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L. The two-dimensional steady motion of a viscous fluid

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so that the equation for ψ becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial\alpha^2}+\frac{\partial^2\psi}{\partial\beta^2}\right)+\frac{\partial(\psi,\nabla^2\psi)}{\partial(\alpha,\beta)}=\nu\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)\nabla^2\psi,$$

or

$$\frac{\partial(\psi,\nabla^2\psi)}{\partial(\alpha,\beta)}=\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)\left(\nu\nabla^2\psi-\frac{\partial\psi}{\partial t}\right).$$

L. *The Two-Dimensional Steady Motion of a Viscous Fluid.* By G. B. JEFFERY, *M.A., B.Sc., Assistant in the Department of Applied Mathematics, University College, London**.

THE object of this paper is to search for some *exact* solutions of the equations of motion of a viscous fluid. Much has been accomplished by assuming that the motion is slow, and that the squares and products of the velocity components may therefore be neglected. It has indeed been held that this is the only useful proceeding, since the equations of motion are themselves formed on the assumption of a linear stress-strain relation, and this is probably only justifiable if the motion is sufficiently slow. On the other hand, there is very little evidence of the breakdown of the linear law in the case of fluids, and in any case it is only possible to test its validity by an investigation of solutions which do not require the motion to be slow. It is, therefore, of some importance to obtain some solutions which are free from this limitation. In the present paper we confine our attention to plane motion. Orthogonal curvilinear coordinates are employed, and we discuss the possibility of so choosing them that either the stream-lines or the lines of constant vorticity are identical with one family of the coordinate curves. The most important solutions obtained are those which correspond to (1) the motion round a canal in the form of a circular arc, (2) the motion between rotating circular cylinders with a given normal flow over the surfaces, as in a centrifugal pump, (3) the flow between two infinite planes inclined at any angle.

If u, v be the components of velocity, p the mean pressure, V the potential of the external forces, ν the kinematic viscosity, and ρ the density of the fluid, the equations of

* Communicated by Prof. Karl Pearson, F.R.S.

motion in two dimensions are

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{\partial V}{\partial x}+\nu\nabla^2 u\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-\frac{\partial V}{\partial y}+\nu\nabla^2 v\end{aligned}\right\} \quad . \quad . \quad (1)$$

Eliminating the pressure from these equations we have

$$\frac{\partial}{\partial t}(\nabla^2\psi)+\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)}=\nu\nabla^4\psi,$$

where

$$u=-\frac{\partial\psi}{\partial y},\quad v=\frac{\partial\psi}{\partial x},$$

ψ being Earnshaw's current function.

Take a system of orthogonal curvilinear coordinates defined by conjugate functions α, β of x, y. The equation for ψ may then be written

$$\frac{\partial(\psi,\nabla^2\psi)}{\partial(\alpha,\beta)}=\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)\left(\nu\nabla^2\psi-\frac{\partial\psi}{\partial t}\right),$$

or if the motion is steady

$$\frac{\partial(\psi,\nabla^2\psi)}{\partial(\alpha,\beta)}=\nu\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)\nabla^2\psi. \quad . \quad . \quad . \quad . \quad (2)$$

§ 1. *Solutions for which the lines of constant vorticity are a possible set of equipotential lines.*

The coordinates can be so chosen that the curves $\alpha=const.$ are identical with any given set of equipotential lines in free space. Hence the characteristic property of this type of solution is that it is possible to choose the system of coordinates α, β so that $\nabla^2\psi$ is a function of α only, say

$$\nabla^2\psi=f(\alpha).$$

Substitute in (2) and we have

$$-\frac{\partial\psi}{\partial\beta}\cdot f'(\alpha)=\nu f''(\alpha),$$

or

$$\frac{\partial\psi}{\partial\beta}=-\nu\frac{d}{d\alpha}(\log f'(\alpha)).$$

Integrating with respect to β

$$\psi=-\nu\beta\frac{d}{d\alpha}(\log f'(\alpha))+F(\alpha).$$

Hence

$$\nabla^2\psi=\left\{\left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\beta}{\partial x}\right)^2\right\}\left\{-\nu\beta\frac{d^3}{d\alpha^3}(\log f'(\alpha))+\mathrm{F}''(\alpha)\right\}=f(\alpha).$$

Using a well-known property of conjugate functions this may be written

$$\left|\frac{d(x+iy)}{d(\alpha+i\beta)}\right|^2=-\nu\beta\frac{1}{f(\alpha)}\frac{d^3}{d\alpha^3}(\log f'(\alpha))+\frac{\mathrm{F}''(\alpha)}{f(\alpha)},$$

so that we have to determine a function of $\alpha+i\beta$ such that the square of its modulus is linear in β. Mr. G. N. Watson, to whom I submitted this problem, has supplied me with the complete solution. If

$$|\phi(\alpha+i\beta)|^2=\mathrm{A}\beta+\mathrm{B},$$

where A, B are real functions of α, then

$$\phi(\alpha+i\beta)=\kappa\, e^{\lambda(\alpha+i\beta)},$$

where λ is real but κ may be complex. In this case

$$\mathrm{A}=0,\qquad \mathrm{B}=|\kappa|^2 e^{2\lambda\alpha}.$$

Applying this result to the problem in hand

$$\frac{d(x+iy)}{d(\alpha+i\beta)}=\kappa\, e^{\lambda(\alpha+i\beta)},$$

$$\frac{d^3}{d\alpha^3}(\log f'(\alpha))=0,\qquad \mathrm{F}''(\alpha)=|\kappa|^2 e^{2\lambda\alpha} f(\alpha).$$

Hence

$$f(\alpha)=\int e^{a\alpha^2+b\alpha+c}\,d\alpha,$$

$$\mathrm{F}(\alpha)=|\kappa|^2\iint(d\alpha)^2 e^{2\lambda\alpha}\int e^{a\alpha^2+b\alpha+c}\,d\alpha.$$

The constants κ,λ merely determine the scales of measurement in the different systems of coordinates, and we have only two distinct solutions (1) $\kappa=1,\ \lambda=0$, (2) $\kappa=1,\ \lambda=1$.

The first case gives

$$\alpha+i\beta=x+iy,$$

and we have a solution in Cartesian coordinates

$$\psi=-\nu y(2ax+b)+\iiint e^{ax^2+bx+c}\,(dx)^3.$$

If $a=0$ this gives

$$\psi = -bvy + \mathrm{A}\, e^{bx} + \mathrm{B}x^2 + \mathrm{C}x, \quad . \quad . \quad . \quad \text{(I.)}$$

while if a is not zero, a shift of origin gives

$$\psi = -2vaxy + \mathrm{A}\iiint e^{ax^2}\,(dx)^3. \quad . \quad . \quad \text{(II.)}$$

(I.) and (II.) are the only distinct solutions for which the lines of constant vorticity are a set of parallel straight lines. In the second case, when $\kappa=1$ and $\lambda=1$

$$\alpha + i\beta = \log(x+iy).$$

Hence, if r, θ be polar coordinates

$$\alpha = \log r, \qquad \beta = \theta,$$

and

$$\psi = -v\theta(2a\log r + b) + \int\frac{dr}{r}\int r\,dr\int e^{a(\log r)^2 + b(\log r) + c}\,\frac{dr}{r}.$$

This also leads to two distinct solutions according as a is or is not zero.

If $a=0$

$$\left.\begin{aligned}\psi &= -bv\theta + \mathrm{A}r^{b+2} + \mathrm{B}r^2 + \mathrm{C}\log r \quad (b\neq 0 \text{ or } -2)\\ &= 2v\theta + \mathrm{A}(\log r)^2 + \mathrm{B}r^2 + \mathrm{C}\log r \quad (b=-2),\end{aligned}\right\}\text{(III.)}$$

while if a is not zero, a change in the scale of r gives

$$\psi = -2va\theta\log r + \mathrm{A}\int\frac{dr}{r}\int rar\int e^{a(\log r)^2}\,\frac{dr}{r}. \quad . \quad . \quad \text{(IV.)}$$

(III.), (IV.) are the only distinct solutions for which the lines of constant vorticity are a set of concentric circles. We have thus obtained all the solutions for which the lines of constant vorticity are the equipotential lines in free space of some possible distribution of matter.

§ 2. *Solutions for which the stream-lines are possible equipotential lines.*

The characteristic property of this type of solution is that it is possible to choose a set of curvilinear coordinates α, β so that $\psi = f(\alpha)$. It has not been found possible to solve this case with the generality of the previous section. Substituting in (2) we have

$$f'(\alpha)\frac{\partial}{\partial\beta}(\nabla^2\psi) = v\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right)\nabla^2\psi,$$

or writing
$$M=\left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\beta}{\partial x}\right)^2,$$

we see that α, β, f are restricted by the condition that M must satisfy the equation

$$\nu f''(\alpha)\left(\frac{\partial^2 M}{\partial\alpha^2}+\frac{\partial^2 M}{\partial\beta^2}\right)+2\nu f'''(\alpha)\frac{\partial M}{\partial\alpha}-f''(\alpha)\,.\,f''(\alpha)\frac{\partial M}{\partial\beta}$$
$$+\nu f''''(\alpha)M=0. \quad . \quad . \quad . \quad (3)$$

This will be satisfied by any system α, β whatever, if $f''=0$, which corresponds to the otherwise obvious fact that any solution of $\nabla^2\psi=0$ is a solution of (2). Thus any irrotational motion is a possible motion of a viscous fluid.

Suppose
$$\alpha+i\beta=(x+iy)^n,$$

then
$$M=n^2(\alpha^2+\beta^2)^{\frac{n-1}{n}},$$

equation (3) gives

$$2\nu(n-1)(n-2)f''(\alpha)+4\nu n(n-1)\alpha f'''(\alpha)-2n(n-1)\beta f'(\alpha)f''(\alpha)$$
$$+\nu n^2 f''''(\alpha)(\alpha^2+\beta^2)=0.$$

If $n=1$, it is sufficient that $f''''(\alpha)=0$, which leads to the well-known solution for the motion between infinite parallel planes:

$$\psi=Ax^3+Bx^2+Cx.$$

Otherwise, equating coefficients of powers of β separately to zero,

$$f''''(\alpha)=0,\quad f'(\alpha)\,f''(\alpha)=0,\quad f'''(\alpha)=0.$$

Hence $f(\alpha)$ is a linear function of α, which corresponds to the case of irrotational motion. It appears, therefore, that for no value of n other than unity is there any new solution of this class.

Next consider polar coordinates r, θ. Equation (2) becomes

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(r, \theta)}=\nu r\nabla^4\psi, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and
$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

First seek a solution of the form

$$\psi=f(r).$$

From (4) $$\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)\right)\right)=0,$$

and hence $$\psi=\mathrm{A}r^2\log r+\mathrm{B}r^2+\mathrm{C}\log r. \quad . \quad . \quad . \quad (\mathrm{V.})$$

Next seek a solution for which ψ is a function of θ only. Let

$$\psi=\Theta$$

then $$\nabla^2\psi=\frac{1}{r^2}\Theta''$$

and $$\nabla^4\psi=\frac{1}{r^4}(4\Theta''+\Theta'''').$$

Substituting in (4)

$$2\Theta'\Theta''=4\nu\Theta''+\nu\Theta''''.$$

Integrating $$\Theta'^2=4\nu\Theta'+\nu\Theta'''+a,$$

and $$\frac{\Theta'^3}{3}=2\nu\Theta'^2+\nu\frac{\Theta''^2}{2}+a\Theta'+b,$$

or $$\theta=\sqrt{\frac{3\nu}{2}}\int\frac{d\Theta'}{\sqrt{\Theta'^3-6\nu\Theta'^2-3a\Theta'-3b}}.$$

This may be written

$$\theta=\frac{\sqrt{6\nu}}{2}\int\frac{d\Theta'}{\sqrt{(\Theta'-\lambda)(\Theta'-\mu)(\Theta'-6\nu+\lambda+\mu)}},$$

where λ, μ are constants.

Write

$$\Theta'=\lambda\sin^2\phi+\mu\cos^2\phi,$$

and the integral becomes

$$\left\{\frac{6\nu-\lambda-2\mu}{6\nu}\right\}^{\frac{1}{2}}(\theta-\theta_0)=\int_0^{\phi}\frac{d\phi}{\sqrt{1-\frac{\lambda-\mu}{6\nu-\lambda-2\mu}\sin^2\phi}}.$$

Finally introduce new constants, k, m

$$k^2=\frac{\lambda-\mu}{6\nu-\lambda-2\mu}, \quad m^2=\frac{6\nu-\lambda-2\mu}{6\nu},$$

and we have

$$\frac{\partial\psi}{\partial\theta}=\Theta'=2\nu(1-m^2-k^2m^2)+6\nu k^2m^2 sn^2\{m(\theta-\theta_0)\} \quad (\mathrm{VI.})$$

where k, m, θ_0 are arbitrary constants.

§ 3. *Solutions which are independent of the degree of viscosity.*

If we consider the motion of a fluid between two concentric circular cylinders rotating with given angular velocities, an increase in the viscosity of the fluid would necessitate an increase in the couples which maintain the rotation of the cylinders, but otherwise the motion would be unchanged. This will be true for any solution for which ψ is independent of ν, *i. e.* the two sides of the equation (2) vanish separately. Taking the equation in its Cartesian form we have

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and
$$\nabla^4\psi=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It will be noted that equation (6) is the usual equation for the two-dimensional motion of a viscous fluid on the assumption that it is so slow that squares and products of the velocities may be neglected. For an exact solution equation (5) must also be satisfied, and hence

$$\nabla^2\psi=f(\psi),$$

that is the vorticity is constant along each stream-line. Substituting in (6),

$$f''(\psi)\left\{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right\}+f'(\psi)f(\psi)=0.$$

Hence either (1) the resultant velocity is a function of ψ, and is therefore constant along each stream-line, or (2) $f''(\psi)$ and $f'(\psi)$ are zero. In the latter case

$$\nabla^2\psi=\text{const.}=4a, \text{ say}$$

of which the general solution is

$$\psi=a(x^2+y^2)+\chi,$$

where χ is any solution of $\nabla^2\chi=0$. Hence any solid body rotation superposed upon any irrotational motion is a possible motion of a viscous fluid. The only other solutions of this class are those for which the velocity and the vorticity are constant along each stream-line.

§ 4. *Discussion of solutions obtained.*

Flow between two planes inclined at any angle.—We will first consider solution VI. We may without loss of generality take $\theta_0=0$, so that

$$\frac{d\psi}{d\theta}=2\nu(1-m^2-m^2k^2)+6\nu k^2m^2sn^2(m\theta, k).$$
$$=2\nu(1+2m^2-m^2k^2)-6\nu m^2dn^2(m\theta, k).$$

The stream-lines are all straight lines passing through the origin. If u be the velocity of the fluid

$$u=-\frac{1}{r}\frac{d\psi}{d\theta}.$$

The constants m, k can be chosen so that the velocity is zero when $\theta=\alpha$, and hence, since u is an even function of θ, when $\theta=-\alpha$, while the values of ψ appropriate to these two stream-lines differ by a given amount. Thus we have a solution for the motion of a fluid between two fixed planes inclined at an angle 2α due to a line source of given strength along their line of intersection, or if we exclude the origin, for the flow along a canal with converging banks. If Q is the total flux of fluid outwards from the origin

$$Q=-4\nu\alpha(1+2m^2-m^2k^2)+12\nu m^2\int_0^{\alpha}dn^2(m\theta, k)d\theta.$$

Now

$$\int_0^{\zeta}dn^2(z, k)dz=E(\zeta, k)$$

where E denotes the elliptic integral of the second kind. Hence

$$Q=-4\nu\alpha(1+2m^2-m^2k^2)+12\nu mE(m\alpha, k).$$

This relation, together with

$$3k^2m^2sn^2(m\alpha, k)+1-m^2-m^2k^2=0,$$

is sufficient to determine m and k. (N.B.—α must be expressed in circular measure.)

If the angle between the planes is small, we may write $sn\,.\,z=z$, and we have as an approximation when the planes are nearly parallel

$$\frac{d\psi}{d\theta}=2\nu\{1-m^2-m^2k^2(1-3m^2\theta^2)\}.$$

If the angle between the planes is $2a$ we have

$$1-m^2-m^2k^2(1-3m^2\alpha^2)=0,$$

or

$$k^2=\frac{1-m^2}{m^2(1-3m^2\alpha^2)},$$

and

$$\frac{d\psi}{d\theta}=\frac{6\nu m^2(1-m^2)}{1-3m^2\alpha^2}(\theta^2-\alpha^2).$$

We see that to this approximation the velocity across any cross section follows the same parabolic law as in the flow between two parallel planes.

Solutions I., II., and IV. lead to some interesting sets of stream-lines. They cannot, however, be realized physically, and they seem to be of little importance. We pass to the consideration of solution III.

A centrifugal pump.—We have

$$\psi = -b\nu\theta + \mathrm{A}r^{b+2} + \mathrm{B}r^2 + \mathrm{C}\log r,$$

and the components of velocity are given by

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{b\nu}{r}$$

$$v = \frac{\partial\psi}{\partial r} = \mathrm{A}(b+2)r^{b+1} + 2\mathrm{B}r + \frac{\mathrm{C}}{r}.$$

From the equations of motion we can determine the mean pressure. If there are no external forces

$$p = -4b\rho\nu\mathrm{B}\theta + \rho f(r),$$

where $f(r)$ can readily be determined if necessary. If we include the whole of the space round the origin, p must be single valued, and hence $\mathrm{B}=0$. In this case the solution corresponds to the motion generated by the rotation of a perforated cylinder, which, as it rotates, either sucks in or ejects fluid uniformly over its surface. The fluid may either flow away to infinity, or it may be absorbed by a coaxial porous cylinder, which may be at rest or may be rotating with any angular velocity. The total flux of fluid will determine b, while the angular velocities of the cylinders will determine A and C. Such an arrangement will be in fact a centrifugal pump. When the fluid is viscous vanes are not absolutely necessary, although, of course, they may increase the efficiency of the machine. If there is no second cylinder so that the fluid extends to infinity, then we must have $\mathrm{A}=0$ if the fluid is flowing outwards, for in that case $b>0$. Let the radius of the cylinder be a, and let it rotate with angular velocity Ω, and eject a volume Q of fluid in unit time; then

$$\mathrm{C} = a^2\Omega, \qquad b = \mathrm{Q}/2\pi\nu,$$

and we find without difficulty,

$$p = p_0 - \frac{1}{8\pi^2\rho r^2}(4\pi^2 a^4\Omega^2 + \mathrm{Q}^2),$$

where p_0 is the pressure at infinity. Thus we have the pressure head created by the pump when it rotates with a given velocity and discharges a given volume of fluid per unit time.

Flow under pressure along a circular canal.—Finally, we will consider solution V.

$$\psi = Ar^2 \log r + Br^2 + C \log r.$$

If u, v are the radial and transverse components of velocity

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0,$$

$$v = \frac{\partial\psi}{\partial r} = 2Ar \log r + (A + 2B)r + \frac{C}{r}.$$

If $A=0$, this is the well-known solution for rotating concentric cylinders. Using the equations of motion in polar coordinates we can without difficulty find the value of the mean pressure p.

$$p = 4\nu\rho A\theta + \rho f(r),$$

where

$$f(r) = 2A(\log r)^2(Ar^2 + C) + 2\log r(ABr^2 + AC + 2BC) + \frac{1}{2}(A^2 + 4B^2)r^2 - \frac{1}{2}\frac{C^2}{r^2}.$$

The constants B, C may be chosen so that the velocity is zero for any two values of r, and we have the solution for the flow of a viscous fluid round a canal bounded by two concentric circular arcs. The pressure will not be constant across a radial cross-section, but will vary in a way which is represented by $f(r)$. It will be noted, however, that $f(r)$ contains only the squares of the coefficients A, B, C, and is therefore of the order of the square of the velocity.

Suppose the canal is bounded by circular arcs of radii a, b, which subtend an angle α at their common centre, and let P be the pressure difference between corresponding points on the two bounding cross-sections. Then

$$P = 4\mu A\alpha,$$

where μ is the coefficient of viscosity and is therefore equal to $\nu\rho$. The condition that the velocity shall vanish when $r=a$, b gives the following equations to determine the constants B, C

$$A(r\log r^2 + r) + 2Br + \frac{C}{r} = 0, \quad (r = a, b),$$

from which we have

$$B = -\frac{A(a^2 \log a^2 - b^2 \log b^2 + a^2 - b^2)}{2(a^2 - b^2)}$$

$$C = \frac{Aa^2b^2(\log a^2 - \log b^2)}{a^2 - b^2}.$$

If Q denotes the volume of fluid which flows through the canal per unit depth in unit time, it is equal to the difference of the value of ψ for $r = a, b$.

$$Q = A(b^2 \log b - a^2 \log a) + B(b^2 - a^2) + C(\log b - \log a)$$

$$= \frac{P}{8\mu\alpha}\left[(a^2 - b^2) - \frac{4a^2b^2}{a^2 - b^2}\left(\log\frac{a}{b}\right)^2\right].$$

If a, b tend to infinity in such a way that $a - b \rightarrow d$ and $a\alpha \rightarrow l$

$$Q \rightarrow \frac{Pd^3}{12\mu l},$$

which agrees with the known result for the flow between parallel planes.

LI. *Theory of Dispersion.* *By* Prof. D. N. Mallik, *Sc.D.*, *F.R.S.E.**

1. It is well-known that the electromagnetic theory, as expressed by the equations of Maxwell and Hertz, cannot account for aberration, dispersion, and allied phenomena. In analysing the reason for this, we note that the theory is based on the following postulates:—

(1) The energy of the electromagnetic field is that of the dielectric medium alone, arising from a certain strained condition of the medium.

(2) The conductors having static charges serve only to limit the dielectric region so that no part of the energy resides on them.

(3) The strained condition of a dielectric is due to electric displacement or polarization f, g, h, subject to the condition

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0. \quad . \quad . \quad . \quad . \quad (1)$$

This displacement is apparently held to involve motion of the æther in the medium, subject to a property akin to elasticity (due to inter-action of matter

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