

SOME APPLICATIONS OF THE TWO-THREE BIRATIONAL SPACE TRANSFORMATION

By T. L. WREN.

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IN the following paper certain properties of the configuration of n straight lines meeting a common transversal have been established with the aid of the two-three birational space transformation. This transformation, first investigated by Cayley*, has been used by Noëther† in mapping certain types of surface point for point upon a plane.

The present paper consists of four parts. In Part I an account is given of the general properties of the transformation used in the subsequent work. All the results are derived from first principles without any reference to the properties of birational transformation in general.

Part II is devoted to the establishment of certain existence theorems for surfaces of order n with $(n-1)$ -ple or $(n-2)$ -ple straight line, and for the rational space curve of order n with an $(n-1)$ -point secant. The results of sections 6–9 may be summarised as follows, the word “line” being understood to mean “straight line” throughout the paper.

(i) A surface of order n with $(n-2)$ -ple line is uniquely determined by $2n-1$ lines meeting the multiple line.

(ii) A surface of order n with $(n-1)$ -ple line is uniquely determined by p lines meeting the multiple line and $3n-2p$ points.

(iii) A curve of order n with an $(n-1)$ -point secant is determined uniquely by p points and $3n+1-2p$ lines meeting the multiple secant.

The proof in each case is by induction, the transformation being used to reduce the order of the surface or curve while preserving its type. The results just related are thus deduced from the corresponding theorems for the quadric surface and straight line. In none of these cases does the

* *Collected Papers*, Vol. VII, p. 230.

† *Math. Ann.*, Vol. III, p. 559.

proof involve consideration of elements of either space corresponding to fundamental elements of the other. In section 10 are discussed some relations between ruled surfaces and rational curves of the type already considered.

The method of sections (6-9) is really equivalent to a succession of applications of the two-three transformation, the principal fundamental line remaining the same throughout. The transformation resulting from such a succession of operations is dealt with in Part III, where it is shewn to lead to the plane constructions used by Noëther* in classifying the curves upon the surfaces of the types here considered.

The results of Parts II and III are essentially given by Noëther in the paper just referred to. They were collected by the present writer with a view to using the two-three transformation in an attempt to extend the line configuration theorem of Part IV indefinitely. For this purpose it was desirable to establish the uniqueness of the surfaces and curves as determined by the given conditions in sections 6-9. Noëther shews that in each case the given conditions determine at least one construct of the type considered, but does not prove definitely that this is the only one. For example, consider a surface of order n having a given $(n-2)$ -ple line. Its equation contains $6n-3$ independent constants, so that the surface may be made to pass through $6n-3$ arbitrary points. If these $6n-3$ points be taken in threes on $2n-1$ lines meeting the given multiple line it appears that there will always be at least one surface of the required type containing the $2n-1$ lines. There may, however, be a linear system of such surfaces, as we have no justification for assuming that the $6n-3$ linear equations to determine the coefficients are linearly independent when the $6n-3$ points lie by threes on straight lines. Noëther appears to assume that this is the case. The proof by elementary methods used in sections 6-9 of this paper establishes with certainty not only the existence, but also the uniqueness of the constructs as determined by the configurations considered. It is interesting to note the mutual relations between the curves and surfaces discussed in sections 10, 12.

Part IV deals with a theorem of line configurations. If a straight line l be taken, any four lines meeting l determine another line which meets all four. Again, if we take five lines meeting l , every four of the five lines determines a new line as above. Then, by the well known theorem of the double-six (incidentally established anew in section 15) these five new lines are all met by a straight line, which we may regard as determined by the original set of five. The double-six plays an important part

* *Math. Ann.*, Vol. III, pp. 183, 195.

in the geometry of the cubic surface. It has also been discussed in relation to a certain quadric by Mr. G. T. Bennett,* and an elementary proof not involving the cubic surface has been given by Prof. Baker.†

Now suppose we start with six lines meeting l ; every five determine in the above manner a new line. The question arises whether these six new lines are all met by a common transversal; and if so is there a similar theorem for seven lines, and so on universally. In section 15 we use the two-three transformation to deduce the five-line theorem from the four-line theorem; and in the succeeding sections 16, 17 we establish the six-line theorem with a single application of the transformation. For the six lines we get in the course of the construction a configuration consisting of two sets of 12 and 32 lines respectively; such that every line of the 32 is met by 6 of the 12, and every line of the 12 by 16 of the 32.

For the next case, that of seven lines with a common transversal, I have so far been unable to establish the theorem. The figure obtained for the construction of the seven new lines determined by every six of the original seven involves 64 lines, and the figures resulting from one or two applications of our transformation are extremely complicated and difficult to visualise.

The theorem for six lines meeting a common transversal is a particular case of the corresponding theorem for six linear complexes having a line in common. Any four of these complexes have another line in common; hence from a set of five of the six complexes we get five lines which determine a linear complex. From the six original complexes we thus get six sets of five, and so obtain six new complexes, which have one common line. This has been proved by Mr. J. H. Grace‡ by considerations of cubic threefolds in four dimensional space. Mr. Grace tells me that he has also considered the cases of seven and of eight lines.

In conclusion, I wish to express my thanks to Prof. Baker for his kindness in reading through the paper and in suggesting improvements.

I. *The Two-Three Transformation.*

1. Consider in a space (X) the system of an arbitrary straight line G and three arbitrary points 1, 2, 3. Through these pass four linearly independent quadric surfaces Θ, Φ, Ψ, F . Any quadric through the system will

* *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 336.

† *Proc. Roy. Soc.*, A, Vol. 84, p. 597.

‡ "Circles, Spheres, and Linear Complexes," *Camb. Phil. Trans.*, Vol. xvi, 1898, p. 163.

then be represented by an equation of form

$$l\Theta + m\Phi + n\Psi + pF = 0.$$

Any two of these quadrics intersect, besides the line G , in a cubic curve passing through the points 1, 2, 3 and having G for a chord. A third quadric of the family meets the cubic curve in six points of which two are on G and three at 1, 2, 3. Thus three of these quadrics intersect in one point other than the fundamental elements. If then we put

$$\frac{x}{\Theta} = \frac{y}{\Phi} = \frac{z}{\Psi} = \frac{t}{F}$$

to any set of ratios $x : y : z : t$ corresponds one point of the X -space other than the fundamental elements. We shall thus be able to deduce from these equations a set of equations of the form

$$\frac{X}{\mathfrak{S}} = \frac{Y}{\phi} = \frac{Z}{\psi} = \frac{T}{f},$$

where \mathfrak{S} , ϕ , ψ , f are polynomials homogeneous in x , y , z , t . The degree of these polynomials will be the number of common solutions of three equations of the form

$$a\mathfrak{S} + b\phi + c\psi + df = 0,$$

$$a'x + b'y + c'z + d't = 0,$$

$$a''x + b''y + c''z + d''t = 0,$$

that is, of

$$aX + bY + cZ + dT = 0,$$

$$a'\Theta + b'\Phi + c'\Psi + d'F = 0,$$

$$a''\Theta + b''\Phi + c''\Psi + d''F = 0;$$

and therefore the order of the residual curve of intersection of any two of the original quadrics, that is 3. Hence \mathfrak{S} , ϕ , ψ , f represent cubic surfaces, such that any three intersect, beyond certain fundamental elements common to all surfaces of the system

$$a\mathfrak{S} + b\phi + c\psi + df = 0,$$

in a single point. To find these fundamental elements we must specify the form of Θ , Φ , Ψ , F in more detail. We may choose the tetrahedron

of reference in the X -space so that the fundamental elements are

$$\begin{array}{ll} \text{the line } G & X = Y = 0, \\ \text{the points } \left\{ \begin{array}{l} 1 \quad X = Z = T = 0, \\ 2 \quad Y = Z = T = 0, \\ 3 \quad X - Z = Y - Z = T = 0. \end{array} \right. \end{array}$$

Then the plane-pairs

$$XT = 0, \quad YT = 0, \quad X(Y - Z) = 0, \quad Y(X - Z) = 0$$

are four linearly independent quadrics through the fundamental elements, and we may take these to be the quadrics Θ, Φ, Ψ, F , putting

$$\left. \begin{array}{l} x : y : z : t = XT : YT : X(Y - Z) : Y(X - Z). \\ \text{We may then readily deduce the reciprocal relations} \\ X : Y : Z : T = x(xt - yz) : y(xt - yz) : xy(t - z) : xy(x - y). \end{array} \right\} \quad (I)$$

The fundamental elements of the x -space are then seen to be the four straight lines

$$\begin{aligned} g \quad & x = y = 0, \\ (1) \quad & x = z = 0, \\ (2) \quad & y = t = 0, \\ (3) \quad & x - y = t - z = 0. \end{aligned}$$

The surfaces of the system $a\mathfrak{S} + b\phi + c\psi + df = 0$ are then seen to be cubic surfaces having g for double line, and (1), (2), (3) for generating lines. Any two of these surfaces meet besides the fundamental elements in a conic meeting g , (1), (2), (3). A third surface of the family meets this conic in a single point not lying on any of the fundamental lines, and this is the point x, y, z, t corresponding to the point X, Y, Z, T of intersection of the corresponding planes in the X -space. In fact the sets of equations (I) determine a birational transformation between the spaces (X) , (x) ; we now proceed to consider the properties of this transformation.

The transformation is such that to the planes of x correspond the quadric surfaces in X which have G for a generator and pass through the points 1, 2, 3: while to the planes of X correspond in x cubic scrolls having g for double line and (1), (2), (3) for generators.

In general, to a point of either space corresponds a unique point of

the other: to an arbitrary straight line in x corresponds a space cubic in X , having G for chord and passing through the points 1, 2, 3: to an arbitrary straight line in X corresponds in x a conic meeting each of the fundamental lines g , (1), (2), (3) in a single point.

We have now to consider what elements of each space may be regarded as corresponding to the fundamental elements of the other, so as to be able to determine the construct in the former corresponding to a construct meeting or containing fundamental elements of the latter. To find the elements of x corresponding to the fundamental elements of X we replace X, Y, Z, T by $X+\epsilon\xi, Y+\epsilon\eta, Z+\epsilon\zeta, T+\epsilon\tau$ in the first set of relations (I): thus obtaining the set

$$\left. \begin{aligned} \rho x &= XT + \epsilon(\xi T + \tau X) + \epsilon^2 \xi \tau, \\ \rho y &= YT + \epsilon(\eta T + \tau Y) + \epsilon^2 \eta \tau, \\ \rho z &= X(Y-Z) + \epsilon\{\xi(Y-Z) + (\eta-\zeta)X\} + \epsilon^2 \xi(\eta-\zeta), \\ \rho t &= Y(X-Z) + \epsilon\{\eta(X-Z) + (\xi-\zeta)Y\} + \epsilon^2 \eta(\xi-\zeta). \end{aligned} \right\} \quad (II)$$

In this set give X, Y, Z, T values corresponding to a point on a fundamental construct, and then let ϵ tend to zero. The limiting ratios of $x : y : z : t$ give the coordinates of a point which may be considered to correspond to the point in question when approached in a definite direction determined by the ratios $\xi : \eta : \zeta : \tau$. Thus we find that

(i) To a given point ($Z = \mu T$) of G corresponds that generator of the quadric $xt - yz = 0$ determined by the equations

$$z + \mu x = t + \mu y = 0.$$

The point in which this generator is met by the plane $x = \lambda y$ corresponds to all directions of approach in the plane $X = \lambda Y$.

(ii) To every direction through the point 1 corresponds a point of the plane $x = 0$ given by $x : y : z : t = 0 : \tau : \xi : \xi - \zeta$.

(iii) To a direction through 2 corresponds a point of $y = 0$ given by $x : y : z : t = \tau : 0 : \eta - \zeta : \eta$.

(iv) To a direction through 3 corresponds a point of $x - y = 0$ given by $x : y : z : t = \tau : \tau : \eta - \zeta : \xi - \zeta$.

Proceeding similarly for the fundamental elements of the x -space, we

replace the second set of relations (I) by

$$\begin{aligned}
 \rho X &= x(xt-yz) + \epsilon [\xi(xt-yz) + x(x\tau+t\xi-y\xi-z\eta)] \\
 &\quad + \epsilon^2 [\xi(x\tau+t\xi-y\xi-z\eta) + x(\xi\tau-\eta\xi)] + \epsilon^3 \xi(\xi t-\eta\xi), \\
 \rho Y &= y(xt-yz) + \epsilon [\eta(xt-yz) + y(x\tau+t\xi-y\xi-z\eta)] \\
 &\quad + \epsilon^2 [\eta(x\tau+t\xi-y\xi-z\eta) + y(\xi\tau-\eta\xi)] + \epsilon^3 \eta(\xi\tau-\eta\xi), \\
 \rho Z &= xy(t-z) + \epsilon [xy(\tau-\xi) + (t-z)(x\eta+y\xi)] \\
 &\quad + \epsilon^2 [(t-z)\xi\eta + (x\eta+y\xi)(\tau-\xi)] + \epsilon^3 \xi\eta(\tau-\xi), \\
 \rho T &= xy(x-y) + \epsilon [xy(\xi-\eta) + (x-y)(x\eta+y\xi)] \\
 &\quad + \epsilon^2 [(x-y)\xi\eta + (x\eta+y\xi)(\xi-\eta)] + \epsilon^3 \xi\eta(\xi-\eta).
 \end{aligned} \tag{III}$$

It appears then that we must regard as corresponding—

(v) To the point $z = \mu t$ of g , the conic

$$\mu YZ - XZ - (\mu - 1)XY = 0, \quad T = 0;$$

so that to every direction of approach in the plane $x = \lambda y$ corresponds the point where the conic is met again by the plane $X = \lambda Y$.

(vi) To the point $y = \mu t$ of (1), the line $X = T + \mu Z = 0$.

(vii) „ „ $x = \mu z$ „ (2), „ „ $Y = T + Z\mu = 0$.

(viii) „ „ $t = \mu x$ „ (3), „ „ $X = Y = Z + \mu T$.

(ix) „ „ $\{g.1\}$, $\{g.2\}$, $\{g.3\}$, respectively the straight lines $[2.3]$, $[3.1]$, $[1.2]$.

2. From the original form of the equations of transformation we see at once that to the pencil of planes through G corresponds the pencil of planes through g : and to the pencils of planes through (1), (2), (3) correspond respectively the pencils through $[2.3]$, $[3.1]$, $[1.2]$.

Thus to a straight line meeting the line (1) corresponds a conic passing through 2, 3 and meeting g : and similarly for lines meeting (2) or (3).

Again to a straight line meeting both (1) and (2) corresponds a straight line through the point 3.

Finally, to a straight line meeting all three lines (1), (2), (3) corresponds a definite point of G [see (i) above].

Further, to an arbitrary line $ay + bz + ct = 0$ in the plane $x = \lambda y$, corresponds a straight line $AY + BZ + CT = 0$ in the plane $X = \lambda Y$; where $A : B : C = (c + b)\lambda : -(c + b\lambda) : a$. Hence it appears that to a

given curve in the first plane corresponds in general a curve of the same order in the second; such that to every singularity of the first corresponds a singularity of the same type on the second curve.

II. Existence Theorems.

3. *The most general surface of order n having an $(n-2)$ -ple straight line contains $3n-4$ pairs of straight lines lying in planes through the multiple line.*

Taking $x = y = 0$ as the multiple line, the equation to the surface may be expressed in the form

$$t^2\alpha + zt\beta + z^2\gamma + t\delta + z\epsilon + \eta = 0, \quad (\text{IV})$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \eta$ are homogeneous polynomials in x, y of orders $n-2, n-2, n-2, n-1, n-1, n$ respectively. Thus the surface appears as a locus of conics in planes $x = \lambda y$, and $3n-4$ of these break up into line-pairs, namely those for which λ satisfies the equation

$$\begin{vmatrix} 2\alpha & \beta & \delta \\ \beta & 2\gamma & \epsilon \\ \delta & \epsilon & 2\eta \end{vmatrix} = 0,$$

where x, y are replaced by $\lambda, 1$ respectively.

4. Take three lines, one from each of three of these $3n-4$ pairs as the lines (1), (2), (3) of the x -space, as defined in section 1. The equation to the surface may now be written in the form

$$t^2xa + tzb + z^2yc + txd + zye + xyf = 0, \quad (\text{V})$$

where a, b, c, d, e, f are homogeneous polynomials in x, y of orders $n-3, n-2, n-3, n-2, n-2, n-2$ respectively, such that

$$f = e + d = a + b + c = 0,$$

when $x = y$. These conditions are equivalent to

$$\left. \begin{aligned} f &\equiv (x-y)A, \\ e+d &\equiv (y-x)B, \\ ye+xd &\equiv (x-y)D, \\ ya+b+xc &\equiv (x-y)C, \\ xa+b+yc &\equiv (x-y)F, \\ 2xy(a+c)+(x+y)b &\equiv (y-x)E, \end{aligned} \right\} \quad (\text{VI})$$

where A, B, \dots, F are polynomials homogeneous in x, y ; A, B, D being perfectly general, while C, E, F are subject to the single condition that $C + E + F = 0$, when $x = y$.

Applying the transformation (I) to (V), we get, in virtue of (VI), after removing common factors,

$$T^2A + ZTB + Z^2C + TD + ZE + XYF = 0. \quad (\text{VII})$$

This is the most general form of equation to a surface of order $n-1$ having G for $(n-3)$ -ple line, and passing through the points 1, 2, 3. This surface will have $3n-7$ pairs of straight lines in planes through G , corresponding to the remaining $3n-7$ pairs on the original surface of order n .

Thus it appears that *the transformation (I) reduces the most general surface of order n having g for $(n-2)$ -ple line and containing the lines (1), (2), (3) to the most general surface of order $n-1$ with G for $(n-3)$ -ple line and passing through the points 1, 2, 3.*

5. If in the preceding section all the coefficients in the polynomials α, b, c be supposed zero, we obtain in place of (V), the equation

$$txd + yze + f = 0, \quad (\text{VIII})$$

where $f = e + d = 0$, when $x = y$. This represents a general scroll of order n with g as $(n-1)$ -ple line, and (1), (2), (3) as generators. A factor T can now be removed from (VII), leaving

$$TA + ZB + D = 0. \quad (\text{IX})$$

This is the most general equation to a scroll of order $n-2$ with G as $(n-3)$ -ple line, not in general passing through the points 1, 2, 3. Thus *the transformation reduces the scroll of order n with g for $(n-1)$ -ple line, and (1), (2), (3) as generators to a scroll of order $n-2$ with G as $(n-3)$ -ple line.*

From the results of this and the preceding section, we derive the property referred to by Noëther, *Math. Ann.*, III, p. 561; namely, that by successive applications of the transformation the surface of the former type may be reduced to a quadric, and the surface of the latter type to a quadric or a plane according as n is even or odd.

6. We are now in a position to prove that—*One and only one surface of order n exists, having a given $(n-2)$ -ple line G and containing $2n-1$ arbitrary lines which meet G but not one another.*

The form of equation (IV) shews that when the multiple line is given the surface may be made to satisfy $6n-3$ independent conditions. Now a line meeting the multiple line will lie wholly on the surface if the equation to the surface be satisfied by the coordinates of some three points on the line, giving three linear relations between the coefficients. Thus given a system of $2n-1$ lines meeting the line G there will always be at least one surface of the type required containing these lines: and only one such surface provided the $6n-3$ conditions for the $2n-1$ lines to lie upon the surface are linearly independent.

Suppose that for some value $n = N$ this is the case, so that our theorem is true for $n = N$. On replacing the three conditions that one of the $2N-1$ lines A, B, C, \dots should lie on the surface by the three conditions, also linear in the coefficients, for the surface to pass through three arbitrary points there can be no loss of independence among the $6N-3$ relations which determine the coefficients. Thus we see that a system of $2N-2$ lines A, B, C, \dots meeting G together with three arbitrary points also determine a unique surface of order N with G for $(N-2)$ -ple line.

Take now at random a system of $2N+1$ lines $(1), (2), (3), a, b, c, \dots$ all meeting a line g . There are then ∞^2 ways of choosing the coordinate system, so that the equations to $g, (1), (2), (3)$ are those given in section 1.* Applying the transformation, to any surface of order $N+1$ with g as $(N-1)$ -ple line and containing the $2N+1$ lines $(1), (2), (3), a, b, c, \dots$, there corresponds in the X -space a surface of order N with G as $(N-2)$ -ple line on which lie the points 1, 2, 3 and the $2N-2$ lines A, B, C, \dots . These $2N-2$ lines correspond respectively to a, b, c, \dots , and therefore have no special relations of position among themselves or with respect to 1, 2, 3; for the lines a, b, c, \dots were only subject to the condition of meeting g . Hence, on our hypothesis, there is in the X -space only one surface of order N satisfying the required conditions: and consequently in the x -space but one surface of order $N+1$ of the type considered.

Thus, if our theorem holds for $n = N$, it is true also for $n = N+1$. But it is well known to be true for $n = 3$. Indeed this result may be deduced as above from the fact that a quadric surface is uniquely determined by two non-intersecting generators and three points. Hence *the theorem is true generally for all values of n .*

7. The analogous theorem for the scroll of order n with $(n-1)$ -ple line takes different forms according to the parity of n . The form of

* $z = t = 0$ may be any line meeting $(1), (2)$, and (3) , and the unit point any point of (3) .

equation (IX) shews that, the multiple line G being given, the surface may be made to satisfy $3n$ independent conditions. A line meeting G will lie wholly on the surface if the equation be satisfied by the coordinates of any two of its points, giving two linear relations between the coefficients.

(i) When n is even we have—*A unique scroll of order n exists having a given $(n-1)$ -ple line G and $3n/2$ arbitrary lines meeting G as generators.*

(ii) When n is odd—*A unique scroll of order n exists having a given $(n-1)$ -ple line G , containing $(3n-1)/2$ arbitrary lines meeting G , and passing also through a single arbitrary point.*

In either case the proof is on the lines of the preceding section: first, that there is at least one surface satisfying the required conditions, since these involve relations linear in the coefficients: secondly, the uniqueness of this surface is established by induction. Case (i) is true when $n = 2$, since a unique quadric is determined by three non-intersecting lines; Case (ii) is true for $n = 1$, since a plane is uniquely determined by a straight line and a point.

Notice that on replacing the conditions for the surface to contain one of the lines by the conditions that it may pass through two arbitrary points, there is no loss of independence among the $3n$ linear equations which determine the coefficients. We may then combine (i) and (ii) into a single theorem which holds for all values of n , odd or even—

A surface of order n having a given $(n-1)$ -ple line G is uniquely determined by p generators meeting G and $3n-2p$ arbitrary points ($p < 3n/2$).

8. We shall now establish analogous existence theorems for the rational space curve of order n having $n-1$ points on a given straight line g . *In general there is one and only one such curve meeting each of $3n+1$ arbitrary lines which intersect g .*

For, with the notation of section 1 the most general rational C_n having $n-1$ points on g and meeting (1), (2), (3), can be represented by the equations

$$x : y : z : t = L_1 P_{n-1} : M_1 P_{n-1} : L_1 M_1 R_{n-2} - L_1 Q_{n-1} : L_1 M_1 R_{n-2} - M_1 Q_{n-1},$$

where $L_1, M_1, P_{n-1}, Q_{n-1}, R_{n-2}$ are polynomials in a single parameter λ , of degrees denoted by their suffixes, and such that no two vanish for the same value of λ . The transformation gives for the corresponding curve

in the X -space the equations

$$X : Y : Z : T = L_1 R_{n-2} : M_1 R_{n-2} : Q_{n-1} : P_{n-1}.$$

These represent the most general curve of order $n-1$, having $n-2$ points on G (and *not* in general passing through the points 1, 2, 3).

Suppose that the theorem is true for a particular value $n = N$, and take a system of $3N+4$ arbitrary lines (1), (2), (3), a, b, c, \dots meeting a line g . Then the tetrahedron of reference may be chosen in one of ∞^2 ways so that the equations to g , (1), (2), (3) are those of section 1. If there is any curve of order $N+1$ with N points on g and meeting (1), (2), (3), a, b, c, \dots the corresponding construct in the X -space will be a curve of order N with $N-1$ points on G . The latter curve must also meet each of the $3N+1$ lines A, B, C, \dots which correspond to a, b, c, \dots . These lines A, B, C, \dots are subject to no further relations of position beyond that of meeting G . Hence in the X -space there is, according to our hypothesis, a unique curve of order N satisfying the required conditions. There will thus be at least one curve of order $N+1$ with N points on g and meeting each of the original $3N+4$ lines. This will indeed be the only such curve: for, although there might be others which by reason of special relations with the coordinate-tetrahedron give simpler corresponding constructs in the X -space, by a change of coordinates one or more of these could be generalised without bringing the first into any special relationship. We should then have more than one curve C_N of type considered in the X -space, which is contrary to the hypothesis that the theorem holds for $n = N$. Thus, if the theorem is true for $n = N$, it is also true for $n = N+1$.

It is well known that a system of four skew straight lines possesses two common transversals. Starting with one of these we may regard the other as uniquely determined by the system of four lines meeting a fifth. This is the form of our theorem for $n = 1$. Hence it is true for $n = 2$ (for an independent proof of this case see Clebsch, *Math. Ann.*, I, p. 258), and so on generally.

9. The theorem of the preceding section may now be used to establish the more general theorem—*Given p arbitrary points and $3n+1-2p$ arbitrary straight lines meeting a line G , there is in general a unique curve of order n with $n-1$ points on G passing through each of the p points and meeting each of the $3n+1-2p$ lines.*

The most general curve of order $n+2$ in the X -space having $n+1$

points on G and passing through the points 1, 2, 3 is represented by

$$X : Y : Z : T = L_1(L_1 P_n - M_1 Q_n) : M_1(L_1 P_n - M_1 Q_n) : L_1 M_1 (P_n - Q_n) \\ : L_1 M_1 (L_1 - M_1) R_{n-1},$$

where $L_1, M_1, P_n, Q_n, R_{n-1}$ are polynomials in a single parameter λ , and no two vanish for the same value of λ . The corresponding curve in the x -space is then given by

$$x : y : z : t = L_1 R_{n-1} : M_1 R_{n-1} : Q_n : P_n,$$

which represent the most general curve of order n with $n-1$ points on g .

From this result and the analogous property used in the preceding section it follows as before that if the theorem is true for N, P it is also true for $N+2, P+3$ and for $N+1, P$. Thus we may reduce our theorem to depend upon those cases for which p and $3n+1-2p$ are each less than 3 (the theorem for $p=0$ having already been established independently). We have then to consider the cases—

(i) $p=1, n=1$. The theorem holds for this case since there is a unique line through a given point meeting each of two given lines.

(ii) $p=2, n=1$. The theorem is true here since a unique straight line passes through two given points.

Since the theorem is true for these simple cases it will be true generally.

10. By comparison between the theorems of the last three sections, we may deduce some characteristic properties of the type of ruled surface of the type considered in section 7. These surfaces fall into two distinct classes according as they are of odd or even order. In this section we shall use S_n to denote a scroll of order n having a given line g as $(n-1)$ -ple line; and C_n a rational curve of order n having $n-1$ points on the line g . These are the only types of surface and curve with which we shall here be concerned.

(i) Start with an arbitrary system of one point and $3m+1$ lines meeting g . This system determines a unique S_{2m+1} , while the $3m+1$ lines alone determine a unique C_m , which lies wholly on the S_{2m+1} , since they have already $2m(m-1)+3m+1 = m(2m+1)+1$ common points. The curve C_m then meets every generator of the surface S_{2m+1} . *It is thus the only curve C_m on the surface*, for any $3m+1$ generators are independent as not sufficing to fix the surface.

Again, any $3m$ generators of S_{2m+1} form an independent system, and so determine a unique S_{2m} which also contains the curve C_m , having common with it $(2m-1)(m-1)+3m=2m^2+1$ points. *The curve C_m thus appears as the residual intersection of the surface S_{2m+1} by every surface S_{2m} having $3m$ generators common with it.*

There are on the surface S_{2m+1} no curves of type C_n of order lower than m : for such a curve would have to meet all generators, and therefore every one of an independent set of $3m+1$, which is contrary to the theorem that a curve C_p is determined uniquely by $3p+1$ independent lines meeting g .

For $m=1$ we have the cubic scroll, all whose generators meet the double line and one other straight line, usually called the directrix. In the general case the curve C_m may be regarded as the directrix of S_{2m+1} .

(ii) Start now with a system of $3m$ straight lines meeting g . This system determines a unique S_{2m} .

Taking $3m-1$ of the $3m$ lines and adding some one point of the remaining line, we have a perfectly independent system which determines a unique C_m lying wholly on the surface S_{2m} . Thus through every point of a generator of S_{2m} passes one such C_m , and there is a pencil of these curves on the surface. For $m=1$, we have for this pencil the opposite system of generators of the quadric.

Again, any $3m-2$ generators of S_{2m} taken with an arbitrary point P of the surface constitute an independent system, and so determine a unique S_{2m-1} whose residual intersection with S_{2m} is the curve C_m through P . *The curves C_m thus appear as residual intersections of S_{2m} by the surfaces S_{2m-1} having common $3m-2$ generators of S_{2m} .*

Just as before there are on the surface S_{2m} no curves of this type of order lower than m .

III. Repeated Application of the Two-Three Transformation.

11. Let us now consider the properties of the birational transformation obtained by N successive applications of the transformation of section 1; at each step the line g_k being taken to be the line G_{k-1} of the last transformation, and the lines $(1)_k, (2)_k, (3)_k$ being any three meeting g_k .

To the properties of the transformation already noted must be added the following:—*To a surface of order n with $(n-1)$ -ple line g for which $(1), (2), (3)$ are not generators corresponds in the X -space a surface of order $n+1$ with n -ple line G passing through the points 1, 2, 3. For the*

scroll of order n in the x -space is represented by

$$tA_{n-1} + zB_{n-1} + C_n = 0,$$

where A_{n-1} , B_{n-1} , C_n are polynomials homogeneous in x, y . The equation to the equivalent construct in the X -space is then

$$Y(X-Z)A_{n-1} + X(Y-Z)B_{n-1} + TC_n = 0,$$

where A, B, C are now expressed in terms of X, Y . This is the most general equation to a scroll of order $n+1$ with n -ple line G , passing through 1, 2, 3.

In virtue of the results already obtained, we find that the result of N successive transformations is a transformation between spaces x_1 , X_N , such that—

(i) To planes of x_1 correspond in X_N surfaces of order $N+1$ with N -ple line G_N and passing through $3N$ fundamental points a_i .

(ii) To straight lines of x_1 correspond space curves of order $2N+1$ with $2N$ points on G_N and passing through the $3N$ points a_i .

(iii) To planes of X_N correspond in x_1 surfaces of order $2N+1$ with $2N$ -ple line g_1 , on which lie $3N$ fundamental lines A_i .

(iv) To straight lines of X_N correspond space curves of order $N+1$ with N points on g_1 meeting each of the lines A_i .

By combining N transformations taken directly with N' taken inversely, so that $G_N _ G_{N'}$, but the points a_i all different from the points a'_i , we obtain a more general transformation in which there correspond—

(i) To planes of x_1 , surfaces in x'_1 of order $N+2N'+1$ with $(N+2N')$ -ple line g'_1 , passing through $3N$ fundamental points β'_i and containing $3N'$ fundamental lines B'_i which meet g'_1 .

(ii) To planes of x'_1 , surfaces in x_1 of order $2N+N'+1$ with $(2N+N')$ -ple line g_1 , passing through $3N'$ points β_i and containing $3N$ lines B_i meeting g_1 .

This transformation gives at once a point for point representation on a plane of the surface of order n with $(n-1)$ -ple line—

(i) For odd $n = 2N+1$, taking $N' = 0$, we get the simplest representation, in which plane sections of the surface are represented by curves of order $N+1$ with fixed N -ple point.

(ii) For even $n = 2N+2$, taking $N' = 1$, plane sections of the surface are represented by curves of order $N+3$ with a fixed $(N+2)$ -ple point and three ordinary points fixed. By a Cremona transformation of the plane this can be reduced to a system of curves of order $(N+2)$, having common an $(N+1)$ -ple and one ordinary point.

These are the plane representations used by Noëther* in classifying the curves on the surface.

12. For the surface of order n with $(n-2)$ -ple straight line a point for point representation upon a quadric is obtained by means of $n-2$ successive transformations. To plane sections of the original surface correspond the curves in which the quadric is cut by surfaces of order $n-1$ with $(n-2)$ -ple line G_{n-2} which pass also through the $3(n-2)$ fundamental points α_i . These curves, of order $2(n-1)$, have $n-2$ branches through each of the points A, B in which the quadric is met by G_{n-2} . They have $n-1$ points on a generator of either system, since the lines of the space X_{n-2} correspond to space curves of order $n-1$ in the x_1 -space.

Now project points of the quadric from A on to an arbitrary plane. The curves in question project into plane curves of order n having fixed an $(n-2)$ -ple and $3n-4$ simple points (the sections of the two generators through A and the projections of the $3n-6$ points α_i). This is the simplest form of plane representation of the surface, as used by Noëther.†

The characteristic property from which Noëther obtains the plane representation may be stated as follows:—*Every set of $3n-5$ lines taken one out of each of $3n-5$ of the $3n-4$ line-pairs on the surface determines on the surface a unique curve of order $n-2$ with $n-3$ points of intersection with G_{n-2} . This curve meets each of the $3n-5$ lines and one line of the remaining pair. There are in all 2^{3n-5} such curves on the surface.* This may be deduced from the property of the general cubic surface by the method of induction used in sections 6-8.

Assuming the property to be true for $n = N$, consider a surface of order $N+1$ with g for $(N-1)$ -ple line, and (1), (2), (3) members of three of its $3N-1$ line-pairs. The corresponding surface in the X -space is of order N with $(N-2)$ -ple line G : its $3N-4$ line pairs correspond to the $3N-4$ on the original surface remaining after those containing (1), (2), (3) are removed. We have assumed that some $3N-5$ lines taken one from each of $3N-5$ of the line-pairs on the surface of order N determine a

* *Math. Ann.*, III, p. 195.

† *Math. Ann.*, III, p. 183.

unique curve of order $N-2$ meeting G in $N-3$ points and lying on the surface. To this corresponds in the x -space a definite curve of order $N-1$ with $N-2$ points on g , intersecting each of the corresponding $3N-5$ lines and the three lines (1), (2), (3); and conversely. It appears then that the curve of order $N-1$ is determined uniquely by the system of $3N-2$ lines on the surface of order $N+1$, and the theorem is true for $N+1$ if it is for N .

In the case of the cubic surface $n = 3$, the theorem is known to be true. Through a chosen line of the surface pass five planes cutting it again in line-pairs; a set of four lines taken one from each of four pairs determines a unique straight line which lies wholly on the surface, and therefore meets one of the lines of the remaining pair. Hence the theorem is true for $n = 4$, and so on universally.

We have already seen (section 8) that a perfectly arbitrary system of $3n-5$ lines meeting a common transversal g determines a unique curve of order $n-2$ having $n-3$ points on g . If now the $3n-5$ lines be not perfectly independent, but lie all on the surface of order n with $(n-2)$ -ple line g which is determined by some $2n-1$ of them, the last theorem shews that they still determine a unique curve of order $n-2$, which however lies on the surface.*

IV. *Straight Line Configurations—an Extension of the Theorem of the Double-Six.*

13. The theorem of the double-six may be conveniently enunciated in the following form:—Suppose we are given an arbitrary system of five straight lines a_1, a_2, a_3, a_4, a_5 , all meeting a straight line g . Then every set of four out of the five lines a is met by another straight line b_k , where the suffix k is that of the fifth line a . The five lines b thus determined, all meet a further straight line d which does not meet g ; the position of the line d being completely determined when the original system of g and the five lines a is given.

We now proceed to establish the analogous theorem for a system of six arbitrary lines $a_1, a_2, a_3, a_4, a_5, a_6$ meeting a line g . Every set of five out of the six lines a determine as above a line d_k , the suffix k being that of the sixth line. It will be proved that there exists a straight line meeting all the six lines d thus obtained.

* See Prof. Baker, "Some Recent Advances in the Theory of Algebraic Surfaces," *Proc. London Math. Soc.*, Ser. 2, Vol. 12, footnote on p. 36.

14. Consider a system of n arbitrary straight lines meeting a straight line g . Any three of these may be taken to be the lines (1), (2), (3) of section 1. The transformation then gives, corresponding to the remaining $n-3$ lines a, b, c, \dots , a system of $n-3$ straight lines A, B, C, \dots , subject only to the condition of meeting G . Every set of four of the n lines of the first set is met besides g by a second transversal, which may be regarded as determined by the four lines; and similarly for the $n-3$ lines of the second set. The following notation will be adopted:—

(i) $\{abcd\}$ to denote the second common transversal determined by the four lines a, b, c, d .

(ii) $\{P.ab\}$ the straight line through a point P meeting both a and b .

(iii) $\{g.a\}$ the point where a meets g .

Then, referring to section 2, we may tabulate the elements of the X -space which correspond to transversals of four of the n lines in the x -space according to the types—

- (i) To a line $\{(1)(2)(3)a\}$ corresponds the point $\{G.A\}$.
- (ii) „ $\{(1)(2)ab\}$ „ line $\{3.AB\}$.
- (iii) „ $\{(1)abc\}$ „ conic passing through the points 2, 3, and meeting lines G, A, B, C .
- (iv) „ $\{abcd\}$ corresponds the space cubic passing through the points 1, 2, 3, meeting A, B, C, D , and having two points on G . (Section 1, p. 149.)

15. Take now the system of five straight lines $a_1 \equiv (1)$, $a_2 \equiv (2)$, $a_3 \equiv (3)$, a_4, a_5 , all meeting g . Applying the transformation the elements corresponding to the five straight lines b_1, b_2, \dots, b_5 are—

$$\begin{array}{ll}
 b_1 \equiv \{(2)(3)a_4a_5\} & \text{straight line } B_1 \equiv \{1.A_4A_5\}, \\
 b_2 \equiv \{(1)(3)a_4a_5\} & „ \quad B_2 \equiv \{2.A_4A_5\}, \\
 b_3 \equiv \{(1)(2)a_4a_5\} & „ \quad B_3 \equiv \{3.A_4A_5\}, \\
 b_4 \equiv \{(1)(2)(3)a_5\} & \text{point } \beta_4 \equiv \{G.A_5\}, \\
 b_5 \equiv \{(1)(2)(3)a_4\} & „ \quad \beta_5 \equiv \{G.A_4\}.
 \end{array}$$

Now the lines B_1, B_2, B_3 determine a unique quadric surface Q which passes through the points 1, 2, 3, β_4, β_5 . Taking on this quadric any generator l of the same system as B_1 , a pencil of quadrics can be drawn

to pass through 1, 2, 3, β_4, β_5 , and have l as common generator. Each of these quadrics meets Q besides l in a space cubic passing through 1, 2, 3, β_4, β_5 , and meeting B_1, B_2, B_3 each in a second point. Conversely, any space cubic satisfying these conditions has eight points in common with Q and therefore lies entirely on Q . It will therefore meet every generator of the same system as B_1 in two points: and consequently lies entirely on every quadric of the pencil determined by the generator l and the five points 1, 2, 3, β_4, β_5 . There is then one and only one such space cubic. To this curve corresponds in the x -space a straight line d which meets each of the five lines b_1, b_2, b_3, b_4, b_5 . We have thus incidentally established the theorem of the double-six.

16. Let us now start with a set of six arbitrary straight lines $a_1, a_2, a_3, a_4, a_5, a_6$ meeting the line g . Every set of five out of the six lines a determines a line d , and we shall find that the six lines d thus obtained are all met by a single straight line. The proof depends upon the fact that there are 15 straight lines c_{ij} ,* such that, for example, c_{12} meets $a_1, a_2, d_3, d_4, d_5, d_6$.

Consider the three sets of five lines obtained by omitting respectively a_4, a_5, a_6 . These determine three lines d_4, d_5, d_6 to which correspond in the X -space the three space cubics D_4, D_5, D_6 : where D_4 is the unique space cubic passing through the points 1, 2, 3, $\{G.A_5\}$, $\{G.A_6\}$, and having the lines $\{1.A_5A_6\}$, $\{2.A_5A_6\}$, $\{3.A_5A_6\}$ as chords; and similarly for D_5, D_6 by cyclic permutation of the suffixes 4, 5, 6. These curves D_4, D_5, D_6 are projected from the point 3 by quadric cones K_4, K_5, K_6 determined respectively by the sets of five generators—

$$K_4: [3.1], [3.2], [3.\{G.A_5\}], [3.\{G.A_6\}], \{3.A_5A_6\},$$

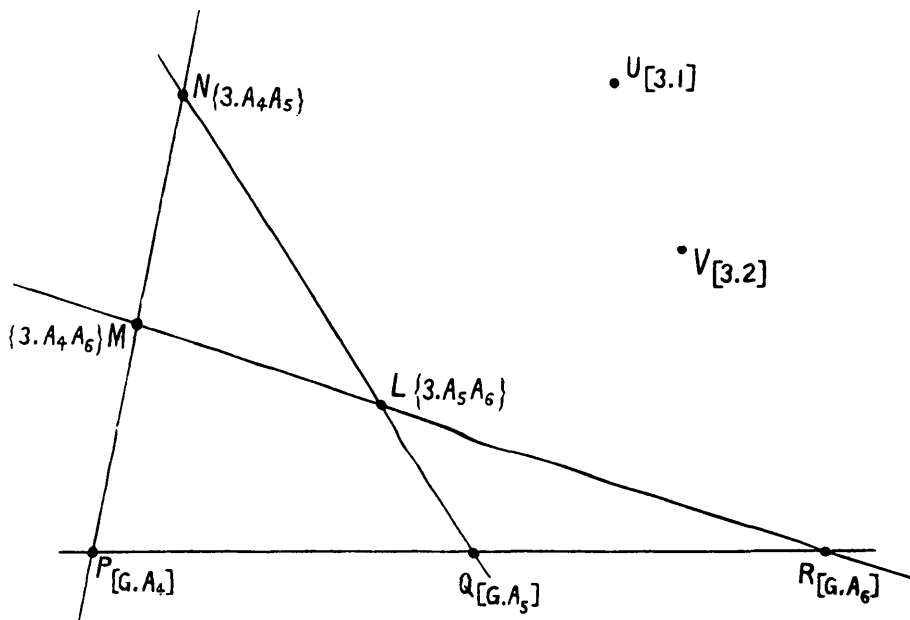
$$K_5: [3.1], [3.2], [3.\{G.A_6\}], [3.\{G.A_4\}], \{3.A_6A_4\},$$

$$K_6: [3.1], [3.2], [3.\{G.A_4\}], [3.\{G.A_5\}], \{3.A_4A_5\}.$$

Projecting from 3 on to an arbitrary plane through G , the lines A_4, A_5, A_6 project into three straight lines PMN, QNL, RLM respectively, and the lines $[3.1], [3.2]$ to the points U, V . Then the space cubics D_4, D_5, D_6 project into conics through U, V circumscribing the triangles QRL, RPM, PQN . These three conics have besides U, V a third common point W ,

* These lines must not be confused with the lines denoted by c_{ij} in the usual notation for the twenty-seven lines on the general cubic surface.

through which passes also the conic through U, V, L, M, N .* The straight line $[3. W]$ is a chord of each of the space cubics D_4, D_5, D_6 , and



therefore meets each in a further point besides 3. To this line there corresponds in the x -space a *straight line meeting each of the lines* (1), (2), d_4, d_5, d_6 . It will be seen that *this line also meets d_3* . For the elements in the X -space corresponding to the five lines $b_{31}, b_{32}, b_{34}, b_{35}, b_{36}$ which meet d_3 are—

b_{31} : the conic through 1, 3 meeting G, A_4, A_5, A_6 ,

b_{32} : ,, 2, 3 ,, G, A_4, A_5, A_6 ,

b_{34} : the straight line $\{3.A_5A_6\}$,

b_{35} : ,, $\{3.A_4A_6\}$,

b_{36} : ,, $\{3.A_4A_5\}$.

Thus to the line d_3 in general corresponds a definite space cubic D_3 passing through 1, 2, 3 and having $G, \{3.A_5A_6\}, \{3.A_4A_6\}, \{3.A_4A_5\}$ as

* If W be the fourth point common to the conics $UVQRL, UVRPM$, we have

$$R(UVLW) = Q(UVLW) \text{ and } R(UVMW) = P(UVMW).$$

But RLM, PMN, QNL are straight lines. Thus $P(UVNW) = Q(UVNW)$, and W lies also on the conic $UVPQN$. Similarly it may be shewn that the conic $UVLMN$ passes through W .

chords. This curve is therefore projected from 3 by the quadric cone K_3 determined by the generators $[3.1], [3.2], \{3.A_5A_6\}, \{3.A_4A_6\}, \{3.A_4A_5\}$. This cone projects the cubic into the conic $UVLMN$ which passes through W . Hence the line $[3.W]$ is a generator of K_3 and meets the space cubic D_3 in a further point besides 3. The corresponding line in the x -space consequently meets $a_1, a_2, d_3, d_4, d_5, d_6$; denote it by the symbol c_{12} . There will be in all 15 such lines c , one being determined by every set of four of the lines d . No two of the lines c intersect, and no line c is met by any of the 14 lines b not having the same suffix—for it is readily seen that if any two of the thirty lines having different suffixes are coplanar their plane must contain two or more of the lines a ; which is contrary to the supposition that the six lines a were taken at random.

17. Consider now the system of five lines $c_{12}, c_{13}, c_{14}, c_{15}, c_{16}$ which meet a_1 . These are met in fours by d_2, d_3, d_4, d_5, d_6 : there is then a straight line e meeting these last five. This line e must also meet d_1 ; otherwise, from the symmetry of the configuration there would be six lines of this type, different from any of the lines c , each meeting five of the lines d —in which case the lines d would all lie on a quadric surface on which the original six lines a would also lie. Thus *there exists a single straight line e meeting the six lines d* .

18. Figure 2 gives a diagrammatic representation of the configuration: the dots denote actual intersections, while the unmarked crossings correspond to apparent intersections. The symmetry of the configuration becomes at once evident. Each of the 32 lines g, b, c, e meets 6 of the 12 lines a, d : while every line of the latter group meets 16 lines of the former. It appears that the whole configuration may be obtained from any one of the 32 lines of the former group and the 6 lines of the latter that meet it—for example, starting from b_{56} and $a_1, a_2, a_3, a_4, d_5, d_6$ we finish with the line c_{56} meeting $d_1, d_2, d_3, d_4, a_5, a_6$. In fact, the 32 lines consist of 16 pairs, such that if we start from one member of a pair we finish with the other.

