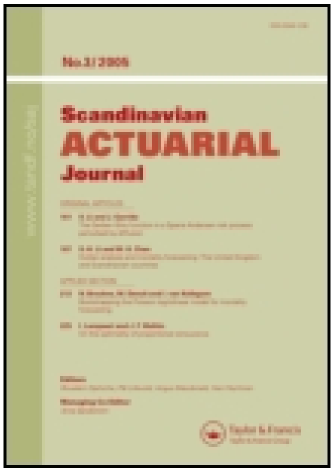

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On numerical integration of differential equations

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On Numerical Integration of Differential Equations.

By **J. F. Steffensen** (Copenhagen).

The value of numerical methods for dealing with differential equations has always been recognised by astronomers; but in recent years these methods have proved ever more useful. As examples we need only remind of the light thrown on the Problem of the three Bodies by the untiring calculations of G. H. Darwin and by the numerical work performed at or in touch with the observatory of Copenhagen. In a different field of research we may mention Störmer's explanation of the Northern Lights.

In the present paper we intend to develop a method of numerical integration on such lines, that no approximate formula is allowed without stating the corresponding *remainder-term* in a form sufficiently simple for taking it into account during the whole of the calculations. The result is a perfectly safe and simple method where we can always indicate a higher limit for the error involved.

We assume that the system of equations under consideration has been reduced to the standard-form

$$\frac{d x_i}{d t} = f_i(x_1, x_2, \dots x_k, t) \qquad (i = 1, 2, \dots k), \tag{1}$$

and that the functions f_i are analytic except for special values of or relations between the x_i and t, so that the existence of the integral is, generally, secured.

If the system is of the first order, there is only one equation, say

$$\frac{dx}{dt} = f(x, t); \tag{2}$$

we shall, for the moment, confine ourselves to this case, as the extension to a greater number of equations presents no difficulty, as we shall see later on.

Introducing the initial conditions, let it be required that $t = t_0$ corresonds to $x = x_0$; in that case we obtain from (2)

$$x = x_0 + \int_{t_0}^{t} f(x, t)\, dt. \tag{3}$$

Now let the interval from t_0 to $t > t_0$ be divided into n equal parts h, so that $t = t_0 + nh$, and let us assume, that the value of x has already been calculated for the following values of t

$$t_0,\ t_0 + h,\ t_0 + 2h, \ldots t_0 + (n - 1)h. \tag{4}$$

Then the problem is to calculate the integral from t_0 to $t_0 + nh$ of a function whose values are known for the values (4) of the argument, and to state the remainder-term in a simple form.

Or, as we may always, by a linear transformation, introduce the limits 0 and 1 in a given integral:

To calculate the integral $\int_0^1 f(x)\, dx$ *and indicate the remainder, the value of* $f(x)$ *being known for*

$$x = 0,\ \frac{1}{n},\ \frac{2}{n}, \ldots \frac{n-1}{n}.$$

This is very nearly the same problem which I have dealt with in my paper »*On the Remainder Form of certain formulas* *of Mechanical Quadrature*»[1], only it was assumed there, that the value of $f(x)$ was also known for $x = 1$. But, this being precisely the missing value on the present occasion, we must retrace part of the analysis on the new assumption. I shall, for brevity, refer to the earlier paper as »I» or »first paper».

We prefer to leave out the value $f(0)$ as well as $f(1)$. Putting, then, in I (1): $n = r - 2$, $a_\nu = \frac{\nu + 1}{r}$, we have

$$P(x) = \left(x - \frac{1}{r}\right)\left(x - \frac{2}{r}\right)\cdots\left(x - \frac{r-1}{r}\right) \qquad (5)$$

and, by I (2),

$$P_\nu(x) = \frac{P(x)}{x - \frac{\nu + 1}{r}}. \qquad (6)$$

Reserving the notation U_ν for the coefficients in the first paper, and writing V_ν for the new coefficients, we obtain by I (8)

$$V_\nu = \int_0^1 \frac{P_\nu(x)}{P_\nu\left(\frac{\nu+1}{r}\right)}\,dx. \qquad (7)$$

It is easy to prove, that $V_{r-\nu-2} = V_\nu$; for, putting in (5) and (6) $1 - x$ for x, we have

$$\left.\begin{aligned} P(1-x) &= (-1)^{r-1} P(x) \\ P_\nu(1-x) &= (-1)^r P_{r-\nu-2}(x) \end{aligned}\right\} \qquad (8)$$

whence, by (7),

$$V_{r-\nu-2} = \int_0^1 \frac{P_{r-\nu-2}(x)}{P_{r-\nu-2}\left(1 - \frac{\nu+1}{r}\right)}\,dx = \int_0^1 \frac{P_\nu(1-x)}{P_\nu\left(\frac{\nu+1}{r}\right)}\,dx$$

[1] Skandinavisk Aktuarietidskrift, 1921, p. 201.

or, if x is replaced by $1 - x$ in the last integral,

$$V_{r-\nu-2} = V_\nu. \tag{9}$$

We now have, by I (10) and I (9),

$$\int_0^1 f(x)\,dx = \sum_{\nu=0}^{r-2} V_\nu f\left(\frac{\nu+1}{r}\right) + R, \tag{10}$$

$$R = \int_0^1 P(x) f\left(x, \frac{1}{r}, \frac{2}{r}, \ldots \frac{r-1}{r}\right) dx. \tag{11}$$

In analogy with the notation in the first paper we put

$$\psi(x) = \int_0^x (t-1)(t-2)\ldots(t-r+1)\,dt, \tag{12}$$

$$Q(x) = \int_0^x P(t)\,dt = \frac{\psi(rx)}{r^r}, \tag{13}$$

$$I_\nu = \int_\nu^{\nu+1} (t-1)(t-2)\ldots(t-r+1)\,dt. \tag{14}$$

If in (14) we put $t = r - \tau$, we find

$$I_\nu = (-1)^{r+1} I_{r-\nu-1}. \tag{15}$$

Assuming now, that r is *even*, or $r = 2m$, it follows from (15), that $\psi(2m) = 0$, consequently $Q(1) = 0$. It may further, following exactly the same line of reasoning as in the first paper, be concluded, for $r = 2m$, that if $\psi(x)$ keeps the same sign for $0 < x < m$, then this function keeps its sign in the whole interval $0 < x < 2m$. But

$$I_{\nu-1} = \int_{\nu-1}^{\nu} (t-1)(t-2)\dots(t-r+1)\,dt$$

$$= \int_{\nu}^{\nu+1} (t-2)(t-3)\dots(t-r)\,dt$$

$$= -\int_{\nu}^{\nu+1} \frac{r-t}{t-1}\cdot(t-1)(t-2)\dots(t-r+1)\,dt;$$

and

$$\frac{r-t}{t-1} > 1 \text{ for } t < \frac{r+1}{2},$$

so that

$$|I_{\nu-1}| > |I_\nu| \text{ for } \nu+1 < \frac{r+1}{2}.$$

It follows, as in the first paper, that $Q(x)$ keeps the same sign for $0 < x < 1$, if r is even.

We now obtain from (11), for $r = 2m$,

$$R = -\int_0^1 f\left(x,\, x,\, \frac{1}{r},\, \frac{2}{r},\, \dots\, \frac{r-1}{r}\right) Q(x)\,dx$$

$$= -\frac{f^{(r)}(\xi)}{r!}\int_0^1 Q(x)\,dx$$

$$= \frac{f^{(r)}(\xi)}{r!}\int_0^1 x\,P(x)\,dx.$$

But

$$\int_0^1 x\,P(x)\,dx = \int_0^1 x\left(x-\frac{1}{r}\right)\cdots\left(x-\frac{r-1}{r}\right)dx$$

$$= \frac{1}{r^{r+1}}\int_0^r t\,(t-1)\ldots(t-r+1)\,dt$$

whence, for $r = 2m$, $t = m + \tau$,

$$(2m)^{2m+1}\int_0^1 xP(x)dx = \int_{-m}^{m}(m+\tau)(m+\tau-1)\ldots(-m+\tau+1)\,d\tau$$

$$= \int_{-m}^{m}(m+\tau)\tau(\tau^2-1)(\tau^2-4)\ldots[\tau^2-(m-1)^2]\,d\tau$$

$$= \int_{-m}^{m}\tau^2(\tau^2-1)\ldots[\tau^2-(m-1)^2]\,d\tau$$

$$= 2\int_0^m \tau^2(\tau^2-1)\ldots[\tau^2-(m-1)^2]\,d\tau,$$

so that finally

$$R = \frac{2f^{(2m)}(\xi)}{(2m)!\,(2m)^{2m+1}}\cdot\int_0^m \tau^2(\tau^2-1)\ldots[\tau^2-(m-1)^2]\,d\tau. \quad (16)$$

As regards the coefficients V_ν we first note that, by (5) and (6)

$$P_r\left(\frac{\nu+1}{r}\right) = (-1)^{r-\nu}.\,\frac{\nu!\,(r-\nu-2)!}{r^{r-2}}. \quad (17)$$

Further, we find, by (6)

$$\int_0^1 P_\nu(x)\,dx = \int_0^1 \left(x-\frac{1}{r}\right)\cdots\left(x-\frac{\nu}{r}\right)\cdot\left(x-\frac{\nu+2}{r}\right)\cdots\left(x-\frac{r-1}{r}\right)dx$$

$$= \frac{1}{r^{r-1}}\int_0^r (t-1)\ldots(t-\nu)\,.\,(t-\nu-2)\ldots(t-r+1)\,dt$$

or, for $r = 2m,\ t = m + \tau$,

$$(2m)^{2m-1}\int_0^1 P_\nu(x)\,dx = \int_{-m}^{m} (\tau + m - 1)\ldots(\tau + m - \nu)\,.$$

$$.\,(\tau + m - \nu - 2)\ldots(\tau - m + 1)\,d\tau$$

$$= \int_{-m}^{m} \tau\,(\tau - m + \nu + 1)\,.\,(\tau^2 - 1)\ldots[\tau^2 - (m-\nu)^2]\,.$$

$$[\tau^2 - (m - \nu - 2)^2]\ldots[\tau^2 - (m-1)^2]\,d\tau$$

$$= 2\int_0^m \tau^2(\tau^2 - 1)\ldots[\tau^2 - (m-\nu)^2]\,.$$

$$.\,[\tau^2 - (m - \nu - 2)^2]\ldots[\tau^2 - (m-1)^2]\,d\tau.$$

We therefore get, by (7),

$$V_\nu = \frac{(-1)^\nu}{m\,.\,\nu!\,(2m - \nu - 2)!}\int_0^m \tau^2(\tau^2 - 1)\ldots[\tau^2 - (m-\nu)^2]\,. \qquad (18)$$

$$.\,[\tau^2 - (m - \nu - 2)^2]\ldots[\tau^2 - (m-1)^2]\,d\tau.$$

The appended table which has been arranged on similar lines as the corresponding table in the first paper shows the coefficients and remainder-terms as calculated by (18) and (16). The three-terms formula, or

$$\int_0^1 f(x)\,dx = \frac{2\,[f(\frac{1}{4}) + f(\frac{3}{4})] - f(\frac{1}{2})}{3} + \frac{7 f^{(4)}(\xi)}{23040} \tag{19}$$

is slightly preferable to SIMPSON's formula where also three terms are used. The five-terms formula may be written

$$\int_0^1 f(x)\,dx = \tfrac{11}{20}\,[f(\tfrac{1}{6}) + f(\tfrac{5}{6})] + 1.3 f(\tfrac{1}{2}) - 0{\cdot}7\,[f(\tfrac{1}{3}) + f(\tfrac{2}{3})] + \frac{1{\cdot}1 f^{(6)}(\xi)}{10^6} \tag{20}$$

where the remainder-term is approximate. This formula is particularly useful on account of the simple coefficients and its great accuracy.

The application to differential equations is best illustrated by means of a numerical example. It would serve no purpose to choose a complicated one, so we prefer a very simple system of two equations which can also be treated by other methods and where the reader can easily check the calculations for himself. The extension to any number of equations is obvious.

Let the equations be

$$\left.\begin{aligned} x' &= x - y + 2t - 1 \\ y' &= 2x - y + 3t + 1 \end{aligned}\right\} \tag{21}$$

and let it be proposed to calculate the integral for which $t = 0$ gives $x = 1$, $y = 0$; that is, x and y must be calculated by the equations

Quadrature Formulas with Remainder Terms.

Number of Terms	r	$f\left(\frac{1}{r}\right)$ $f\left(\frac{r-1}{r}\right)$	$f\left(\frac{2}{r}\right)$ $f\left(\frac{r-2}{r}\right)$	$f\left(\frac{3}{r}\right)$ $f\left(\frac{r-3}{r}\right)$	$f\left(\frac{4}{r}\right)$ $f\left(\frac{r-4}{r}\right)$	$f\left(\frac{5}{r}\right)$ $f\left(\frac{r-5}{r}\right)$	$f\left(\frac{6}{r}\right)$ $f\left(\frac{r-6}{r}\right)$	Common Divisor	Remainder Term
1	2	1						1	$\frac{f^{(2)}(\xi)}{24}$
3	4	2	— 1					3	$\frac{7 f^{(4)}(\xi)}{23040}$
5	6	11	— 14	26				20	$\frac{41 f^{(6)}(\xi)}{39191040}$
7	8	460	— 954	2196	— 2459			945	$\frac{989 f^{(8)}(\xi)}{475634073600}$
9	10	4045	— 11690	33340	— 55070	67822		9072	$\frac{16067 f^{(10)}(\xi)}{5987520000000000}$
11	12	9626	— 35771	123058	— 266298	427956	— 494042	23100	$\frac{1364651 f^{(12)}(\xi)}{562276042568368128000}$.

$$\left.\begin{aligned} x &= 1 + \int_0^t x'\,dt \\ y &= \int_0^t y'\,dt. \end{aligned}\right\} \qquad (22)$$

We assume that the first few values of x and y have already been calculated by any other method; the most generally adopted one is to start the calculation at a point where x and y can be developed in powers of t, this quantity being sufficiently small. In the appended table the values of x and y thus given are the framed ones, corresponding to $t = 0$, 0·1, 0·2, 0·3, 0·4, 0·5.

t	x	x'	$x^{(6)}$	y	y'	$y^{(6)}$
0·0	1·000000	0·000000	—1·00	0·000000	3·000000	0·00
0·1	0·994837	—0·104830	—1·09	0·299667	2·990007	—0·20
0·2	0·978736	—0·218603	—1·18	0·597339	2·960133	—0·40
0·3	0·950856	—0·340184	—1·25	0·891040	2·910672	—0·59
0·4	0·910479	—0·468358	—1·31	1·178837	2·842121	—0·78
0·5	0·857007	—0·601844	—1·36	1·458851	2·755163	—0·96
0·6	0·789978	—0·739306	—1·39	1·729284	2·650672	—1·13
0·7	0·709060	—0·879376	—1·41	1·988436	2·529684	—1·29
0·8	0·614062	—1·020649	—1·41	2·234711	2·393413	—1·43
0·9	0·504937	—1·161718	—1.40	2·466655	2·243219	—1·57
1·0	0·381772		—1.38	2·682941		—1·68

We first calculate, by (21), the values of x' and y' corresponding to these arguments and insert them in their places.

The next question to consider is the choice of quadrature formula. For this purpose we want an estimate of the first few differential coefficients of x and y. These can always be obtained by successive differentiation of a system like (1), inserting after each differentiation the given values of the x'_j in the right-hand side of the new equation. In the present simple special case we find

$$\left.\begin{aligned} x^{(6)} &= -(x+t) \\ y^{(6)} &= -(y-t) \end{aligned}\right\} \qquad (23)$$

from which may be concluded, that the five-terms formula is amply accurate enough for our purpose. The interval chosen for t being 0·1, the range of integration for the five-terms formula will be 0·6. Putting, for abbreviation,

$$f_\nu = f\left(t + \frac{\nu}{10}\right),$$

this formula may for our purpose be written

$$\int_t^{t+0·6} f(t)\,dt = \frac{0·6}{20}\left[11\,(f_1+f_5) + 26 f_3 - 14\,(f_2+f_4)\right] + \frac{41\,(0·6)^7 f^{(6)}(\xi)}{39191040}$$

or, with approximate remainder-term,

$$\int_t^{t+0·6} f(t)\,dt = 0·33\,(f_1+f_5) + 0·78 f_3 - 0·42\,(f_2+f_4) + \frac{3 f^{(6)}(\xi)}{10^8} \qquad (24)$$

from which appears that as long as $x^{(6)}$ and $y^{(6)}$ do not exceed 16 the error caused by the quadrature formula cannot exceed $\frac{1}{2}$ unit of the sixth place of decimals. An examination of the three-terms formula shows, on the other hand, that this would not secure accuracy in the sixth place.

The columns headed $x^{(6)}$ and $y^{(6)}$ have been added in order to follow the variations in the remainder-term during the process of integration. These columns have only been taken to two places of decimals, as all we want is a rough estimate of the remainder.

The only element of uncertainty still left is, that the remainder-term depends on a quantity ξ which may be any value between the limits of integration, and is not confined to the values of t for which $x^{(6)}$ and $y^{(6)}$ are known. But as these values of t have not been chosen with a view to obtaining particularly small values of the corresponding $x^{(6)}$ and $y^{(6)}$, we may consider the columns $x^{(6)}$ and $y^{(6)}$ as containing fair trial samples of the sixth differential coefficient.

It is, however, easy to render the process perfectly rigorous. Let us consider a system of two equations

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x, y, t) \\ \frac{dy}{dt} &= \varphi(x, y, t). \end{aligned}\right\} \tag{25}$$

The nature of the functions f and φ, if for a moment we consider them as functions of three *independent* variables $x, y, t,$ will generally be such, that if these variables assume arbitrary values within a not too large region $|x - x_0| \leq k$, $|y - y_0| \leq k$, $|t - t_0| \leq h$, is constantly $|f| \leq M$ and $|\varphi| \leq M$ where M denotes a constant depending on the constants x_0, y_0, t_0, k and h.

If now we examine the solution of (25) that, for $t = t_0$, reduces to $x = x_0$, $y = y_0$, this solution is evidently defined by

$$\left.\begin{aligned} x - x_0 &= \int_{t_0}^{t} f(x, y, t)\, dt \\ y - y_0 &= \int_{t_0}^{t} \varphi(x, y, t)\, dt. \end{aligned}\right\} \tag{26}$$

It is clear that by choosing $|t - t_0|$ *sufficiently small* we can always obtain that $|x - x_0| \leq k$, $|y - y_0| \leq k$; if further $|t - t_0| \leq h$, we have $|f| \leq M$, $|\varphi| \leq M$, so that according to (26)

$$\left.\begin{aligned} |x - x_0| &\leq |t - t_0| M \\ |y - y_0| &\leq |t - t_0| M. \end{aligned}\right\} \qquad (27)$$

These inequalities have, therefore, been established on the sole assumption, that $|t - t_0|$ is sufficiently small and at any rate $\leq h$. If now we allow $|t - t_0|$ gradually to increase, yet without exceeding h, it is clear from (27), that as long as $|t - t_0| \leq \frac{k}{M}$ we have still $|x - x_0| \leq k$, $|y - y_0| \leq k$ as at the outset.

We have therefore proved, that *if the variation of t does not exceed the smaller of the two quantities h and* $\frac{k}{M}$, *the variation of x and y cannot exceed k.*

This theorem which is easily extended to any number of equations, is usually proved in connection with the existence-theorem for differential equations.

In the numerical example under consideration we have

$$\left.\begin{aligned} f &= x - y + 2t - 1 = (x + t) - (y - t) - 1 \\ \varphi &= 2x - y + 3t + 1 = 2(x + t) - (y - t) + 1. \end{aligned}\right\}$$

If we give x, y and t the increments Δx, Δy and Δt, the increments of f and φ will be respectively $\Delta x - \Delta y + 2\Delta t$ and $2\Delta x - \Delta y + 3\Delta t$, so that

$$\left.\begin{aligned} |f| &\leq |x + t| + |y - t| + 1 + |\Delta x| + |\Delta y| + 2|\Delta t| \\ |\varphi| &\leq 2|x + t| + |y - t| + 1 + 2|\Delta x| + |\Delta y| + 3|\Delta t| \end{aligned}\right\}$$

If $|\Delta x| \leq k$, $|\Delta y| \leq k$, $|\Delta t| \leq h$, we may therefore evidently take

$$M = 2|x_0 + t_0| + |y_0 - t_0| + 1 + 3k + 3h \qquad (28)$$

as a common limit for f and φ, if the point of departure is denoted by x_0, y_0, t_0.

As regards the above table, the interval for t is 0·1, and we therefore put $h = 0{\cdot}1$. If we take $k = 1$, we shall have

$$M = 2\,|x_0 + t_0| + |y_0 - t_0| + 4{\cdot}3 \tag{29}$$

and it is easily verified step by step, inserting in (29) for t_0, x_0, y_0 the successive tabular values, that $0{\cdot}1 < \frac{1}{M}$, that is $h < \frac{k}{M}$, so that x and y cannot vary with more than 1 when t does not vary with more than 0·1. Consequently, $x^{(6)}$ and $y^{(6)}$ cannot, owing to (23), vary with more than 1·1 when t does not vary with more than 0·1. It follows, that the remainder-term in (24) can safely be neglected, as regards the part of the table shown above.

The details of the process of integration are, for the rest, as follows. The values of x' and y' having been calculated as far as $t = 0{\cdot}5$, we calculate by (24)

$$\left.\begin{aligned} x(0{\cdot}6) &= 1 + \int_0^{0{\cdot}6} x'\,dt \\ y(0{\cdot}6) &= \int_0^{0{\cdot}6} y'\,dt \end{aligned}\right\}$$

and thereafter $x'(0{\cdot}6)$ and $y'(0{\cdot}6)$ by (21).

Next, we calculate in the same way

$$\left.\begin{aligned} x(0{\cdot}7) &= x(0{\cdot}1) + \int_{0{\cdot}1}^{0{\cdot}7} x'\,dt \\ y(0{\cdot}7) &= y(0{\cdot}1) + \int_{0{\cdot}1}^{0{\cdot}7} y'\,dt \end{aligned}\right\}$$

and then $x'(0{\cdot}7)$ and $y'(0{\cdot}7)$ by (21).

Generally, we calculate by (24)

$$\left.\begin{aligned} x(t+0{\cdot}6) &= x(t) + \int_t^{t+0{\cdot}6} x'\,dt \\ y(t+0{\cdot}6) &= y(t) + \int_t^{t+0{\cdot}6} y'\,dt \end{aligned}\right\} \quad (30)$$

and thereafter $x'(t+0{\cdot}6)$ and $y'(t+0{\cdot}6)$ by (21). It is recommended to the reader to calculate the values of x and y from $t = 0{\cdot}6$ to $t = 1{\cdot}0$ after which he will be perfectly familiar with the method.

A check on the successive values of x and y may be obtained by applying, at suitable periods, the five-terms formula I (24) which may, for our purpose, be written

$$\int_t^{t+0{\cdot}4} f(t)\,dt = \frac{14\,(f_0+f_4)+64\,(f_1+f_3)+24\,f_2}{450} - \frac{f^{(6)}(\xi)}{10^9} \quad (31)$$

where the remainder-term is approximate.

In the simple special case we are considering the complete integration of the differential system is easy. It will be found, that the particular integral-curve we have been tracing is

$$\left.\begin{aligned} x &= \cos t + \sin t - t \\ y &= 2 \sin t + t. \end{aligned}\right\} \quad (32)$$

It is found by (32) that x and y do not differ more than a unit in the last place of decimals from the values found by the numerical integration, as might be expected.

The most important field of applications of numerical integration is, of course, the cases where the equations (1) cannot be integrated by other methods. But no matter how complicated the given functions f_i on the right hand side of (1) are, the method of numerical integration remains, in principle, the same. Only the choice of interval and the way of obtaining the first few initial values of x and y must be considered specially in each particular case. Also it often occurs, that an interval which was suitable at the beginning of the calculation proves too large at a later stage. In that case we must either resort to one of the quadrature-formulas of higher order or — which is the usual remedy — diminish the interval. In the latter case the required number of initial values of the x_i are found by interpolation in the table already at hand, of course using an interpolation-formula with remainder-term.

Generally speaking, the heaviest part of the work is the calculation of the functions f_i from given values of the x_j and t, and no method of numerical integration can ever be invented by which this trouble is avoided. But facilities may sometimes be introduced in dealing with the remainder-term, as all we want is a suitable *higher limit* for the numerical value of $f^{(2m)}(\xi)$. What is obtainable in this respect depends, however, on the nature of the functions f_i in each particular case.

In order to make the principle of the method as clear as possible, we have confined ourselves to the broad outlines without going too much into such details that claim attention when we have to do with more complicated differential equations. One point, however, may still be mentioned. In the course of a long calculation it is unavoidable, that the small errors arising from working with only a certain number of decimals accumulate slowly, and having started, for instance, with six decimals, we may, perhaps, at the end of the work only be able to rely on four, although we still possess a limit for the error involved. The natural remedy is to start with a couple of decimals more than are required in the end. But this may sometimes be found inconvenient, especially when auxiliary tables are not available to a sufficient number of decimals. In such cases, where strict economy is necessary, we may use the controlling formula (31) step by step, adopting as final values for x and y the values furnished by this formula, these values being generally more accurate than the values first obtained by (24). Supposing, to take an extreme case, that f_0, f_1, f_2 and f_3 are correct to 6 places but f_4 only to 5, then the error in x and y, caused by the erroneous value of f_4, is less than

$$\frac{0{\cdot}5}{10^5} \cdot \frac{14}{450} < \frac{1{\cdot}56}{10^7},$$

so that we obtain correct values for x and y by which f_4 may be adjusted.

Recalculation is, however, not a necessary feature of the method, but only a matter of convenience under certain circumstances.