

down the dotted line to 204, then to 203, 15, 16, 14 (in place of 200), 199; then across the diagram and upward, observing the same methods, back to 216. This gives us the numbers which constitute our square No. I, written from left to right in successive rows. In like manner the diagrams in column II give us square No. II, and so on to the end. It is worthy of notice that in the fourth column of diagrams the numbers are written in the reverse of their natural order. This is because it was necessary in writing the fourth square to begin with the number 145 (which naturally would be at the bottom of the diagram) in order to give the initial numbers the desired sum of 651.

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A NEW METHOD FOR MAKING MAGIC SQUARES OF AN ODD DEGREE.

In an endeavor to discover a general rule whereby all forms of magic squares might be constructed, and thereby to solve the question as to the possible number of squares of the fifth order, a method was devised whereby squares may be made, for whose construction the rules at present known to the writer appear to be inadequate.

A *general rule*, however, seems as yet to be unattainable; nor does the solution of the possible number of squares of an order higher than four seem to be yet in sight, though, because of the discovery, so to speak, of hitherto unknown variants, the goal must, at least, have been brought nearer to realization.

The new method now to be described does not pretend to be other than a partial rule, i. e., a rule by which most, but possibly not *all* kinds of magic squares may be made. It is based on De La Hire's method, i. e., on the implied theory that a normal magic square is made up of two primary squares, the one superimposed on the other and the numbers in similarly placed cells added together. This theory is governed by the fact that a given series of numbers may be produced by the consecutive addition of the terms of two or more diverse series of numbers. For example, the series of natural numbers from one to sixteen may be regarded (*a*) as a single series, as stated, or (*b*) as the result of the addition, successively, of all the terms of a series of eight terms to those of another series of two terms. For example, if series No. 1 is composed of 0-1-2-3-4-5-6

and 7 and series No. 2 is composed of 1 and 9, all the numbers from 1 to 16 may be thus produced. Or (c) a series of four numbers, added successively to all the terms of another series of four numbers, will likewise produce the same result, as for example 0-1-2 and 3, and 1-5-9 and 13.

Without undertaking to trace out the steps leading up to the rule to be described, we will at once state the method in connection with a 5×5 square. First, two primary squares must be made, which will hereafter be respectively referred to as the A and B primary squares. If the proposed magic square is to be *regular*, that is, if its complementary couplets are to be arranged geometrically equidistant from the center, the central cell of each square must naturally be occupied by the central number of the series of which the square is composed. The two series in this case may be 1-2-3-4-5 and 0-5-10-15-20. The central number of the first series being 3 and of the second series 10, these two numbers must occupy the central cells of their respective squares.

	3			
3				
		3		
				3
			3	

Fig. 1.

			10	
				10
		10		
10				
	10			

Fig. 2.

				3
	3			
		3		
			3	
3				

Fig. 3.

In each of these squares, each of the terms of its series must be represented five times, or as many times as the series has terms. Having placed 3 and 10 in their respective central cells, four other cells in each square must be similarly filled. To locate these cells, any geometrical design may be selected which is *balanced* about the central cell. Having done this in primary square A the *reverse* of the same design must be taken for primary square B, two examples being shown in Figs. 1 and 2 and Figs. 3 and 4.

Having selected a design, the next step will be to fill the *central* row, which may be done by writing in any of the four empty cells in this row, any of the four remaining terms of the series. The opposite cell to the one so filled, must then be filled with the complementary number of the one last entered. Next, in either of the two remaining empty cells, write either of the remaining two terms

of the series, and, in the last empty cell the then remaining number, which will complete the central row as shown in Fig. 5. All the other rows in the square must then be filled, using the same *order*

10				
			10	
		10		
	10			
				10

Fig. 4.

	3			
3				
4	1	3	5	2
				3
			3	

Fig. 5.

1	3	5	2	4
3	5	2	4	1
4	1	3	5	2
5	2	4	1	3
2	4	1	3	5

Fig. 6.

of numbers as in this *basic* row, and the square will be completed as shown in Fig. 6. The second square can then be made up with the numbers of its series in exactly the same way, as shown in Fig. 7.

5	15	0	10	20
20	5	15	0	10
15	0	10	20	5
10	20	5	15	0
0	10	20	5	15

Fig. 7.

6	18	5	12	24
23	10	17	4	11
19	1	13	25	7
15	22	9	16	3
2	14	21	8	20

Fig. 8.

				3
	3			
		3		
			3	
3				

Fig. 9.

Adding together the terms of Figs. 6 and 7, will give the regular 5×5 magic square shown in Fig. 8, which can not be made by any previously published rule known to the writer. Another example

4	5	1	2	3
2	3	4	5	1
1	2	3	4	5
5	1	2	3	4
3	4	5	1	2

Fig. 10.

10	5	0	20	15
0	20	15	10	5
20	15	10	5	0
15	10	5	0	20
5	0	20	15	10

Fig. 11.

14	10	1	22	18
2	23	19	15	6
21	17	13	9	5
20	11	7	3	24
8	4	25	16	12

Fig. 12.

may be given to impress the method on the student's mind, Fig. 9 showing the plan, Figs. 10 and 11 the A and B primary squares, and Fig. 12 the resulting magic square. Any odd square can be readily

made by this method, a 7×7 being shown. Fig. 13 shows the plan, Figs. 14 and 15 being the primary squares and 16 the complete example. Returning to the 5×5 square, it will be seen that in filling out the central row of the A primary square Fig. 5, for the first of the four empty cells, there is a choice of 16, and next a choice

	4					
		4				
4						
			4			
						4
				4		
					4	

Fig. 13.

7	4	1	2	3	5	6
6	7	4	1	2	3	5
4	1	2	3	5	6	7
5	6	7	4	1	2	3
1	2	3	5	6	7	4
3	5	6	7	4	1	2
2	3	5	6	7	4	1

Fig. 14.

of four. Also for the B primary square there are the same choices. Hence we have

$$(16 \times 4)^2 = 4096 \text{ choices.}$$

In addition to this, by *reversing* the *patterns* in the two primary squares, the above number can be doubled.

35	14	28	7	42	21	0
14	28	7	42	21	0	35
0	35	14	28	7	42	21
28	7	42	21	0	35	14
21	0	35	14	28	7	42
7	42	21	0	35	14	28
42	21	0	35	14	28	7

Fig. 15.

42	18	29	9	45	26	6
20	35	11	43	23	3	40
4	36	16	31	12	48	28
33	13	49	25	1	37	17
22	2	38	19	34	14	46
10	47	27	7	39	15	30
44	24	5	41	21	32	8

Fig. 16.

It is therefore evident that with any chosen geometrical plan, 8192 variants of regular 5×5 squares can be produced, and as at least five distinct plans can be made, 40,960 different 5×5 *regular* squares can thus be formed. This however is not the limit, for the writer believes it to be a law that all "*figures of equilibrium*" will

produce magic squares as well as *geometrically* balanced diagrams or plans.

Referring to Fig. 17, if the circles represent equal weights connected as by the dotted lines, the system would balance at the center of the square. This therefore is a "figure of equilibrium" and it may be used as a basis for magic squares, as follows: Fill the marked cells with a number, as for example 1 as in Fig. 18; then

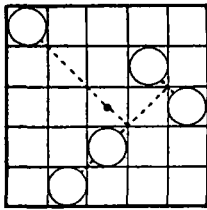


Fig. 17.

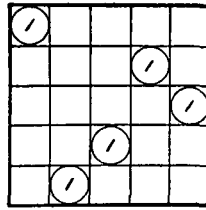


Fig. 18.

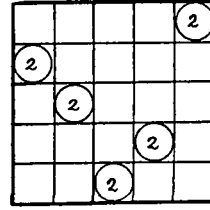


Fig. 19.

with the other numbers of the series, (excepting only the central number) make three other similar "figures of equilibrium" as shown separately in Figs. 19, 20 and 21, and collectively in Fig. 22. The four cells remaining empty will be geometrically balanced, and must be filled with the middle terms of the series (in this instance 3) thus completing the A primary square as shown in Fig. 23. Fill the B primary square with the series 0-5-10-15-20 in the same manner as

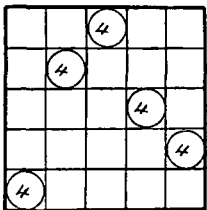


Fig. 20.

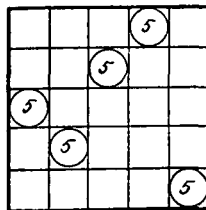


Fig. 21.

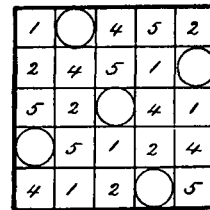


Fig. 22.

above described and as shown in Fig. 24. The combination of Figs. 23 and 24 produces the regular magic square given in Fig. 25.

There are at least five different "figures of equilibrium" that can be drawn in a 5×5 square, and these can be readily shown to give as many variants as the geometrical class, which as before noted yield 40,960 different squares. This number may therefore now be doubled raising the total to 81,920 regular 5×5 magic

squares, that are capable of being produced by the rules thus far considered.

The student must not however imagine that the possibilities of this method are now exhausted, for a further study of the subject

1	3	4	5	2
2	4	5	1	3
5	2	3	4	1
3	5	1	2	4
4	1	2	3	5

Fig. 23.

5	0	15	10	20
10	20	0	15	5
20	15	10	5	0
15	5	20	0	10
0	10	5	20	15

Fig. 24.

6	3	19	15	22
12	24	5	16	8
25	17	13	9	1
18	10	21	2	14
4	11	7	23	20

Fig. 25.

will show that a geometrical pattern or design may often be used not only with its own reverse as shown, but also with another *entirely*

0	5	15	20	10
5	15	20	10	0
15	20	10	0	5
20	10	0	5	15
10	0	5	15	20

Fig. 26.

3	1	2	4	5
5	3	1	2	4
4	5	3	1	2
2	4	5	3	1
1	2	4	5	3

Fig. 27.

2	4	1	3	5
3	5	2	4	1
4	1	3	5	2
5	2	4	1	3
1	3	5	2	4

Fig. 28.

different design, thus rendering our search for the universal rule still more difficult.

3	6	17	24	15
10	18	21	12	4
19	25	13	1	7
22	14	5	8	16
11	2	9	20	23

Fig. 29.

2	9	16	23	15
8	20	22	14	1
19	21	13	5	7
25	12	4	6	18
11	3	10	17	24

Fig. 30.

For example the pattern shown in Fig. 26 may be combined in turn with its reverse shown in Fig. 27 and also with Fig. 28, making the two regular magic squares shown in Figs 29 and 30.

In consideration of this as yet unexplored territory, therefore, the rules herein briefly outlined can only be considered as partial, and fall short of the "universal" rule for which the writer has been seeking. Their comprehensiveness however is evidenced by the fact that *any square* made by any other rule heretofore known to the

4	2	5	3	/
3	/	4	2	5
2	5	3	/	4
/	4	2	5	3
5	3	/	4	2

Fig. 31.

2	3	4	5	/
4	5	/	2	3
/	2	3	4	5
3	4	5	/	2
5	/	2	3	4

Fig. 32.

3	/	4	2	5
5	3	/	4	2
2	5	3	/	4
4	2	5	3	/
/	4	2	5	3

Fig. 33.

writer, may be made by these rules, and also a great variety of other squares, which may only be made with great difficulty, if at all, by the older methods.

To show the application of these rules to the older methods, a few squares given by Mr. Andrews in his recent book on *Magic Squares and Cubes* may be analyzed.

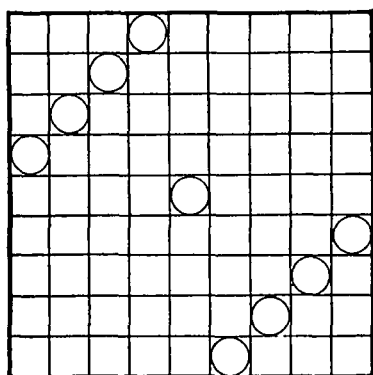


Fig. 34.

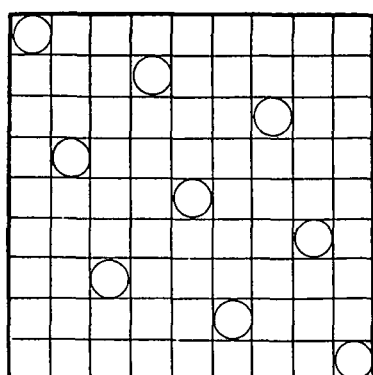


Fig. 35.

Figs. 31, 32 and 33 show the plans of 5×5 squares given in Figs. 22, 23 and 41 in the above mentioned book.

Their comprehensiveness is still further emphasized in squares of larger size, as for example in the 7×7 square shown in Fig. 16, which can not be constructed by any of the older methods known

to the writer. Two final examples are shown in Figs. 34 and 35 which give plans of two 9×9 squares which if worked out will be found to be unique and beyond the power of any other rule to produce. In conclusion an original and curious 8×8 square is submitted in Fig. 39. This square is both "regular" (in the sense of being centrally balanced) and "continuous" or "Nasik," inasmuch as all constructive diagonals give the correct summation, a combination of two qualities which is believed to be new in squares of 8×8 .

The theory upon which the writer proceeded in the construction of this square was to consider it as a compound square composed of four 4×4 squares, the latter being in themselves "continuous" but not "regular." That the latter quality might obtain in the 8×8

1	14	7	12	<i>B</i>			
15	4	9	6				
10	5	16	3				
8	11	2	13				
<i>A</i>				4	15	6	9
				14	1	12	7
				11	8	13	2
				5	10	3	16

Fig. 36.

1	14	7	12	3	16	5	10
15	4	9	6	13	2	11	8
10	5	16	3	12	7	14	1
8	11	2	13	6	9	4	15
2	13	8	11	4	15	6	9
16	3	10	5	14	1	12	7
9	6	15	4	11	8	13	2
7	12	1	14	5	10	3	16

Fig. 37.

square, each *quarter* of the 4×4 square is made the exact counterpart of the similar *quarter* in the diagonally opposite 4×4 square, but turned on its axis 180 degrees.

Having in this manner made a "regular" and continuous 8×8 square composed of four 4×4 squares, each containing the series 1 to 16 inclusive, another 8×8 square, made with similar properties, with a proper number series and added to the first square term to term will necessarily yield the desired result.

Practically, the work was done as follows: In one quarter of an 8×8 square, a "continuous" (but not "regular") 4×4 square was inscribed, and in the diagonally opposite quarter another 4×4 square was written in the manner heretofore described and now illustrated in Fig. 36. A simple computation will show that in the unfilled parts of Fig. 36, if it is to be "continuous," the contents of the cells

C and D must be 29 and A and B must equal 5. Hence A and B may contain respectively 1 and 4, or else 2 and 3. Choosing 2 and 3 for A and B, and 14 and 15 for D and C, they were located as marked by circles in Fig. 37, the "regular" or centrally balanced idea being thus preserved.

The other two quarters of the 8×8 square were then completed in the usual way of making nasik 4×4 squares, thus producing the A primary square shown in Fig. 37, which, in accordance with our theory must be both "regular" and "continuous" which inspection confirms.

As only the numbers in the series 1 to 16 inclusive appear in this square, it is evident that they must be combined term by term, with another square made with the series 0-16-32-48 in order that the final square may contain the series 1 to 64 inclusive. This is accom-

0	48	32	16
48	0	16	32
16	32	48	0
32	16	0	48

Fig. 38.

1	14	55	60	35	48	21	26
15	4	57	54	45	34	27	24
58	53	16	3	28	23	46	33
56	59	2	13	22	25	36	47
18	29	40	43	52	63	6	9
32	19	42	37	62	49	12	7
41	38	31	20	11	8	61	50
39	44	17	30	5	10	57	64

Fig. 39.

plished in Fig. 38, which shows a 4×4 square both "regular" and "continuous," composed of the numbers in the above mentioned series.

At this point, two courses of operation seemed to be open, the first being to expand Fig. 38 into an 8×8 square, as in the case of the A primary square, Fig. 37, and the second being to consider Fig. 37 as a 4×4 square, built up of sixteen subsquares of 2×2 regarded as units.

The latter course was chosen as the easier one, and each individual term in Fig. 38 was added to each of the four numbers in the corresponding quadruple cells of Fig. 37, thus giving four terms in the complete square as shown in Fig. 39. For example 0 being the term in the upper left-hand cell of Fig. 38, this term was added to 1-14-15-4 in the first quadruple cell of Fig. 37, leaving these numbers

unchanged in their value, so they were simply transferred to the complete magic square Fig. 39. The second quadruple cell in Fig. 37 contains the numbers 7-12-9-6, and as the second cell in Fig. 38 contains the number 48, this number was added to each of the last mentioned four terms, converting them respectively into 55-60-57 and 54, which numbers were inscribed into the corresponding cells of Fig. 39, and so on throughout.

Attention may here be called to the "figure of equilibrium" shown in Fig. 38 by circles and its quadruple reappearance in Fig. 39 which is a complete "regular" and "continuous" 8×8 magic square, having many unique summations.

The writer wishes to express his gratitude to his friend, and fellow student, Mr. W. S. Andrews, of Schenectady, New York, for having executed the diagrams illustrating this article and other incidental assistance. It is exceedingly doubtful whether this contribution to the literature of magic squares would ever have seen the light of day without his generous aid.

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OVERLAPPING MAGIC SQUARES.*

A peculiar species of Compound Squares may be called overlapping magic squares. In these the division is not made as usual by some factor of the root into four, nine, sixteen or more subsquares of equal area, but into several subsquares or panels not all of the same size, some lying contiguous, while others overlap. The simplest specimens have two minor squares of equal measure apart in opposite corners, and in the other corners two major squares which overlap at the center, having as common territory a middle square 2×2 , 3×3 , or larger, or only a single cell. Such division can be made whether the root of the square is a composite or a prime number, as 4-5-9; 4-6-10; 5-6-11; 6-9-15; 8-12-20 etc. The natural series 1 to n^2 may be entered in such manner that each subsquare shall be magic by itself, and the whole square also magic to a higher or lower degree. For example the 9-square admits of division into two minor squares 4×4 , and two major squares 5×5 which overlap in the center having one cell in common. For convenience, the process of construction may begin with an orderly arrangement of materials.

* The diagrams have been drawn by Mr. W. S. Andrews of Schenectady, New York.