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632. The Ambiguous Case

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631. [K¹. 11. e.] *Note on a Porism of Lord Brougham.*

Recently I have come across Lord Brougham's *Tracts Mathematical and Physical*, wherein is reprinted a paper entitled "General Theorems, chiefly Porisms, in the Higher Geometry," from the *Philosophical Transactions* of 1798. Among others he gives the following statement of a Porism:

"Two points in a circle being given, but not in one diameter, another circle may be described, such, that if from any point of it to the given points straight lines be drawn, and a line touching the given circle, the tangent shall be a mean proportional between the lines so inflected. Or, more generally, the square of the tangent shall have a given ratio to the rectangle under the inflected lines."

Treated as substantive propositions, Brougham's statements are capable of easy proof by elementary geometry. Considered from the point of view of a Porism they are incomplete. If A and B be two fixed points, and OT the tangent from a moving point to a fixed circle; then the locus, defined by the condition $OA \cdot OB = k \cdot OT^2$, breaks down into two circles in two different ways; firstly when A and B are on the circumference of the fixed circle, as in Brougham's statement, and secondly when A and B are inverse points with respect to the said circle.

The second of these cases is capable of generalisation. Suppose that three circles are given, and that we desire to know in what way the locus of the point O , which moves in such a manner that the lengths of the tangents from it to the three circles are connected by the relation $OT_2 \cdot OT_3 = k \cdot OT_1^2$, may break down into a pair of circles, we find a case controlled by two conditions. Firstly, the three centres must be collinear. Secondly, if we denote the positions of the centres by A, B, C , and the radii of the corresponding circles by a, b, c , the other condition may, according to the conventions of modern geometry, be written in the form

$$a^2 \cdot BC + b^2 \cdot CA + c^2 \cdot AB + BC \cdot CA \cdot AB = 0.$$

This is a truly Poristic case, as the conditions do not involve k . It is to be remarked that, although the second condition is symmetrical, we cannot associate our fixed circles indiscriminately in the formation of the condition $OT_2 \cdot OT_3 = k \cdot OT_1^2$.

When the fixed circles satisfy these conditions, the three polars of any point with respect to them are concurrent.

The only alternative to this, supposing the conditions satisfied irrespective of the value of k , is that two of the circles should be reduced to points. There would seem to be other cases, defined by relations among the circles, for which the locus breaks down into a pair of circles, but only for particular values of k dependent on the special conditions satisfied by the original circles.

J. BRILL.

632. [K¹. 20. e.] *The Ambiguous Case.*

(i) The case of congruence of triangles which agree as to two sides and a right angle not contained is at present regarded by examiners as essentially distinct from the other cases.

If two triangles agree as to three sides, when the equalities of the three sides have been stated and justified nothing remains for the examinee to do before stating that the triangles are congruent; but if they agree as to two sides and a right angle not contained, he must, if he is to avoid losing half his marks, either have commenced by saying "in the right-angled triangles" or finish by adding "these are two sides and a right angle" before he can conclude that the triangles are congruent.

[It is true that some examiners draw a subtle distinction between the statements " $A = B$, right angles" and " A and B are right angles," by which the examinee may, probably by pure luck, be saved from loss.]

Why this severity? There is no ambiguity about the congruence of two triangles which agree in this way. Why should it be necessary to say "these are two sides and a right angle" when in another case it is not necessary to say "these are two angles and a side suitably placed"? Might not the examinee who has stated and justified equalities which ensure congruence be permitted to infer congruence without adding (by implication) that he is aware that if the conditions had been different there might have been ambiguity?

(ii) Is it not time that the ambiguous case in congruence of triangles should be more freely used? If two triangles agree as to two sides and an angle not contained, then another pair of angles are either equal or supplementary, so that if they are known not to be supplementary they must be equal and the triangles congruent.

It is not hard for beginners to learn which the angles are. One pair of equal sides have the given angles opposite them. It is the angles opposite the other given sides which are either equal or supplementary.

Probably the best rider which uses this case of congruence is "to prove that if the bisector of the vertical angle of a triangle also bisects the base, then the triangle is isosceles."

This rider is often done by producing the bisector an equal distance, but it is quite unnecessary to make this construction. From the data we infer that the base angles of the triangle are equal or supplementary, and as the latter is impossible unless the vertex is at infinity, the former is true and the triangle is isosceles.

In a recent School Certificate paper a line was drawn through the internal centre of similitude O of two circles (the technical term not being used), and met the circles for the first time at P and Q , a figure being given. It had to be proved that $PO:OQ$ = ratio of radii. This is the ambiguous case in similarity, but if the circles were equal would become that in congruence.

As a last example: ABC is equilateral, $BDEC$ a square on BC externally, AD cuts BE at O . Prove angles OAC , OEC equal, and hence, using the ambiguous case, prove triangles AOC , EOC congruent. C. O. TUCKEY.

633. [L¹. 14. a.] C. Smith, "Conics (Coord. Geom.)," chap. xiii. no. 16.

A conic is inscribed in a triangle so that its centre is at the circumcentre. To prove that its axes are $R+d$ and $R-d$, where d is the distance between the circumcentre and the orthocentre of the triangle.

(Proof by pure geometry.)

Let C and O be respectively the circumcentre and orthocentre of the triangle STU , and P, Q, R the points of contact of the inscribed conic. Let SO cut TU in K and the circumcircle in L ; let OH perpendicular to OU cut TU in H ; and let E be foot of perpendicular CE on TU . Then $\triangle CPT = \triangle CTR$, and similarly for the other vertices.

$$\therefore \triangle CPT + \triangle CSU = \frac{1}{2} \triangle STU.$$

But quadrilateral $SCKU = \frac{1}{2} \triangle STU$, for $\triangle SCK = \triangle SEK$.

$$\therefore \triangle CPT = \triangle CKU; \therefore TP = KU.$$

If $RT, R'T'$ (T' on TU) are parallel tangents to the conic, the projections perpendicular to ST of TT' and HU (CG perpendicular to ST) are respectively $2CG$ and OU , which are equal.

$$\therefore TT' = HU; \therefore PT' = HK;$$

$\therefore CD$ being the semidiameter conjugate to CP ,

$$CD^2 = PT \cdot PT' = HK \cdot KU = OK^2; \therefore CD = OK.$$

Now it is a known theorem that if, on the normal at P , points J and J' be taken so that $PJ = PJ' = CD$, $CJ = a+b$ and $CJ' = a-b$.

But since $TP = KU$ and $CD = OK = KL$, J is evidently on the circumcircle and CJ' is symmetrically equal to CO .

Hence $a+b = R$, $a-b = d$; whence the result.

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