maximum (about 0·67) occurs when

$$z=n+\cdot51\,n^{\frac{1}{3}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the function sinks to insignificance (0·01) when

$$z=n-1\cdot5\,n^{\frac{1}{3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus in the membrane problem the practical range is only about $2\cdot7\,n^{\frac{1}{3}}$.

In like manner

$$z_1'=n+\frac{1\cdot0845}{2\cdot1123}n^{\frac{1}{3}}=n+\cdot51342n^{\frac{1}{3}}; \quad . \quad . \quad . \quad (8)$$

so that in the aerial problem the practical range given by (7) and (8) is about $2\cdot1\,n^{\frac{1}{3}}$.

To take an example in the latter case, let $n=1000$, representing approximately the radius of the reflecting circle. The vibrations expressed by (1) are practically limited to an annulus of width 20, or one fiftieth part only of the radius. With greater values of n the concentration in the immediate neighbourhood of the circumference is still further increased.

It will be admitted that this example fully illustrates the observed phenomena, and that the clinging of vibrations to the immediate neighbourhood of a concave reflecting wall may become exceedingly pronounced.

Another example might be taken from the vibrations of air within a *spherical* cavity. In the usual notation for polar coordinates (r, θ, ϕ) we have as a possible velocity-potential $\psi=(kr)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(kr)\sin^n\theta\cos(kat-n\phi)$, and the discussion proceeds as before.

So far as I have seen, the ultimate form of $J_n(z)$ when n is very great and z a moderate multiple of n has not been considered. Though unrelated to the main subject of this note, I may perhaps briefly indicate it.

The form of (2) suggests the application of the method employed by Kelvin in dealing with the problem of water waves due to a limited initial disturbance. Reference may also be made to a recent paper of my own*.

When n and z are great the only important part of the range of integration in (2) is the neighbourhood of the place or places, where $z\sin\omega-n\omega$ is *stationary* with respect to ω. These are to be found where

$$\cos\omega_1 = n/z, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

from which we may infer that when z is decidedly less than n, the total value of the integral is small, as we have already seen to be the case. When $z>n$, ω_1 is real, and according to (9) would admit of an infinite series of values. Only one, however, of these comes into consideration, since the actual range of integration is from 0 to π. We suppose that z is so much greater than n that ω_1 has a sensible value.

The application of Kelvin's method gives at once

$$J_n(z)=\sqrt{\left(\frac{2}{\pi z}\right)}\frac{\cos\{z\sin\omega_1-n\omega_1-\frac{1}{4}\pi\}}{\sqrt{\{\sin\omega_1\}}}. \quad . \quad (10)$$

We may test this by applying it to the familiar case where z is so much greater than n as to make $\omega_1=\frac{1}{2}\pi$. We find

$$J_n(z)=\sqrt{\left(\frac{2}{\pi z}\right)}.\cos\{z-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi\}, \quad . \quad . \quad (11)$$

the well known form.

As an example of (10),

$$J_n(2n)=\sqrt{\left(\frac{2}{n\pi\sqrt{3}}\right)}.\cos\{(\sqrt{3}-\tfrac{1}{3}\pi)\,n-\tfrac{1}{4}\pi\}. \quad (12)$$

Although in (2) n is limited to be integral, it is not difficult to recognize that results such as (3), (5), (12), applicable to large values of n, are free from this restriction.

* Phil. Mag. xviii. p. 1, immediately preceding Nicholson's paper just quoted.