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THE USES OF ALGEBRA IN STUDY AND READING.¹

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I. THE USES OF ALGEBRA AS SHOWN BY AN INVENTORY OF HIGH SCHOOL TEXTBOOKS.

As a crude but impartial method of estimating which algebraic abilities are useful and how great the need for them is, we have examined from two to four high-school textbooks under each of the titles of Table 1, noting the nature and extent of each demand or opportunity for the use of algebraic processes.

The books examined were as follows:

- Robinson and Breasted: History of Europe, 1914.
- Hazen: Modern European History, 1919.
- Beard and Bagley: History of the American People, 1920.
- Muzzey: American History, Revised Edition, 1921.
- Ely and Wicker: Elementary Principles of Economics (Revised), 1917.
- Marshall and Lyon: Our Economic Organization, 1921.
- Giles: Vocational Civics, 1919.
- Hart: Community Organization, 1920.
- Hughes: Community Civics, 1917.
- Hodgdon: Elementary General Science, 1918.
- Barber, Fuller, Pricer, Adams: First Course in General Science, 1916.
- Snyder: Every Day Science with Projects, 1919.
- Salisbury: Physiography; Briefer Course, 1919.
- Wright: Manual in Physical Geography, 1906.
- Peabody and Hunt: Elementary Biology, 1912.
- Gruenberg: Elementary Biology, 1919.
- Hodge and Dawson: Civic Biology, 1918.
- Hegner: Practical Zoology, 1915.
- Packard: Zoology, 1904.
- Allen and Gilbert: Botany, 1917.
- Bergen and Caldwell: Introduction to Botany, 1914.
- Jackson and Daugherty: Agriculture, 1908.
- Warren: Elements of Agriculture, 1909.
- Conn: Elementary Physiology and Hygiene, 1910.
- Fitz: Physiology and Hygiene, 1908.
- Calkins: First Book in Psychology, 1910.
- Pillsbury: Essentials of Psychology, 1920.
- Kalenberg and Hart: Chemistry, 1913.
- Brownlee, Fuller, Hancock, Sohon, Whitsett: Elementary Principles of Chemistry, 1921.

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McPherson and Henderson: First Course in Chemistry, 1915.
 Henderson and Woodhull: Elements of Physics, 1900.
 Carhart and Shute: Physics, 1920.
 Millikan and Gale: Practical Physics (Pyle's Revision), 1920.
 Matteson and Newlands: Foods and Cookery, 1916.
 Greer: Textbook of Cooking, 1915.
 Greer: School and Home Cooking, 1920.
 Kinne and Cooley: Foods and Household Management, 1914.
 Kinne and Cooley: Shelter & Clothing, 1913.
 Baldt: Clothing for Women, 1916.
 Matthews: Sewing and Textiles, 1921.
 Cooley and Spohr: Household Arts for Home and School, 1920.
 Stilwell: School Print Shop, 1919.
 Matthews: Notes for Mechanical Drawing, 1904.
 Griffith: The Essentials of Woodworking, 1908.

TABLE 1.

Subject	Total Number Linear Inches Examined	Total Number Linear Inches Using Algebra	Per cent Linear Inches Using Algebra	Number Separate Times Algebra Was Used	Average Number Uses of Algebra Per hun- dred In. (4)
<i>Social Sciences</i>					
American History.....	7,070	210	3.0	56	0.8
Community Civics.....	4,010	6	0.1	4	0.1
Economics.....	5,220	146	2.8	38	0.7
European History.....	8,200	48	0.6	11	0.1
Total: Social Sciences.....	24,500	410	1.7	109	0.4
<i>Physical and Biological Sciences</i>					
Agriculture.....	4,690	30	0.6	10	0.2
Biology.....	5,200	154	3.0	44	0.8
Botany.....	3,990	6	0.2	1	0.03
Chemistry.....	7,540	} 161 } 332	} 2.1 } 4.4	} 424 } 6,079	} 5.6 } 80.5 (1)
Total Chemistry.....					
		499	6.5	6,503	86.1
General Science.....	10,440	304	2.9	111	1.1
Physics.....	7,550	} 307 } 4	} 4.0 } .1	} 837 } 132	} 11.0 } 2.8 (2)
Total Physics.....					
		311	4.1	969	12.8
Physiography.....	4,290	513	12.0	127	3.0
Physiology.....	3,730	10	0.3	5	0.1
Psychology.....	4,410	23	0.5	8	0.2
Zoology.....	4,330	6	0.1	2	0.05
Total: Physical and Biological Sciences.....	56,170	1,856 1,520	3.3 2.7	7,780 1,559	13.8 2.8 (3)
<i>Practical Arts</i>					
Cookery.....	8,920	89	1.0	47	0.5
Mechanical Drawing.....	340	8	2.4	141	41.5
Sewing.....	7,680	1	.01	1	0.01
Woodworking.....	1,029	0	0.0	0	0.0
Total: Practical Arts.....	17,960	98	.5	189	1.1
Total for all subjects.....	98,630	2,364 2,028	2.4 2.1	8,078 1,857	8.2 1.9 (3)

The amount of text examined in each subject, the frequency of the uses of algebra, and the total use of algebra are reported in Table 1.

The units of measure—"an inch examined," "an inch of algebra used," and "a use"—are, of course, extremely crude, but entirely impartial.

1. In Chemistry 332 inches, 6,079 separate uses, are chemical formulae and reaction equations, each writing of either being counted.

2. In Physics 4 inches, 132 separate uses, are chemical formula and reaction equations, each writing of either being counted.

3. This line gives the data with the chemical formulae and reaction equations discarded.

4. In the ordinary high school text 100 inches is approximately eighteen pages.

A detailed study was made of the kind of algebra called for under the five following heads:

- (1) Manipulation of complicated polynomials;
- (2) Formation and solution of identities and equations;
- (3) Formation and evaluation of formulae;
- (4) Development and use of the notion of function;
- (5) Construction and interpretation of graphs for
 - (a) statistics
 - (b) functions.

Table No. 2 presents the facts as found.

The following conclusions from a survey of these tables would appear to be justifiable:

(1) Omitting from consideration courses in mathematics itself, there is no need in high school studies for facility in complex manipulation of polynomials.

(2) In present textbooks there is no use made of the mathematical concept of function.

(3) Except in chemistry and physics and agriculture the study of equations has at present no utilization in high school work.

(4) The making of formulae is practically not required of the high school student.

(5) The comprehension and evaluation of formulae is required only in physics, chemistry, and in the physical and chemical parts of general science.

(6) The mathematical graph, either as illustration of the scope of the formula, or as vivifying the concept of function, practically does not occur in any high school work.¹

(7) The statistical graph is used to a greater or less degree in all high school subjects investigated, the number of linear inches given to graphs (in cuts and explanations) being twice as great as for all other types of the utilization of algebra combined.

It is important to note to what extent the algebra of which use is made is necessary to a comprehension of the texts considered.

To read a statistical graph is within the power of the high school student untrained in algebra. Pictograms and cartograms have become part of the common language of newspapers and magazines. But to make them offers such difficulty that

¹Its only utilization in high school physics lies in the reading and construction of graphs illustrating the resolution of forces.

only the exceptionally gifted child might be expected to master it without specific instruction. Still less would one anticipate that a critical attitude concerning the veracity of a graph and ability to detect falseness and graphic misstatement would develop without drill in that precise function.

TABLE 2.

	Manipulation of Symbols		Equations		Formulae		Function		Graphs	
			Identities	Conditions	Formation	Evaluation			Statistical	Mathematical
	Inches	Occurrences	Inches	Occurrences	Inches	Occurrences	Inches	Occurrences	Inches	Occurrences
European History									48	11
American History									210	56
Economics									146	38
Civics									6	4
General Science						13	4		287	107
Physiography									513	127
Biology									144	44
Zoology									6	2
Botany									6	1
Agriculture				6	1				24	9
Physiology									10	5
Psychology									23	8
Chemistry		235	906	124	276	138	132	5281	7	2
Physics				240	570		63	254		413
Cookery				3	5				27	6
Sewing									1	1
Wood Work										
Mechanical Drawing						8	141			

In chemistry the formula is not the equivalent of the formula in algebra. To quote from a recent text (Brownlee, Fuller, Hancock, Sohon, Whitset, "Elementary Principles of Chemistry," pp. 114, 115, 123) "The formula of a molecule is formed by grouping together the symbols of the atoms composing it.

. . . When a molecule contains more than one atom of the same kind the symbol is not usually repeated, but the number of atoms is written as a subscript to the symbol. . . . The only thing that decides the formula of a compound is a chemical analysis. The symbol of an element and the formula of a compound represent more than the name. The symbol of an element stands for one atom of that element . . . and for a definite quantity of the substance" (by weight).

Further, in the chemistry texts reviewed the chemical reaction identities in the two newer books are written with arrows, while in the older one the sign of equality is used. For example, the identity written $\text{NaCl} + \text{H}_2\text{SO}_4 = \text{Na H SO}_4 + \text{HCl}$ in the older books, in the newer is written $\text{NaCl} + \text{H}_2\text{SO}_4 \rightarrow \text{Na H SO}_4 + \text{HCl}$. The student is instructed to read such an equation "NaCl and H_2SO_4 will yield NaH SO_4 and HCl." If the reaction is reversible two arrows are used thus: $2\text{NH}_3 \rightleftharpoons \text{N}_2 + 3\text{H}_2$.

To quote again from the same source (p. 133): "These equations are not like mathematical equations. They are brief statements of experimental facts and should not be written unless it is known that the reactions actually take place. If we know by experiment:

- (1) That the substances do react;
- (2) The composition of each of these substances;
- (3) All the products formed;
- (4) The composition of each product;

we can represent the reaction by an equation, and calculate the relative quantities involved."

The treatment of formulae and equations in algebra may then do as much harm as good to the student's learning of chemistry, especially if "emphasis" is put on the equation of condition rather than on the equation of identity.

It is, therefore, only in the "problems" of the chemistry text that the student would find his elementary algebra of advantage. The equations, which will be made, are, in the large fraction of cases, of the form of a proportion in which three terms are known numerical quantities. A child thoroughly grounded in proportion in arithmetic would be able to make and solve most of these without recourse to algebra, though the student who has had equation drill in algebra might fairly be expected to form and to solve such proportions with increased readiness and smaller percentage of error.

A similar conclusion regarding the equation problems of agriculture is legitimate. The problems deal chiefly with balanced

rations and are similar to the algebra problems of alloy or mixture. A knowledge of the proportion equation suffices for the solution.

In high school physics the significant demand for algebra lies in the reading and evaluation of formulae. Typical formulae are:

$$P_1/P_2 = D_1/D_2; S = 1/2gt^2; I = E/(R_e + R_i/n); V = \sqrt{2as}; \\ E = R/N; (P_1V_1)/(P_2V_2) = T_1/T_2.$$

These formulae are selected from those stated in Millikan and Gale, *Practical Physics*, (Pyle's Revision); 1920; pp. 36, 76, 77, 109, 137, 280.

To read these and such other formulae as the student will probably encounter in high school physics requires knowledge of the signs +, -, = ; the fraction line, the algebraic expression of multiplication, the use of subscripts and exponents,² the radical sign and the parenthesis, in addition to the fundamental notion of the expression of quantity by letters as well as by digits.

The student well prepared in arithmetic probably knows all of these except the expression of multiplication, and the subscript and literal notations, before he studies algebra.

It would be only just, however, to expect that one who had studied algebra would bring to these formulae a richness of content and a clarity and certainty of meaning that should make his comprehension of what he reads in them more vivid.

The evaluation of such formulae from an algebraic point of view means the substitution (in an algebraic expression) of numerical values for the various literal quantities and the performing of the indicated operations. The answer sought will require no more than addition, subtraction, multiplication, division, squaring, and square root, in various combinations. These do not demand mastery of algebra, but it is likely that the student of algebra would be spared the labor and ignominy of certain errors in the manipulation of his quantities.

For instance, in $S = 1/2 gt^2$ the student unacquainted with algebra is in danger of multiplying the values of g and t , dividing by 2, and squaring the result, or of multiplying, squaring, and then dividing. Any teacher made familiar with like errors by their continual recurrence could supply equivalent illustrations. A very small amount of properly learned algebra would free the student from such practices.

Further, the student unacquainted with algebra would find

himself quite at a loss in the derivation of new formulae from those given.

To illustrate, by the algebra student $S = 1/2 gt^2$ is transformed into $g = 2s/t^2$ or $t = \sqrt{2s/g}$ with a small amount of labor, but to the uninitiated each new form is a totally new experience. He can neither make it for himself nor readily understand how it was obtained by the other student. The foundation formula acquired through reasoning or experimentation is thus deprived of the richness of application legitimately to be expected from it.

In high school physics as in chemistry a large number of problems is given in the texts examined. These problems present almost exclusively data to be dealt with by means of formulae found in the body of the text. The student must select the formula appropriate to the problem, make the needed substitutions, and solve the resulting equation.

The complexity of the operations required in such solutions rarely goes beyond the clearing of fractions in a proportion, transposing, combining terms, and extracting a numerical square-root.

The following formulae present from the algebraic point of view the maximum of difficulty:

$1/p + 1/q = 1/f$ with numerical quantities given for p and q , to solve for f ;

$K = (l_2 - l_1)/l_1 (t_2 - t_1)$, with K, l_1, l_2, t_1 , known, to solve for t_2 ;

$I = E/(r + r^{-1})$, I, E, r , being known, to solve for r^{-1} ;

$I = nE/(r + nr^{-1})$, I, E, r, r^{-1} given, to solve for n ;

$f = mv^2/r$, f, m, r , being known, to solve for v ;

$t = \pi\sqrt{l/g}$, t and l being given, to solve for g .

The algebra required in these is certainly much less than that ordinarily taught in the first year course.

To recapitulate: I. As a preparation for high school subjects outside the mathematics courses as they are at present taught, so far as may be judged from textbooks, the most extensive need for algebra is in the reading of statistical graphs, and to a far lesser degree, in the making of them.

II. As a preparation for physics the chief value lies in the mastery of the formula, particularly in handling the transformation known as "changing the subject of the formula." The stu-

²It is useful for the student to understand both positive and negative exponents for the expression of standard numbers, though rarely beyond 6, and the fractional exponent $1/2$, and, rarely, $1/3$ and $1/4$ as an alternative for the use of the radical sign.

³From Carhart & Chute: Practical Physics; 1920, pp. 254, 290, 383, 384, 134, 139.

dent unacquainted with algebra is entirely at a loss in proceeding with this type of work.⁴

III. The solution of physics problems demands the ability to choose the appropriate formula, to make substitutions, and to solve the resulting equations. This involves skill in clearing of fractions (chiefly in proportions), transposing, combining and taking numerical square root.

IV. For the solution of chemistry problems, the ability to form a correct proportion is of high value. The algebra problems which can be applied with advantage in this field are those dealing with proportions, mixtures and alloys.

The next question that presents itself is: Can algebra be utilized by other high school subjects to greater degree than at present with profit to these subjects, and with reinforced vitality in the algebra itself?

Let it be assumed for convenience of classification that possible utilization of algebra may occur under the same five heads already noted:

- (1) Manipulation of Complicated polynomials;
- (2) Formation and solution of equations;
- (3) Formation and evaluation of formulae;
- (4) Development and use of the mathematical concept of function;
- (5) Construction, interpretation and criticism of graphs—both statistical and functional.

In no subject of the high school curriculum outside the mathematics courses is any need for the manipulation of complicated polynomials discovered, nor does any such need seem probable.

By the formation and solution of equations certain science courses might be much enriched. However, the need would seem to be for extreme facility in the handling of easy equations rather than the formation and solution of complicated ones. Especially is this true of the ordinary proportion equation. Charles' Law, Boyle's Law, the laws of the five simple machines (usually given in first-year general science as well as in physics and chemistry) changing recipes in cooking; adjusting patterns in sewing; obtaining lengths for use in mechanical drawing, or in scientific free-hand drawing; plan-making and model-making in wood and metal work and other shop work; all these demand ready use of the proportion.

A pseudo-use of the identity occurs in such dictionary and word-analysis forms as "biology = bios, life+logos, study of."⁵ It is probable that such a form more readily conveys its meaning

⁴Nunn: *The Teaching of Algebra*, 1914, p. 78, also pp. 14, 15.

to the child who has studied algebra than "Biology is derived from two words 'bios' and 'logos' meaning 'life' and 'study of'," while if fifty such derived words are to be taught the advantage of symbolism from the mere consideration of space required for printing is patent. The identity in chemistry already has been exploited. Reaction away from the algebraic identity is the present trend.

This form of work readily passes over into the formation and evaluation of formulae. It seems certain that added usefulness could be found in certain subjects were the child equipped to make a formula, or to use one when given. Differing minds, to grasp and retain with ease, require differing forms. In cookery it might be profitable to the student if he were required to transform rules into equivalent formulae. For instance, the length of time necessary to cook a roast of beef—"a quarter of an hour to the pound and twenty minutes extra"—or the amount of coffee to be used—"a spoonful for each cup and a spoonful for the pot"—are doubtless more easily remembered if presented in formula as well as in rule. Formulae for balanced rations, for infant feeding, for the use of leavening agents, might profitably occur. A move in this direction is the present custom of writing recipes by presenting first a table of the ingredients and their amounts, then a paragraph describing the method of combining them. The advantage here is the advantage of all algebraic symbolism, such compactness as to make possible a complete survey of the "elements of the problem" with a minimum expenditure of time and thought.

In sewing, formulae on the allowance of goods for the making of ruffles, for accordion and other kinds of plaiting, length of the bias strip for goods of given width, are devices for storing needed facts in the memory in an economical form. The allowance of embroidery threads to cover certain spaces are usually made nowadays by the experienced intuition of the worker or the sales woman, with consequent over or under supply. Perhaps in time formulae for them will be developed that will prove of advantage.

In physiography it would be of interest and profit to present such formulae as $d = 1.22\sqrt{h}$, where h = the height of the observer above sea level and d the greatest distance at which an object on the sea is visible to him.

The utilization of the statistical or pictorial graph has already

^aCertain texts use this form "biology, bios = life + logos = study of," a form distinctly reprehensible from the algebra teacher's viewpoint.

been general in all high school subjects. What is needed is the development of the critical faculty in reading and in evaluating such graphs. Especially is this true of the pictograms which utilize natural three-dimension forms. In determining lengths for proportional facts due regard should be paid to formulae for surface, volume and perspective. The student needs to be taught to avoid the false inferences which are natural when a pictogram naturally interpreted in units of volume is presented in linear units.

Many subjects would find it advantageous if they could utilize the mathematical graph, that is, the graphic presentation of a law or a function. It is the aim of science to formulate its conclusions not as mere statistics but as mathematical laws. But at present expressing such laws for the student in a mathematical form is rarely done. Whether this is because the student has not been qualified by his study of algebra to grasp such statement, so that the vernacular must be used instead, or because the writers of the texts are unable to put their conclusions in such form, is not pertinent to the present discussion.

Practically the only use of this highly refined tool of exposition discovered in our textbook survey is to be found in Ely & Wicker's *Elementary Principles of Economics* (revised edition, 1917, pp. 183-201).

Certainly, many of the laws of economic dependency, certain basic notions of continuity and sequence and causation in history; facts of growth versus soil conditions and the composition of soils in agriculture; laws of growth increases in botany; interdependence of animal and plant life in biology; time and rate in interest, and cost-price relations in commercial work; all these might be made clearer to the student by correct and appropriate use of the mathematical graph.

The advantage of such graphic representation to supplement data by interpolation and extrapolation probably needs no emphasis with high-school students, the rather a caution against reading unfounded conclusions into graphs is needed. The fallacy of coordinates, the dangers inherent in optical delusions, the misunderstanding occasioned by the omission of zero reference points, should be pointed out. Also, warning is needed against over-use. It would be easy in such work to go beyond the point of diminishing returns for high school students who are little equipped for scientific generalization, but a small

amount of such work might be made a source of illumination and progress.

For most high school students any larger use of the function concept than that suggested in the mathematical graph is of doubtful value. It is one of those fundamentally powerful conceptions whose elaboration has been one of the half dozen significant achievements of the race, but to the high school student it is vague and tantalizing and stimulating rather than clarifying. To sum up:

(1) Involved manipulation of polynomial expressions is not a justifiable way of using the high school student's time.⁶

(2) Since the application of equations in other high school subjects is chiefly in the proportion form, mastery of that form and other easy equation forms should be secured.

(3) It would be profitable to extend the field of application of the construction of formulæ as well as their evaluation.

(4) There is need for the careful development of the art of criticism as applied to graphs.

(5) The presentation of laws by means of mathematical graphs should be encouraged.

(6) The function concept should be used when advantageous, but with economy.

⁶This statement is meant to cover addition and subtraction of long expressions; multiplication and division of polynomials; manipulation of fractions with polynomial terms; squares and cube root of polynomials resulting in answers of more than two terms; involved factorization of polynomials; reduction of radicals (or fractional exponents) whose index is greater than three; operations with polynomials containing radicals; the factor and remainder theorems; and the binomial theorem, when the exponent is larger than five.

(To be continued)

MANGANESE IN 1921.

The domestic shipments of high-grade manganese ore—containing 35 per cent or more metallic manganese—amounted to about 13,000 gross tons in 1921, of which more than 10,000 tons were shipped from Montana, according to H. A. C. Jenison, of the United States Geological Survey, Department of the Interior. The shipments of ore containing 10 to 35 per cent of manganese amounted to about 72,000 tons, most of which was shipped from Minnesota. The shipments of mangiferous and ferruginous manganese ore amounted to about 14,000 tons.

The net imports for the first eleven months of the year amounted to 386,405 tons of high-grade ore and oxide, valued at \$3,288,595. Of this Brazil contributed 247,568 tons, valued at \$7.58 per ton, and India 113,730 tons, valued at \$6.46 per ton.

The most important event that may affect the future of the domestic industry was a favorable report by the House of Representatives on a proposed tariff on imports of manganese ore of 1 cent per pound of metallic manganese content of ore or of concentrates containing more than 30 per cent of metallic manganese. The measure has not been reported on by the Senate committee.