ON THE GENERAL THEORY OF MULTIPLE CONTINGENCY WITH SPECIAL REFERENCE TO PARTIAL CONTINGENCY.

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(1) Let there be l variates or characteristics $A, B, C, \dots, L$, each of these variates or characteristics being subdivided into categories $A_1, A_2, \dots, A_\alpha, B_1, B_2, \dots, B_\beta, C_1, C_2, \dots, C_\gamma, \dots, L_1, L_2, \dots, L_\lambda$, where $\alpha, \beta, \gamma, \dots, \lambda$ are arbitrary numbers. Then if N be the total population, and $n_{a_1}, n_{a_2}, \dots, n_{a_\alpha}$ the number of individuals in the A -categories; $n_{b_1}, n_{b_2}, \dots, n_{b_\beta}$ those in the B -categories and so on, we have relations

$$S_1^\alpha (n_{a_i}) = S_1^\beta (n_{b_j}) = S_1^\gamma (n_{c_k}) = \dots = N.$$

Further, if there be no relationship whatever between the variates or characteristics, we should anticipate that the frequency of the group $A_\alpha, B_\beta, C_\gamma, \dots, L_\lambda$ in a sample of M would on the average be

$$M \cdot \frac{n_{a_\alpha}}{N} \cdot \frac{n_{b_\beta}}{N} \cdot \frac{n_{c_\gamma}}{N} \dots \frac{n_{l_\lambda}}{N}.$$

Actually we find in the sample M the number $m_{\alpha\beta\gamma\dots\lambda}$, and the problem arises whether the system represented by $m_{\alpha\beta\gamma\dots\lambda}$ is so improbable that in the selected population M the characteristics $A, B, C, \dots, L$ cannot be considered independent, i.e. M is really not a random sample from the supposed population N . Clearly the answer to this problem has already been given. We have to find the value of χ^2 :

$$= S_{\alpha\beta\gamma\dots\lambda} \left[\frac{\left(m_{\alpha\beta\gamma\dots\lambda} - M \cdot \frac{n_{a_\alpha}}{N} \cdot \frac{n_{b_\beta}}{N} \dots \frac{n_{l_\lambda}}{N} \right)^2}{M \cdot \frac{n_{a_\alpha}}{N} \cdot \frac{n_{b_\beta}}{N} \dots \frac{n_{l_\lambda}}{N}} \right] \dots\dots\dots(i),$$

and apply the tables for "goodness of fit." Of course in many cases the sampled population is not known and accordingly we can only put for $\frac{n_{a_\alpha}}{N}, \frac{n_{b_\beta}}{N}, \dots, \frac{n_{l_\lambda}}{N}$ the values given by the sample itself, i.e. $\frac{m_{a_\alpha}}{M}, \frac{m_{b_\beta}}{M}, \dots, \frac{m_{l_\lambda}}{M}$, and test from this substitution the degree of divergence from independence. If we take the mean value of χ^2 , i.e. $\phi^2 = \chi^2/M$, ϕ^2 is termed the mean square contingency, and $C_s = \sqrt{\phi^2/(1 + \phi^2)}$

gives a measure of the divergence from independence*. This is a multiple contingency coefficient.

Another case not infrequently arises; the population N has the characteristics $A, B, C, \dots$ not independent, but related; the cell $uvw \dots \psi$ contains $n_{uvw \dots \psi}$ and the question arises how far it is safe to consider the population M as a sample of this population. In this case†

$$\chi^2 = S_{uvw \dots \psi} \left[\frac{m_{uvw \dots \psi} - \frac{M}{N} n_{uvw \dots \psi}}{\frac{M}{N} n_{uvw \dots \psi}} \right] \dots \dots \dots (ii).$$

In both these cases we have the relation

$$S_{uvw \dots \psi} (m_{uvw \dots \psi}) = M \dots \dots \dots (iii),$$

and accordingly the number of cell frequencies is one more than the number of independent variates. Thus in using the tables‡ of "goodness of fit" the n' of the argument is $\alpha\beta\gamma \dots \lambda$, but the value of P , the probability, has actually been determined from $n' - 1$.

(2) Now there are a number of cases in which not only do the cell-contents of the sample obey the linear relation (iii), but also other linear relations are imposed on the cell-contents. In the most general case we can suppose q linear relations between the cell-contents $m_{uvw \dots \psi}$, and obtain the probability P corresponding to a value of χ^2 limited by these q relations. The theory of sampling, when such conditions are introduced, I term the theory of partial contingency. The reason for this terminology will be clearer as we develop the theory.

As far as we are concerned at present, it is of no importance whether we are dealing with one or other of our two cases, i.e. whether we are questioning the possibility of our material being a sample from a population with independent $A, B, C, \dots L$ characteristics, i.e. determining a coefficient of mean squared contingency, or are investigating the possibility of its being a probable sample from a population with any associations between these characteristics. We can accordingly write χ^2 in the form

$$\chi^2 = S_{uvw \dots \psi} \left\{ \frac{(m_{uvw \dots \psi} - \bar{m}_{uvw \dots \psi})^2}{\bar{m}_{uvw \dots \psi}} \right\} \dots \dots \dots (iv),$$

or for convenience we may even drop the descriptive subscripts and, numbering the cells in some sequence 1, 2, 3, $\dots s$, $\dots$ ($\alpha\beta\gamma \dots \lambda$), write

$$\chi^2 = S_s \frac{(m_s - \bar{m}_s)^2}{\bar{m}_s} \dots \dots \dots (v).$$

* If the characteristics may be assumed to be continuous variates, certain corrections for units of grouping can be made. There is also a correction due to the necessarily positive value of ϕ^2 . These corrections, which have been for some time in use, will be considered elsewhere.

† We must in this case of course actually know the value of $n_{uvw \dots \psi}$, it cannot be judged from the sample.

‡ For discussion of the deduction of P from χ^2 : see *Phil. Mag.* Vol. L. p. 157 (July 1900); and for Tables: see *Tables for Statisticians and Biometricians*, Table XII (Cambridge University Press).

Here $\bar{m}_{uvw\dots} = \bar{m}_s$ will be either

$$M \cdot \frac{n_{a_s}}{N} \cdot \frac{n_{b_s}}{N} \dots \frac{n_{l_s}}{N} \quad \text{or} \quad \frac{M}{N} n_{uvw\dots},$$

as the case may be.

Now we shall find it convenient to write

$$X_s = (m_s - \bar{m}_s)/\sqrt{\bar{m}_s} \dots\dots\dots(\text{vi}),$$

and thus

$$\chi^2 = S_s (X_s^2) \dots\dots\dots(\text{vii}).$$

Further, if our q linear equations be of type

$$h_{t1}m_1 + h_{t2}m_2 + \dots + h_{ts}m_s + \dots = H_t \dots\dots\dots(\text{viii}),$$

where h and H are constants and t takes every value from 1 to q , we can write our conditions in the form

$$\left. \begin{aligned} k_{t1}X_1 + k_{t2}X_2 + \dots + k_{ts}X_s + \dots &= K_t \\ k_{ts} &= \frac{h_{ts}\sqrt{\bar{m}_s}}{\sqrt{h_{t1}^2\bar{m}_1 + h_{t2}^2\bar{m}_2 + \dots + h_{ts}^2\bar{m}_s + \dots}} \\ \text{and} \quad K_t &= \frac{H_t - h_{t1}\bar{m}_1 - h_{t2}\bar{m}_2 - \dots - h_{ts}\bar{m}_s - \dots}{\sqrt{h_{t1}^2\bar{m}_1 + h_{t2}^2\bar{m}_2 + \dots + h_{ts}^2\bar{m}_s + \dots}} \end{aligned} \right\} \dots\dots\dots(\text{ix}).$$

We shall speak of the first of (ix) as the *prepared condition*. Clearly it corresponds to a plane in n -dimensional space in which the constants $k_{t1}, k_{t2}, \dots, k_{ts}, \dots$ are the direction-cosines and K_t the perpendicular from the origin on the plane. It is convenient to use the notation

$$k_{t1}k_{t'1} + k_{t2}k_{t'2} + \dots + k_{ts}k_{t's} + \dots = \cos(\mu') \dots\dots\dots(\text{x}),$$

for (μ') is now the angle between the t th and t' th planes.

Assuming that the frequency surface with which we have to deal may be taken as

$$z = z_0 e^{-\frac{1}{2}\chi^2} \dots\dots\dots(\text{xi}),$$

we may suppose before applying equations of condition (ix) that $K_1, K_2, \dots, K_t, \dots, K_q$ are variates and that we eliminate $X_1, X_2, \dots, X_q$, expressing our χ^2 in terms of $K_1, K_2, \dots, K_q, X_{q+1}, X_{q+2}, \dots, X_n$. We shall have then

$$z = z_0 \text{ expt. } -\frac{1}{2} (\text{quadratic function of these } n \text{ new variables}).$$

We now proceed to put $K_1, K_2, \dots, K_q$ constant, but leave the other $n - q$ quantities to vary; we are therefore seeking the value of χ^2 for certain variates constant. This is the essence of partial contingency, and the analogy in contingency to partial correlation.

(3) As a rule in partial contingency we do not seek to discuss χ^2 when single cells of our multiple contingency solid are constant in frequency, although our theory covers that case. What we require usually is the value of χ^2 when we make the contents of certain marginal total cells constant. For example, let us consider

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a population of uniform sex and let there be three characteristics: (i) frequency of age groups, (ii) frequency of occupational categories and (iii) frequency of survival and of death, the latter classified by various special disease classes—the whole, say, representing the returns for one year of a large area or country. We then require to determine, whether the like contingency solid for a sub-district, or for another population entirely, may be considered as significantly different from the above general population, i.e. we require to find the probability of its being a random sample of this general population. Now we may do this in the most universal manner, by assuming that not only survivals and deaths, but that age groups and occupations all have frequencies, which are random samples of the above general population. But this is not very often what we require; we admit that the age distribution is differentiated, we admit that the occupational frequencies are peculiar to the locality, and we ask whether, notwithstanding these differences, the death distribution is to be considered as a random sample.

In other words, we do not only fix the size of our sample M ; we fix all one face—that of age groups and occupational categories of our contingency solid—and ask what is the distribution of samples of M taken from this solid, subject to the linear conditions that the totals of age-occupational categories are constant. For example, if A be age and B occupation, we make $n_{a,b}$, constant for all values of u and v , but

$$n_{a,b,c} = n_{a,b,c_1} + n_{a,b,c_2} + n_{a,b,c_3} + \dots,$$

where $n_{a,b,c}$ is the frequency of the uv s cell, c denoting the category C of type of death and survival.

Now clearly in making this investigation we shall be studying the mean square contingency and the resulting probability of a *partial* sample—a sample of survival and death-type in a population of constant age groups and occupational classes. Again, we might treat a population as a sample with only constant age groups or only constant occupational frequencies and again investigate its probability as a sample with regard to deaths and occupations or with regard to deaths and age groups respectively. These would be partial contingencies of the order α or β , while the previous partial contingency was of the order $\alpha + \beta$, α being the number of categories in A (i.e. age groups) and β being the number in B (i.e. mortality and survival classes).

(4) Now the value of χ^2 given above, and of the frequency surface (xi), was discussed by me in the year 1900* from the general normal frequency surface by evaluation of determinants. The demonstration depends on two hypotheses:

- (i) The approach of the binomial $M(p+q)^n$ to the normal curve

$$y = \frac{M}{\sqrt{2\pi} \sqrt{npq}} e^{-\frac{x^2}{2npq}}.$$

We know that this is true if n be considerable and neither p nor q very small.

* *Phil. Mag.* Vol. L. p. 157, 1900. I have recently given a more elementary proof with the probable error of P : see *Phil. Mag.* April, 1916.

The hypothesis is justified therefore if no cell be taken so small that its contents are very small compared with the size of the sample.

(ii) That the sampling takes place out of a population indefinitely greater than the sample. If this be not true, then the distribution of frequency of any given cell for a series of samples follows not a binomial but a hypergeometrical series. The necessary modifications in the formulae are not very substantial and have been discussed elsewhere*

Supposing the above two conditions to be fulfilled, then in true random samples the mean of a cell frequency will be

$$\bar{m}_s = M \frac{n_s}{N} \dots\dots\dots(\text{xii}),$$

where M is size of sample, n_s the contents of the s th cell in the sampled population N , and condition (i) amounts to saying that no cell is to be chosen so that n_s/N is indefinitely small.

Further, the frequency of the s th cell will follow the binomial

$$\left\{ \frac{n_s}{N} + \left(1 - \frac{n_s}{N} \right) \right\}^M,$$

and thus have a standard deviation given by

$$\sigma_{n_s}^2 = M \frac{n_s}{N} \left(1 - \frac{n_s}{N} \right) \dots\dots\dots(\text{xiii}).$$

Lastly the correlation $r_{ss'}$ between deviations in the s th and s' th cells is given by

$$\sigma_{n_s} \sigma_{n_{s'}} r_{ss'} = - M \frac{n_s}{N} \frac{n_{s'}}{N} \dots\dots\dots(\text{xiv}).$$

(5) The following deduction of the value of χ^2 is a variant from my *Phil. Mag.* proof. I owe the suggestion of it to Mr H. E. Soper, although I have deviated somewhat from his track. Let an indefinitely large population N consist of the classes $C_0, C_1, \dots C_i$ in the quantities $n_0, n_1, n_2, \dots n_i$ respectively. Then $p_s = n_s/N$ = chance of drawing a member of the class C_s , and the standard deviation of the distribution of frequency in samples of M drawn from the population will in this class C_s be as above

$$\sigma_s = \sqrt{M p_s (1 - p_s)} \dots\dots\dots(\text{xiii})^{\text{bis}}.$$

Further, the mean of samples for this class will be $M p_s$ by (xii).

In the next place the correlation between deviations from the means in classes C_s and $C_{s'}$ will be in our present notation

$$\sigma_s \sigma_{s'} r_{ss'} = - M n_s n_{s'} / N^2 \dots\dots\dots(\text{xiv})^{\text{bis}},$$

or by (xiii)^{bis}

$$r_{ss'} = - \frac{\sqrt{p_s p_{s'}}}{\sqrt{1 - p_s} \sqrt{1 - p_{s'}}} \dots\dots\dots(\text{xv}).$$

* *Phil. Mag.* p. 239, 1899 and *Biometrika*, Vol. v. p. 174.

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The distribution of frequency of the different classes in these samples of M will be given by the terms of the multinomial

$$(p_0 C_0 + p_1 C_1 + p_2 C_2 + \dots + p_l C_l)^M \dots\dots\dots(\text{xvi}),$$

the general term being

$$\frac{M!}{u_0! u_1! u_2! \dots u_l!} p_0^{u_0} p_1^{u_1} p_2^{u_2} \dots p_l^{u_l} C_0^{u_0} C_1^{u_1} \dots C_l^{u_l} \dots\dots\dots(\text{xvii}),$$

where $C_0, C_1, \dots C_l$ are only logical symbols to denote that this general term is the frequency of the group, where the class C_s occurs u_s times in the sample. Clearly $u_0 + u_1 + \dots + u_l = M$.

But (xvi) may be put into the form of a binomial, it is

$$= \sum_{m=0}^{m=M} \frac{M!}{m! (M-m)!} (p_0 C_0)^{M-m} (p_1 C_1 + p_2 C_2 + \dots + p_l C_l)^m \dots\dots\dots(\text{xviii}).$$

Let $(p_1 + p_2 + \dots + p_l) = \lambda$, then the m th term of the above series may be read as

$$\frac{M!}{m! (M-m)!} (p_0 C_0)^{M-m} \lambda^m (p_1' C_1 + p_2' C_2 + \dots + p_l' C_l)^m \dots\dots\dots(\text{xix}),$$

where

$$p_1' + p_2' + \dots + p_l' = 1,$$

and

$$\frac{p_1}{p_1'} = \frac{p_2}{p_2'} = \frac{p_3}{p_3'} = \dots = \frac{p_l}{p_l'} = \lambda = 1 - p_0 \dots\dots\dots(\text{xx}).$$

Now it is clear that the factor

$$(p_1' C_1 + p_2' C_2 + \dots + p_l' C_l)^m$$

gives the frequency distribution of samples of m drawn from a population of indefinitely large size of which the proportions of the classes $C_1, C_2, \dots C_l$ are $p_1', p_2', \dots p_l'$ and in which no class C_0 occurs. But by (xx) these proportions are the same as in the original population which contains C_0 .

Hence if we take samples of M from an indefinitely large population with classes $C_0, C_1, C_2, \dots C_l$, those that contain m of the classes $C_1, C_2, \dots C_l$ will be distributed in the same proportions as if we had extracted m from an indefinitely large population consisting only of those l classes in the same proportions.

Now thus far the nature of the class C_0 is at our choice. In the original population N it appears with the total frequency n_0 . Let $n_0 + n' = N$, and suppose n_0 is indefinitely greater than n' , then p_0 will be indefinitely greater than $p_1, p_2, \dots p_s$. It follows that p_s if s be not zero is very small compared to unity, because

$$p_0 + p_1 + p_2 + \dots + p_s = 1.$$

Hence in such a system from (xiii)^{bis}, if s be not zero,

$$\sigma_s = \sqrt{M p_s} \dots\dots\dots(\text{xxi}),$$

and from (xv) to the same degree of approximation $r_{s1} = 0$. That is to say, if p_0 be large in taking samples of size M from an indefinitely large population, there will be no correlation in deviations in the frequency of the classes $C_1, \dots C_l$.

On the other hand r_{x_0} is not zero, but equals by (xv)

$$-\sqrt{p_s p_0}/\sqrt{1-p_0} = -\sqrt{p_s}/\sqrt{1-p_0}$$

to the same degree of approximation. Hence by (xx)

$$r_{x_0} = -\sqrt{p_s'} \dots\dots\dots(\text{xxii}).$$

But if we form the partial correlation of deviations in classes C_s and C_t for constant frequency of number in C_0 , we have

$$\begin{aligned} \sigma_{st} &= (r_{st} - r_{s0}r_{t0})/\sqrt{(1-r_{s0}^2)(1-r_{t0}^2)} \\ &= (0 - \sqrt{p_s'}\sqrt{p_t'})/\sqrt{(1-p_s')(1-p_t')}, \end{aligned}$$

or

$$\sigma_{st} = -\frac{\sqrt{p_s'p_t'}}{\sqrt{1-p_s'}\sqrt{1-p_t'}} \dots\dots\dots(\text{xxiii}),$$

which agrees with (xv), the classes being now reduced by unity. Further, the reduced standard deviation must now be

$$\begin{aligned} \sigma\sigma_s &= \sigma_s\sqrt{1-r_{s0}^2} \\ &= \sigma_s\sqrt{1-p_s'} \\ &= \sqrt{Mp_s(1-p_s')} \\ &= \sqrt{M\lambda p_s'(1-p_s')} \dots\dots\dots(\text{xxiv}). \end{aligned}$$

Now take the mean value $\sigma\bar{x}_s$ of x_s the frequency in class C_s for a constant x_0 or for constant frequency in class C_0 ; in our case, if the sample is to be m this will be $M-m$, we have

$$\begin{aligned} \sigma\bar{x}_s - \bar{x}_s &= r_{s0}\frac{\sigma_s}{\sigma_0}(x_0 - \bar{x}_0), \\ \sigma\bar{x}_s - Mp_s &= \frac{-\sqrt{p_s'}\sqrt{Mp_s}}{\sqrt{Mp_0(1-p_0)}}(M-m-Mp_0), \\ \sigma\bar{x}_s &= p_s \left\{ M - \frac{1}{1-p_0} [M(1-p_0) - m] \right\} \end{aligned}$$

or

$$\sigma\bar{x}_s = \frac{mp_s}{1-p_0} = m\eta_s' \dots\dots\dots(\text{xxv}).$$

The partial values $\sigma\bar{x}_s$, σ_{st} of the means and correlations of classes for constant number in class C_0 are given by (xxiii) and (xxv), and are what we might anticipate. But (xxiv) should be $\sqrt{mp_s'(1-p_s')}$. It is accordingly needful to take

$$\lambda = m/M,$$

or

$$p_0 = 1 - m/M \dots\dots\dots(\text{xxvi}).$$

These results have been reached on the assumption that p_0 is very large as compared with $p_1, p_2, \dots p_n$. It follows accordingly that the sample M must be large as compared with m , and further the sum of the classes $C_1, C_2, \dots C_i$ must

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be to that of the class C_0 in the total population in the ratio of the partial sample m to the total sample M . Without this condition it is not possible to replace Mp_s by mp_s' . Assuming these conditions to be satisfied, then samples of the size m in classes $C_1, C_2, \dots, C_l$ picked out of very large samples of M will reproduce the same distribution of frequencies in those classes as samples of m picked out of an indefinitely large population with the same relative frequencies in those classes.

But in the case of samples of M , the deviations have their correlations zero for the classes $C_1, C_2, \dots, C_l$, or they will be approximately distributed by the product of their independent probabilities. The standard deviation being $\sqrt{mp_s'}$ and the mean mp_s' , we see that the frequency distribution would really follow a Poisson's binomial limit, but as shown by L. Whitaker* this binomial limit is approximately Gaussian with fairly low values of mp_s' ; see the Diagrams for $mp_s' = 10$ and $= 30$ in the plate of her memoir. We may accordingly therefore take the distribution of the frequency in the s th class or cell to be given by

$$z_s = \frac{1}{\sqrt{2\pi}\sqrt{\bar{m}_s}} e^{-\frac{1}{2} \frac{(m_s - \bar{m}_s)^2}{\bar{m}_s}} \dots\dots\dots(\text{xxvii})$$

and the general distribution to be

$$z = \left(\frac{1}{\sqrt{2\pi}} \right)^l \frac{1}{\sqrt{m^l p_1' p_2' \dots p_l'}} e^{-\frac{1}{2} \chi^2} \left. \dots\dots\dots(\text{xxviii}). \right\}$$

$$\text{where } \chi^2 = S_s \left\{ \frac{(m_s - \bar{m}_s)^2}{\bar{m}_s} \right\}$$

Here, if the size of the sample only be fixed, we shall have

$$S(m_s) = m = mS(p_s') = S(\bar{m}_s),$$

or

$$S(m_s - \bar{m}_s) = 0.$$

If we take

$$X_s = \frac{m_s - \bar{m}_s}{\sqrt{\bar{m}_s}},$$

we have:

$$\chi^2 = X_1^2 + X_2^2 + \dots + X_l^2 \dots\dots\dots(\text{xxix})$$

subject to the condition:

$$\sqrt{\bar{m}_1} X_1 + \sqrt{\bar{m}_2} X_2 + \dots + \sqrt{\bar{m}_l} X_l = 0 \dots\dots\dots(\text{xxx}).$$

It is clear that χ^2 equal to a constant gives a sphere in l -fold space, and that (xxx) is a plane passing through its centre, and therefore cutting the sphere in l -fold space in a sphere of the same radius in $(l-1)$ -fold space. Hence if we desire to find the volume of the frequency surface (xxviii) which lies outside a value of $\chi = \chi_0$ subject to the condition (xxx), all we have to do is to transfer to polar

* *Biometrika*, Vol. x. p. 36. If $\bar{m}_s = mp_s'$ be the mean, $\beta_1 = 1/\bar{m}_s = \beta_2 - 3$, so that $\beta_1 = .03$, $\beta_2 = 3.03$ already for $\bar{m}_s = 33$.

coordinates and integrate the value of z for the $(l-1)$ -fold surface beyond the value χ_0^* . Accordingly

$$P = \frac{\int_0^\infty e^{-\frac{1}{2}\chi^2} \chi^{l-1} d\chi}{\int_0^\infty e^{-\frac{1}{2}\chi^2} \chi^{l-1} d\chi} \dots\dots\dots(\text{xxx})$$

is the chance of a sample occurring with as great or greater deviation as the χ_0 sample from the general population. This is the expression from which the Tables of "Goodness of Fit" were calculated, the arguments being χ_0^2 and l , i.e. the value of χ^2 for the sample and the total number of categories in the sample. Thus far there is only difference of method of deduction, not of results.

(6) We now propose to replace condition (xxx) by a series of q linear equations of form (viii). These in the case of sampling will, if the size of the sample be fixed, either directly or indirectly involve (xxx).

The type of these equations in their prepared form is

$$k_{11}X_1 + k_{12}X_2 + \dots + k_{1n}X_n + \dots - K_1.$$

Each such plane will intersect

$$\chi^2 = X_1^2 + X_2^2 + \dots + X_n^2 + \dots$$

in a sphere of lower order. For example, if there be n variates X , the first plane gives a sphere of the $(n-1)$ th order, this will be intersected by the second plane in a sphere of the $(n-2)$ th order, so that ultimately we find ourselves reduced to a sphere of the $(n-q)$ th order, by the intersection of the q th plane. If $K_1, K_2, \dots, K_q$ were all zero, the radius of the sphere of the $(n-q)$ th order would be the same, i.e. χ^2 as the radius of the sphere of the n th order. But since these quantities are usually not zero we have to determine the radius of this sphere. The centre of this sphere must lie in every one of the q planes of the n th order, and accordingly on the plane of order $n-(q-1)$ in which they intersect. But the centre of the sphere of $n-q$ order is where the perpendicular, K , from the origin meets this plane of the $(n-q-1)$ th order, and the radius χ' of the sphere of the $(n-q)$ th order is given by $\chi'^2 = \chi^2 - K^2$. To determine χ' we must find P .

Now P will be the minimum distance from the origin to the plane of the $(n-q-1)$ th order in which the q planes intersect. In other words to find P we must make

$$D^2 = X_1^2 + X_2^2 + \dots + X_n^2 + \dots + X_m^2$$

a minimum subject to the q conditions of type

$$k_{11}X_1 + k_{12}X_2 + \dots + k_{1n}X_n + \dots + k_{1m}X_m = K_1,$$

where

$$k_{11}^2 + k_{12}^2 + \dots + k_{1n}^2 + \dots + k_{1m}^2 = 1.$$

Using the method of indeterminate multipliers we find n equations of the form

$$X_i + \lambda_1 k_{1i} + \lambda_2 k_{2i} + \dots + \lambda_q k_{qi} = 0.$$

* *Phil. Mag.* Vol. L. p. 158.

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Multiply by X , and add the series, and we find

$$-K^2 = \lambda_1 K_1 + \lambda_2 K_2 + \dots + \lambda_q K_q \dots\dots\dots(\text{xxxii}).$$

Multiply by k_1 , and add the series, and we find, by aid of (x),

$$-K_1 = \lambda_1 + \lambda_2 \cos(12) + \lambda_3 \cos(13) + \dots + \lambda_q \cos(1q).$$

Similarly:

$$-K_2 = \lambda_1 \cos(21) + \lambda_2 + \lambda_3 \cos(23) + \dots + \lambda_q \cos(2q),$$

$$\dots\dots\dots -K_t = \lambda_1 \cos(t1) + \lambda_2 \cos(t2) + \dots + \lambda_t + \dots + \lambda_q \cos(tq),$$

$$\dots\dots\dots -K_q = \lambda_1 \cos(q1) + \lambda_2 \cos(q2) + \dots + \dots + \dots + \lambda_q.$$

These are q equations to find $\lambda_1, \lambda_2, \dots, \lambda_q$, and we can then substitute in (xxxii) to find the required P^2 .

Now consider the determinant

$$R = \begin{vmatrix} 1, & K_1, & K_2, & K_3, & \dots & K_q \\ K_1, & 1, & \cos(12), & \cos(13), & \dots & \cos(1q) \\ K_2, & \cos(21), & 1, & \cos(23), & \dots & \cos(2q) \\ K_3, & \cos(31), & \cos(32), & 1, & \dots & \cos(3q) \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ K_q, & \cos(q1), & \cos(q2), & \cos(q3), & \dots & 1 \end{vmatrix} \dots(\text{xxxiii}),$$

and let us call the first row and the first column the 0 row and 0 column, then clearly

$$-\lambda_t = -R_{0t}/R_{00},$$

where R_{st} is the minor of the s th row and t th column, and

$$-K^2 = \frac{K_1 R_{01} + K_2 R_{02} + \dots + K_t R_{0t} + \dots + K_q R_{0q}}{R_{00}},$$

$$1 - K^2 = R/R_{00},$$

or

$$K^2 = 1 - R/R_{00} \dots\dots\dots(\text{xxxiv}).$$

If we call Δ the minor R_{00} , we have:

$$R_{0t} = -K_1 \Delta_{1t} - K_2 \Delta_{2t} - \dots - K_t \Delta_{tt} - \dots$$

Thus

$$K^2 = S\left(K_t^2 \frac{\Delta_{tt}}{\Delta}\right) + 2S\left(K_t K_r \frac{\Delta_{tr}}{\Delta}\right) \dots\dots\dots(\text{xxxv}).$$

From this we deduce that the probability of a sample which gives $\chi^2 = \chi_0^2$ with q linear conditions must be obtained from

$$P = \frac{\int_{\chi_0^2 - K^2}^{\infty} e^{-\frac{1}{2}\chi^2} \chi^{n-q-1} d\chi}{\int_0^{\infty} e^{-\frac{1}{2}\chi^2} \chi^{n-q-1} d\chi} \dots\dots\dots(\text{xxxvi}).$$

We must therefore in order to find P enter the Tables of "Goodness of Fit" with $\chi^2 = \chi_0^2 - K^2$ and with $n' = n - q + 1^*$.

The reader who is familiar with the theory of the multiple total and partial correlation coefficients will note how closely analogous the formulae (xxxiii) to (xxxv) are to results in that theory. The fundamental determinant of the correlation may be made to agree with Δ , if we merely write $r_{ss'} = \cos(ss')$. In fact both theories really reduce to the discussion of the formulae of spherical trigonometry in multiple space.

(7) A simple illustration of the above formulae may be taken from the case of the distribution of mortality in two districts, where the problem is to ascertain the probability that the difference of mortality observed allowing for the frequency of the age groups could be due to random sampling.

Let the population sampled be represented by

$$(a) \quad \begin{array}{c|c|c|c|c|c|c|c} D_1 & D_2 & D_3 & \dots & D_s & \dots & D_u & \Delta \\ \hline L_1 & L_2 & L_3 & \dots & L_s & \dots & L_u & \Lambda \\ \hline A_1 & A_2 & A_3 & \dots & A_s & \dots & A_u & P \end{array}$$

where D_s are the dead, L_s the living and A_s the exposed to risk in the s th age group, there being u such groups in the total population P , of whom in the given period Δ die and Λ survive.

Let the districts be represented by

$$(b) \quad \begin{array}{c|c|c|c|c|c|c|c} d_1 & d_2 & d_3 & \dots & d_s & \dots & d_u & \delta \\ \hline l_1 & l_2 & l_3 & \dots & l_s & \dots & l_u & \lambda \\ \hline a_1 & a_2 & a_3 & \dots & a_s & \dots & a_u & p \end{array}$$

and

$$(c) \quad \begin{array}{c|c|c|c|c|c|c|c} d'_1 & d'_2 & d'_3 & \dots & d'_s & \dots & d'_u & \delta' \\ \hline l'_1 & l'_2 & l'_3 & \dots & l'_s & \dots & l'_u & \lambda' \\ \hline a'_1 & a'_2 & a'_3 & \dots & a'_s & \dots & a'_u & p' \end{array}$$

respectively. Then the problem is to ascertain the probability that the last two distributions could both be samples of the same first population.

The general formula has been given by me†; we have to evaluate

$$\chi^2 = S \left\{ \frac{\left(\frac{f_i}{p} - \frac{f'_i}{p'} \right)^2}{\left(\frac{1}{p} + \frac{1}{p'} \right) \frac{F_i}{P}} \right\},$$

* The tables are constructed for $n' - 1$ independent variables; in our case there are $n - q$ such variables, hence $n' - 1 = n - q$.

† *Biometrika*, Vol. VIII. p. 252.

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where f_i must take the value of every frequency in the cells of (β) , f'_i the corresponding cell frequency for (γ) , and F_i that for (α) . We have to seek for P under the argument $\pi' = 2u$ in our tables, if we have no restrictions on our variates. Actually we have such restrictions, for we are going to seek the *partial* χ^2 when we suppose the age groups in each sample to be *constant*. Now

$$d_i + l_i = a_i \text{ and } d'_i + l'_i = a'_i.$$

Thus it follows that

$$\frac{d_i}{p} - \frac{d'_i}{p'} + \frac{l_i}{p} - \frac{l'_i}{p'} = \frac{a_i}{p} - \frac{a'_i}{p'} \dots\dots\dots(\text{xxxvii}).$$

We have accordingly u equations of condition or π' as argument will be reduced to $u + 1$. Now we take

$$X_i = \sqrt{\frac{pp'}{p+p'}} \frac{\left(\frac{d_i}{p} - \frac{d'_i}{p'}\right)}{\sqrt{D_i/P}},$$

$$Y_i = \sqrt{\frac{pp'}{p+p'}} \frac{\left(\frac{l_i}{p} - \frac{l'_i}{p'}\right)}{\sqrt{L_i/P}}.$$

Thus

$$\chi^2 = S_1^2 (X_i^2) + S_1^2 (Y_i^2),$$

with u conditions of form

$$\sqrt{D_i/P} X_i + \sqrt{L_i/P} Y_i = \sqrt{\frac{pp'}{p+p'}} \left(\frac{a_i}{p} - \frac{a'_i}{p'}\right),$$

or in the prepared form

$$\sqrt{\frac{D_i}{A_i}} X_i + \sqrt{\frac{L_i}{A_i}} Y_i = \sqrt{\frac{pp'}{p+p'}} \frac{\left(\frac{a_i}{p} - \frac{a'_i}{p'}\right)}{\sqrt{\frac{A_i}{P}}}.$$

Thus

$$K_i = \sqrt{\frac{pp'}{p+p'}} \left(\frac{a_i}{p} - \frac{a'_i}{p'}\right) / \sqrt{\frac{A_i}{P}}.$$

Further, all the cosines like $\cos(ss')$ are zero, for no equations of condition involve the same variates*. Thus

$$\begin{aligned} K^2 &= K_1^2 + K_2^2 + \dots + K_u^2 \\ &= \frac{pp'}{p+p'} S_1^2 \frac{P}{A_i} \left(\frac{a_i}{p} - \frac{a'_i}{p'}\right)^2 \dots\dots\dots(\text{xxxviii}). \end{aligned}$$

Accordingly

$$\chi_0^2 = \chi^2 - K^2 = \frac{pp'}{p+p'} \left\{ S_1^2 \frac{P}{D_i} \left(\frac{d_i}{p} - \frac{d'_i}{p'}\right)^2 + S_1^2 \frac{P}{L_i} \left(\frac{l_i}{p} - \frac{l'_i}{p'}\right)^2 - S_1^2 \frac{P}{A_i} \left(\frac{a_i}{p} - \frac{a'_i}{p'}\right)^2 \right\}.$$

We shall now substitute from the relation (xxxvii), getting rid of $\frac{l_i}{p} - \frac{l'_i}{p'}$.

* This will necessarily be true if our equations of condition refer to parallel rows or columns, not if they refer to certain rows and columns.

We find

$$\begin{aligned}\chi_0^2 &= \frac{pp'}{p+p'} \left\{ S_1^2 \frac{PA_s}{D_s L_s} \left(\frac{d_s}{p} - \frac{d'_s}{p'} \right)^2 - \frac{2P}{L_s} \left(\frac{d_s}{p} - \frac{d'_s}{p'} \right) \left(\frac{a_s}{p} - \frac{a'_s}{p'} \right) + \frac{PD_s}{A_s L_s} \left(\frac{a_s}{p} - \frac{a'_s}{p'} \right)^2 \right\} \\ &= \frac{pp'}{p+p'} S_1^2 \left\{ \frac{PA_s}{D_s L_s} \left[\frac{d_s}{p} - \frac{d'_s}{p'} - \frac{D_s}{A_s} \left(\frac{a_s}{p} - \frac{a'_s}{p'} \right) \right]^2 \right\}.\end{aligned}$$

Let $\frac{D_s}{A_s} = p_s$, $\frac{L_s}{A_s} = q_s$, then we may write

$$\chi_0^2 = \frac{pp'}{p+p'} S_1^2 \left\{ \frac{P}{A_s p_s q_s} \left(\frac{d_s - p_s a_s}{p} - \frac{d'_s - p'_s a'_s}{p'} \right)^2 \right\}.$$

Now if we know the population sampled, we have only to insert the values of P , A_s , and p_s , q_s to obtain the value of χ_0^2 . But if we do not, and this is usually the case, then our problem is: Are the two districts samples of the same unknown general population? For example, we might enquire whether the death distributions in Bradford and Leeds were, correcting for age, significantly different. It might be supposed that it would be correct to give A_s/P , p_s and q_s , the values found for all England and Wales. But it may be doubted whether this would be satisfactory. The populations of Bradford and Leeds might fairly be considered as samples of a general population which is very far indeed from being that of all England. Accordingly it seems much more reasonable to suppose that they are samples of a population whose mortality characters will be best represented by the combination of those of the two districts themselves, or we take

$$p_s = \frac{d_s + d'_s}{a_s + a'_s}, \quad \frac{A_s}{P} = \frac{a_s + a'_s}{p + p'}.$$

Substituting the first of these relations we find

$$\chi_0^2 = S_1^2 \left(\frac{\frac{a_s a'_s}{p p'}}{\frac{a_s + a'_s}{p + p'} \frac{A_s}{P}} \right) \left(\frac{a_s a'_s \left(\frac{d_s}{a_s} - \frac{d'_s}{a'_s} \right)^2}{(d_s + d'_s) \left(1 - \frac{d_s + d'_s}{a_s + a'_s} \right)} \right) \dots\dots (xxxix).$$

If we substitute the second relation, (xxxix) becomes

$$\chi_0^2 = S_1^2 \left\{ \frac{\frac{a_s a'_s}{p p'}}{\left(\frac{a_s + a'_s}{p + p'} \right)^2} \frac{a_s a'_s \left(\frac{d_s}{a_s} - \frac{d'_s}{a'_s} \right)^2}{(d_s + d'_s) \left(1 - \frac{d_s + d'_s}{a_s + a'_s} \right)} \right\} \dots\dots\dots (xl),$$

and this, I take it, is the best measure of significance in the difference of the death distributions allowing for age groups in the two districts.

It is clear that the first factor will in many cases differ but little from unity. For we should anticipate that approximately

$$a_s = \frac{p}{P} A_s, \quad a'_s = \frac{p'}{P} A_s.$$

Hence it would follow that

$$\frac{a_i a_i'}{p p'} = \left(\frac{A_i}{P}\right)^2 = \left(\frac{a_i + a_i'}{p + p'}\right)^2,$$

or the first factor is approximately unity. The value of χ_0^2 then becomes

$$Q^2 = S_1^* \left\{ \frac{a_i a_i' \left(\frac{d_i}{a_i} - \frac{d_i'}{a_i'} \right)^2}{(d_i + d_i') \left(1 - \frac{d_i + d_i'}{a_i + a_i'} \right)} \right\} \dots\dots\dots (xli),$$

a quantity which I have shown elsewhere* has an important relation to the question of the significant difference of corrected death *rates*.

It is obvious that the present method is of wide generality and I propose to illustrate it later by considering whether the general death distributions of various national groups allowing for age distribution, occupation distribution and character of mortality are or are not significantly different.

Meanwhile (xxxix) provides us with a formula which enables us readily in the case of districts, special diseases or class groups to assert whether mortality experiences corrected for age distribution are or are not significantly different. As far as I can see (xxxix) and its extensions to occupation groups provide the proper means of ascertaining whether the populations at risk in various insurance offices or friendly societies are or are not materially different in character. It should be a guide to the actuary also as to which classes of the population it is most desirable to cater for.

I have to thank my friends Mr H. E. Soper and Mr A. W. Young for suggestions and help at several points. In the following paper numerical illustrations of (xl) are provided.

* See *Biometrika*, Vol. **xx** p. 164.