XXVII. *The Lateral Vibration of Loaded Shafts in the Neighbourhood of a Whirling Speed.—The Effect of Want of Balance.* *By* H. H. Jeffcott *.

1. WHEN a shaft whose centre of mass does not lie on its axis of figure (the shaft then being said to be "out of balance"), is rotated, its geometrical axis ceases to remain straight and in coincidence with the axis of the bearings, but becomes bent and rotates round the latter axis. The amount of this bending depends on, among other things, the relation of the speed of rotation to that of any of the free lateral vibrations of the shaft. So long as the speed of rotation is not too near that of any of these vibrations the shaft remains nearly straight unless it is badly "out of balance." If, however, the speed of rotation increases so as to approach that of any transverse vibration the bending becomes noticeable,—sooner or later according to the perfection of balance of the shaft—and increases rapidly; and the shaft bends in one or more loops according to the particular transverse vibration whose speed is being approached.

If the speed is maintained constant at such a value the bending may become excessive and is ultimately restrained by the action of the bearings. And if the speed changes from one slightly below to one slightly above that of the transverse vibration the character of the motion changes.

In particular, for the slowest transverse vibration of a shaft put out of balance by a single mass, the motion changes from one in which the centre of mass is nearer to the axis of the bearings than is the elastic centre to one in which the reverse is true. In this change the shaft appears to shiver.

Well above the speed of the transverse vibration the shaft settles down again to more or less steady running with the axis nearly straight until the speed approaches that of another transverse vibration.

When the shaft is considerably bent it is said to "whirl." The speeds of transverse vibration are spoken of as the "whirling speeds."

In the foregoing when the shaft is horizontal, the straight form, or the axis of the bearings, is to be understood to refer to the gravitationally deflected form.

It is proposed in this note to discuss to some extent how want of balance causes these phenomena, and to what extent in good practice balancing should be carried. It should be

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pointed out, however, that an adequate mathematical treatment of the general case of a loaded shaft is a matter of considerable difficulty, and that the discussion which follows does not pretend to be more than illustrative.

2. For the sake of simplicity in illustrating the character of these vibrations we will consider the case of a light uniform shaft supported freely in bearings at its ends and carrying a mass m at the centre of its span, the mass centre, however, being slightly eccentric by a distance a from the elastic centre of the shaft.

We will suppose the load to be of the nature of a thin pulley or disk of negligible moment of inertia. Accordingly we are dealing with simple lateral vibration, and are not concerned with oscillatory vibration about a diameter.

The conditions given permit a statement of the problem in a simple form. The motion of the cross-section at the centre of the span in its own plane need alone be considered, subject to (a) a restoring force varying as the distance of a point known as the elastic centre from the axis of the bearings, (b) a damping force, and (c) the disturbing effect produced by an impressed rotation in its own plane combined with the fact that the centre of mass is placed eccentrically with respect to the elastic centre. Thus the problem may be viewed as the motion in the plane XY of a disk on which is impressed a constant angular velocity ω.

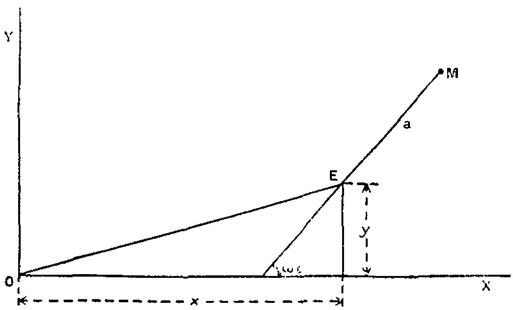

Fig. 1.

In the cross-section in fig. 1, let O be the intersection of the axis of bearings with the plane XY. Let x and y be the co-ordinates of E the elastic centre (which is also supposed to be the centre of figure) of the shaft at any instant ; and let M be the position of the centre of mass at the same instant.

It is given that the shaft rotates with angular velocity ω. This angular velocity determines the angle (ωt) which any line in the disk such as EM makes with a line fixed in space such as OX. It would not be correct in the statement of the problem to assume that the inclination of the radii vectores OE or OM changes as ωt. It will be shown later that the path of E, when steady motion is established, is a circle described with angular velocity ω. Hence O is the instantaneous centre, and the inclinations of the lines OE and OM change as ωt.

The equation of motion of m parallel to OX is

$$m\frac{d^2}{dt^2}(x+a\cos\omega t)+b\frac{dx}{dt}+cx=0,$$

where b is the coefficient of damping due to viscous resistances, and c is the elastic force of restitution at unit displacement.

Hence

$$m\ddot{x}+b\dot{x}+cx=ma\omega^2\cos\omega t.$$

The solution of this equation is well known to be

$$x=\mathrm{A}e^{-\frac{bt}{2m}}\sin(qt+\alpha)+\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}\cos(\omega t-\beta),$$

where

$$\tan\beta=\frac{b\omega}{c-m\omega^2};\quad q=\frac{\sqrt{4mc-b^2}}{2m};$$

and A and α are arbitrary constants.

In like manner the equation of motion parallel to OY is

$$m\frac{d^2}{dt^2}(y+a\sin\omega t)+b\frac{dy}{dt}+cy=0,$$

and the solution is

$$y=\mathrm{A}'e^{-\frac{bt}{2m}}\sin(qt+\alpha')+\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}\sin(\omega t-\beta),$$

where A' and α' are new arbitrary constants.

The first term in the solution represents an oscillatory motion of amplitude

$$\mathrm{A}e^{-\frac{bt}{2m}}\text{ or }\mathrm{A}'e^{-\frac{bt}{2m}}.$$

This amplitude diminishes with increase of t, so that the term becomes negligible.

The second term persists, and is a vibration of amplitude

$$\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}.$$

This forced vibration is caused by the disturbing action of the eccentric mass during rotation.

It is seen from the foregoing values of x and y that the terms representing the forced motion satisfy the relation

$$x^2+y^2=\text{constant}.$$

Accordingly the path of the elastic centre, when the oscillatory motion subsides, is a circle round the axis of the bearings of radius

$$\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}.$$

The co-ordinates of M being $x+a\cos\omega t$ and $y+a\sin\omega t$, and x and y having the values corresponding to the forced motion as already given, it follows in the same way that the path of M is also a circle round the axis of the bearings when the steady motion is established. Thus

$$x'=\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}\cos(\omega t-\beta)+a\cos\omega t,$$

$$y'=\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}\sin(\omega t-\beta)+a\sin\omega t,$$

$$\text{OM}^2=x'^2+y'^2=x^2+y^2+a^2+2a(x\cos\omega t+y\sin\omega t).$$

Now

$$x\cos\omega t+y\sin\omega t=\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}\cos\beta$$

and
$$\tan\beta=\frac{b\omega}{c-m\omega^2}.$$

Hence
$$\text{OM}=a\sqrt{\frac{c^2+b^2\omega^2}{(c-m\omega^2)^2+b^2\omega^2}}.$$

If all the quantities save ω are given the amplitude is maximum, or

$$\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}}$$

is maximum, when

$$\omega=\frac{2c}{\sqrt{4mc-2b^2}}.$$

This value of ω approximates to the speed

$$\frac{\sqrt{4mc-b^2}}{2m}$$

corresponding to the free period, when b is small.

At speeds in the neighbourhood of this value of ω the vibration, caused by a given eccentricity of mass centre, may be excessively great. This is the whirling speed of the shaft.

3. Looking next at the value of β which determines the phase of the displacement of the mass centre relatively to that of the elastic centre, we see that since

$$\tan\beta = \frac{b\omega}{c - m\omega^2},$$

the value of β is 0 when ω is zero. The value of β changes with increase of ω, until when $m\omega^2 = c$, $\beta = \frac{\pi}{2}$, or the phase of the mass displacement has now shifted by $\frac{\pi}{2}$ from that corresponding to very slow rotation.

Above this value of ω, the phase changes further, and at very high values of ω, $\beta \doteq \pi$.

Now the value of b is usually small, and on closer examination it will be seen that the whole phase-change through an angle π takes place practically entirely between a speed very slightly below and another very slightly above the critical speed. In other words, this change of phase takes place mainly within a comparatively small number of revolutions per minute on either side of the critical speed.

Thus at speeds appreciably below the whirling speed the shaft rotates with the mass centre farther from the axis of rotation than the elastic centre, while at speeds appreciably above the critical speed the mass centre is closer to the axis.

4. This result can easily be obtained directly by considering the steady motion in these extreme cases and omitting the damping altogether, as is well known.

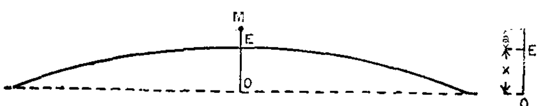

Fig. 2.

Thus in fig. 2, let $OE = x$, $EM = a$, where O, E, M represent the axis of rotation, the elastic centre, and the centre of mass. Equating the centrifugal force to the restoring elastic force acting on m in the assumed steady motion, we obtain

$$m\omega^2(x + a) = cx, \quad \text{or} \quad x = \frac{m\omega^2 a}{c - m\omega^2},$$

so that while $m\omega^2$ is less than c, we have the condition illustrated in fig. 2.

When $m\omega^2=c$ we have x indefinitely great, and the value of ω is a critical speed.

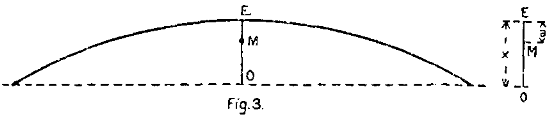

Fig. 3.

When $m\omega^2$ is greater than c, we have the condition illustrated in fig. 3. Then

$$m\omega^2(x-a)=cx \quad \text{and} \quad x=\frac{m\omega^2 a}{m\omega^2-c}.$$

The transition from one condition to the other in passing through the critical speed is clearly brought out above when the damping is included.

5. The phenomena observed at balancing, as to the relation of the (marked) point of maximum deflexion on the shaft's periphery to the position of the mass centre, at speeds below, in the neighbourhood of, or above the whirling speed are well known to concord with the above phenomena due to damping, and a reversal of the direction of rotation gives results confirmatory of this.

Thus in machines designed for testing the dynamic balance of rotors, the part to be balanced is revolved in bearings which are free to move to and fro between controlling spring or rubber buffers which limit the amount of their motion in a horizontal plane when the shaft vibrates.

A scriber or pencil is held near the shaft at the section being examined, and thus it is marked at that part of its circumference that is most deflected.

It is observed that at all speeds below the critical speed an unbalanced body will be marked on the heavy side ; and conversely above the critical speed it will be marked on the light side.

The mark, however, is not at the exact heavy or light spot, but it is displaced angularly round the shaft, more or less according to the proximity to the critical speed. The direction of this angular displacement depends on the direction of revolution of the body, and if displaced in one direction by revolving clockwise, it will be equally displaced in the opposite direction by revolving counter clockwise at the same speed. Thus the true heavy or light spot lies midway between the centres of the markings in these two cases.

In most cases, when balancing, the machine is operated by running above the critical speed, thereby marking on the light side.

It is to be noted, however, that the cases of balancing met with in practice are usually complicated by having several masses along the shaft instead of only one as considered above, so that the resultant displacement at any section then depends on the joint action of all the loads, according to their several effects at that section.

6. It should be pointed out that, at speeds appreciably below the critical value, the phase-change due to damping is small. The introduction of the coefficient b in the equation of motion is mainly of interest in showing how the mass centre changes from being outside to being inside the elastic centre, relatively to the axis of rotation, as the speed rises from a little below to a little above the whirling speed; and also that the amplitude of vibration at the critical speed in the ideal case under consideration, in which no restraint to excessive vibration is assumed to be imposed by the bearings, would not become indefinitely great.

Returning now to the amplitude of the vibration, which is

$$\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}},$$

its value is not seriously altered by the omission of b except in the close neighbourhood of the value $\omega=\sqrt{\frac{c}{m}}$.

At this critical value of ω, the effect of b is to diminish the amplitude and prevent it increasing to an indefinitely great value, as has just been stated.

7. If $b=0$, and also $m\omega^2=c$, the equation of motion is

$$\ddot{x}+\omega^2 x=a\omega^2\cos\omega t,$$

of which the solution is

$$x=\mathrm{A}\sin(\omega t+\alpha)+\tfrac{1}{2}a\omega t\sin\omega t,$$

where A and α are arbitrary constants.

The amplitude of the forced vibration would in this case increase continually with t. But the damping b is not zero, and the amplitude therefore does not increase indefinitely.

It is worthy of note, however, that if a shaft be run quickly through its whirling speed to a higher working speed, there may not be time for serious vibration to take place; in fact, the value of t may be so small that the amplitude of this forced vibration $\frac{1}{2}a\omega t$ may never be great.

8. Apart from a knowledge of the numerical value of b in any particular calculation, we cannot obtain the exact value of the amplitude of vibration; but, as we have seen, it is only in the close proximity of the whirling speed that the damping seriously modifies the amplitude, and we may omit it at all except such speeds.

Also when not too close to the whirling speed we may regard the displacements of the mass centre and elastic centre as taking place along the same radius-vector from the axis of rotation, which likewise may be obtained from the general solution by putting $b=0$.

Let u be the ultimate amplitude of the vibration at the elastic centre, then

$$u=\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}} \doteq \frac{ma\omega^2}{\pm(c-m\omega^2)}.$$

Put $k=\sqrt{\frac{c}{m}}=$speed of free vibration.

If $\omega<k$, we have

$$u=\frac{\frac{a}{k^2}}{\frac{1}{\omega^2}-\frac{1}{k^2}}.$$

The centrifugal force is

$$F=m\omega^2(u+a)=\frac{ma}{\frac{1}{\omega^2}-\frac{1}{k^2}}=mk^2u.$$

If $\omega>k$, we have

$$u=\frac{\frac{a}{k^2}}{\frac{1}{k^2}-\frac{1}{\omega^2}}$$

and

$$F=m\omega^2(u-a)=\frac{ma}{\frac{1}{k^2}-\frac{1}{\omega^2}}=mk^2u.$$

Plotting these results against the speed ω, we see that, with a suitable alteration in scale, the same curves represent both u and F.

The curves (fig. 4) have not been drawn close to the position $\omega = k$, as in that region the damping prevents the values becoming indefinitely great, as would appear to be the case if the formulæ used for u and F were fully plotted.

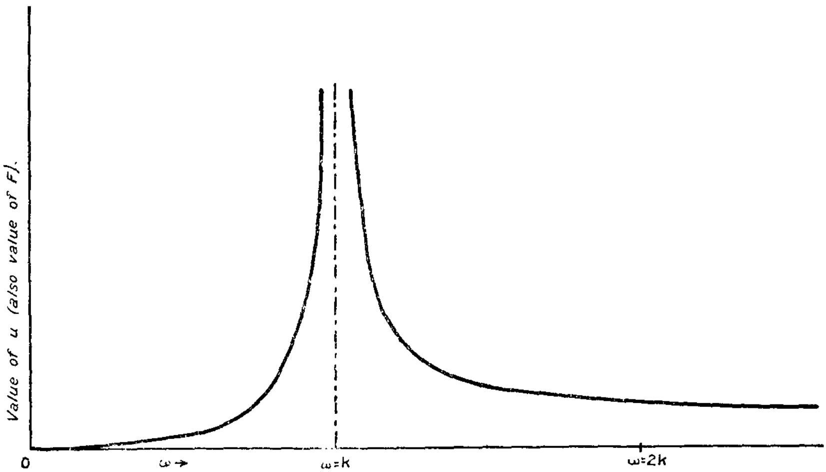

Fig. 4.

The curves exemplify clearly how that the critical speed is one that is particularly favourable to vibration, or one at which the shaft is very sensitive to lateral disturbing forces.

If in the choice of the working speed the critical speed is avoided by 10 per cent. on either side, *i. e.* if ω does not lie between $\cdot 9k$ and $1 \cdot 1k$, then the amplitude of vibration will not be more than about five times the eccentricity of the mass centre.

If the shaft is in good balance this choice of working speed will therefore give good results and sufficiently steady running.

The better the balance the closer to the critical speed may the shaft be worked. For while keeping the amplitude of vibration u within a definite limit, we may approach k the more closely the smaller the value of a; and further, the damping will keep the amplitude rather lower than that plotted.

On the other hand, if the balance is bad the critical speed must be avoided by a large margin, and it may even be dangerous to run through the critical speed to a higher working speed.

9. If we include damping we have the displacement of the shaft $u=\mathrm{OE}$, and $\mathrm{F}=m\omega^2 . \mathrm{OM}$; see fig. 1.

But
$$\mathrm{OE}=\frac{ma\omega^2}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}},$$

and
$$\mathrm{OM}=a\times\sqrt{\frac{c^2+b^2\omega^2}{(c-m\omega^2)^2+b^2\omega^2}}.$$

So that we now have after putting $k=\sqrt{\frac{c}{m}}$ as before,

$$u=\frac{ma\omega^2}{\sqrt{m^2(k^2-\omega^2)^2+b^2\omega^2}},$$

and
$$\mathrm{F}=ma\omega^2\times\sqrt{\frac{m^2k^4+b^2\omega^2}{m^2(k^2-\omega^2)^2+b^2\omega^2}}.$$

It will be noticed that the previous results are a particular case of this.

On assuming a particular value for $\frac{b}{m}$ these results may be plotted similarly to those in fig. 4.

10. Reverting now to the formulæ for centrifugal force written above (§ 8) we may compare the forces and vibrations due to a given eccentricity of mass at speeds above and below the whirling speed.

Consider the design of a shaft for a given duty and operating at a given working speed.

Then the preceding formulæ indicate that it is better from the vibration point of view to design the shaft with its critical speed below the working speed rather than to have a critical speed the same proportion above the working speed. In the latter case the shaft will of course be thicker and somewhat heavier. This result, indeed, is well illustrated in the behaviour of the De Laval steam turbines.

Thus comparing the centrifugal forces in the two cases in the particular example with which we have been dealing, let F be the centrifugal force when the critical speed k is greater than the working speed ω; and let F′ be the centrifugal force when the critical speed k' is less than the working speed; the eccentricity of mass centre a is supposed to be the same in two cases, but the masses m and m' are not quite the same, the former being somewhat greater owing to the shaft being heavier.

Then we have $$F=\frac{ma\omega^2k^2}{k^2-\omega^2}$$

and $$F'=\frac{m'a\omega^2k'^2}{\omega^2-k'^2},$$

so that $$\frac{F'}{F}=\frac{m'}{m}\times\frac{k'^2(k^2-\omega^2)}{k^2(\omega^2-k'^2)}.$$

If now $k=(1+p)\omega$, $k'=(1-p)\omega$, we find

$$\frac{F'}{F}=\frac{m'}{m}\times\frac{2-p(3-p^2)}{2+p(3-p^2)},$$

which is less than 1.

Hence F' is less than F, or the centrifugal force is less when the whirling speed is below the working speed rather than the same amount above.

11. To get a numerical idea of the degree of balance desirable in any machine, we will consider the case of a rotor consisting of a single heavy mass of 5 tons concentrated at the span centre.

Further, suppose the working speed to be 1000 r.p.m. and the critical speed 1250 r.p.m.

Then $$\frac{u}{a}=\frac{1}{1{\cdot}25^2-1}=1{\cdot}8,$$

thus $u=1{\cdot}8a$. If, therefore, the amplitude is to be limited to say ·0001 inch, a must be not more than about half that figure.

Also $$F=mk^2u=\frac{5\times 2240}{32}\times\left(\frac{1250}{9{\cdot}55}\right)^2\times\frac{1{\cdot}8}{12}a$$
$$=900{,}000a,$$

where a is inches, and F is lb. Putting $a=$ ·0001 inch, we find F=90 lb.

If the rotor be out of balance to the extent of ½ oz. at a radius of 10 inches, this corresponds to an eccentricity of the mass centre of ·00003 inch. Good balancing would realize this figure.