PROBLEM DEPARTMENT.

By. E. L. BROWN,

Principal North Side High School, Denver, Colo.

Readers of this magazine are invited to send solutions of the problems in which they are interested. Problems and solutions will be duly credited to their authors. Address all communications to E. L. Brown, 3435 Alcott Street, Denver, Colo.

Algebra.

323. *Proposed by Nelson L. Roray, Metuchen, N. J.*

In a series of equal ratios the sum of the mean proportionals of each antecedent to its consequent is equal to the mean proportional between the sum of the antecedents and the sum of the consequents.

I. *Solution by Nelson L. Roray, Metuchen, N. J., and Agnes MacNeish, Chicago, Ill.*

Put $a/b = c/d = e/f = \dots = r^2$. Then

$$ab = b^2 r^2 \text{ and } \sqrt{ab} = br$$

$$cd = d^2 r^2 \quad \sqrt{cd} = dr$$

$$ef = f^2 r^2 \quad \sqrt{ef} = fr, \text{ etc. Whence}$$

$$\sqrt{ab} + \sqrt{cd} + \sqrt{ef} + \dots = (b+d+f+\dots)r.$$

Again

$$(a+c+e+\dots)/(b+d+f+\dots) = r^2$$

$$(a+c+e+\dots)(b+d+f+\dots) = (b+d+f+\dots)^2 r^2$$

$$\text{Hence } \sqrt{(a+c+e+\dots)(b+d+f+\dots)} = (b+d+f+\dots)r \\ = \sqrt{ab} + \sqrt{cd} + \sqrt{ef} + \dots$$

II. *Solution by John M. Gallagher, Boston, Mass.*

$$\text{Let } a/b = c/d = e/f = \dots \therefore bc = ad, be = af, de = cf.$$

$$\begin{aligned} (a+c+e)(b+d+f) &= ab+ad+af+bc+cd+cf+be+de+ef. \\ &= ab+cd+ef+2ad+2af+2cf. \\ &= ab+cd+ef+2\sqrt{(ad)^2}+2\sqrt{(af)^2}+2\sqrt{(cf)^2}. \\ &= ab+cd+ef+2\sqrt{adbc}+2\sqrt{afbe}+2\sqrt{cfde}. \\ &= [\sqrt{ab}+\sqrt{cd}+\sqrt{ef}]^2. \end{aligned}$$

$$\therefore [(a+c+e)(b+d+f)]^{\frac{1}{2}} = \sqrt{ab} + \sqrt{cd} + \sqrt{ef}.$$

NOTE: This theorem can be used to find the formula for the volume of the frustum of any pyramid, when the formula for the volume of the frustum of a triangular pyramid is known.

324. *Proposed by Otto J. Ramler, Buffalo, N. Y.*

If ax^2+bx+c and $a'x^2+b'x+c'$ have a common factor of the form $x+f$, prove that

$$(ac'-a'c)^2 = (bc'-b'c)(ab'-a'b).$$

I. *Solution by T. M. Blakslee, Ames, Iowa, and H. C. McMullin, Lawrence, Kansas.*

Dividing and putting each of the remainders equal to zero we have

$$af^2-bf+c=0 \quad \text{and} \quad a'f^2-b'f+c'=0$$

$$\text{Eliminating } f \quad f^2 = (bc'-b'c)/(ab'-a'b)$$

$$\text{Eliminating } f^2 \quad f = (ac'-a'c)/(ab'-a'b)$$

$$\text{Hence } f^2 = (ac' - a'c)^2 / (ab' - a'b)^2 = (bc' - b'c) / (ab' - a'b), \\ (ac' - a'c)^2 = (ab' - a'b)(bc' - b'c)$$

II. *Solution by A. H. Hughey, El Paso, Texas, I. L. Winckler, Cleveland, Ohio, and Julia M. Bligh, Batavia, N. Y.*

Factoring each expression we have

$$ax^2 + bx + c = a \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \right) \left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \right) \\ a'^2 + b'x + c' = a' \left(x + \frac{b'}{2a'} + \frac{\sqrt{b'^2 - 4a'c'}}{2a'} \right) \left(x + \frac{b'}{2a'} - \frac{\sqrt{b'^2 - 4a'c'}}{2a'} \right) \\ \therefore \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{b'}{2a'} + \frac{\sqrt{b'^2 - 4a'c'}}{2a'}$$

Reducing we have

$$(ac' - a'c)^2 = (bc' - b'c)(ab' - a'b)$$

III. *Solution by Elmer Schuyler, Brooklyn, N. Y., Walter C. Eells, Tacoma, Wash., and W. J. Risley, Decatur, Ill.*

Using Sylvester's Dialytic Method, we have as the resultant of the two equations the determinant

$$\begin{vmatrix} 0 & a & b & c \\ a & b & c & 0 \\ 0 & a' & b' & c' \\ a' & b' & c' & 0 \end{vmatrix}$$

which must be zero if the equations have a common factor $x + f$. Expanding in terms of determinants of the second order formed from the first two columns.

$$a^2c'^2 - aa'cc' + a'^2c^2 - (ab' - a'b)(bc' - b'c) = 0$$

$$\text{Whence } (ac' - a'c)^2 = (bc' - b'c)(ab' - a'b).$$

See Bôcher's Introduction to Higher Algebra, p. 199.

Geometry.

315. Construct a triangle, given its pedal triangle.

NOTE, by R. M. Mathews, Riverside, Cal. In the two solutions of this problem published in the February number I notice an omission. The solutions are not unique, for a given triangle can be pedal triangle for four different triangles. If for the given pedal triangle ABC we construct the triangle of excenters XYZ we have one solution. But ABC is pedal triangle for any one of the three other triangles formed by two excenters and the incenter.

325. *Proposed by Levi S. Shively, Mount Morris, Ill.*

Construct a triangle, given its circumcenter, its orthocenter, and the mid-point of one of its medians.

Solution by A. M. Harding, Fayetteville, Ark., H. E. Trefethen, Waterville, Me., and I. L. Winckler, Cleveland, Ohio.

Let P, Q, R be the given points, O the centroid of the required triangle ABC. Then O is in PQ, and PO = OQ/2.

Hence RO = 1/3 the median through R. For R bisects this median, and O trisects it.

Therefore to construct the triangle; take PO = PQ/3; draw the line AROF, making AR = 3RO, OF = 2RO. A is a vertex of the triangle, F is the mid-point of its side BC. About P with radius AP describe a circle and through F perpendicular to FP draw the chord BC.

326. *Proposed by A. Babbitt, State College, Pa.*

In the triangle ABC find a point O such that the product of the perpendiculars from it upon the sides shall be a maximum.

Solution by A. H. Hughey, El Paso, Tex., Elmer Schuyler, Brooklyn, N. Y., and A. Babbitt, State College, Pa.

Put x, y, z for the several perpendiculars, K = area of ABC. Then $ax+by+cz = 2K$ (a, b, c, K being constant in a given triangle).

Hence $ax \cdot by \cdot cz = abc \cdot xyz$ = a maximum when $ax = by = cz$; that is, xyz = a maximum when $ax/2 = by/2 = cz/2 = K/3$.

To find the point O, draw a parallel to BC at $\frac{1}{3}$ the distance from BC to A, also a parallel to AC at $\frac{1}{3}$ the distance from AC to B.

These parallels, each cutting off the third of a median, meet in the centroid of the triangle, which is the required point O.

For the areas of triangles on the same base are in the same ratio as their altitudes, and hence area BOC = $K/3$ = area AOC = area AOB.

327. *Proposed by H. E. Trefethen, Waterville, Me.*

In any given triangle ABC,

(a) Find a point O such that its distances from the sides are proportional to these sides respectively;

(b) Through this point O draw parallels EF to AB, GH to AC, IK to BC, and prove that the points E, G, I, F, H, K are concyclic;

(c) Prove also $HF : EK : GI = a^3 : b^3 : c^3$.

Solution by I. L. Winckler, Cleveland, Ohio, Nelson L. Roray, Metuchen, N. J., and H. E. Trefethen, Waterville, Maine.

(a) Let p_1, p_2, p_3 be the perpendiculars from O on a, b, c , respectively, and h_1, h_2, h_3 the altitudes on these sides.

$$\text{Then } \frac{p_1}{a} = \frac{p_2}{b} = \frac{p_3}{c}. \quad (1)$$

$$\text{Also } p_1a + p_2b + p_3c = ch_3, \quad (2)$$

since each side of this equation is twice the area of ABC.

From (1) and (2)

$$\frac{p_3}{h_3} = \frac{c^3}{a^2 + b^2 + c^2}. \quad (3)$$

$$\text{Similarly } \frac{p_2}{h_2} = \frac{b^3}{a^2 + b^2 + c^2}, \quad (4)$$

$$\text{and } \frac{p_1}{h_1} = \frac{a^3}{a^2 + b^2 + c^2}. \quad (5)$$

Find m the side of a square equivalent to $a^2 + b^2 + c^2$.

Construct a right triangle with legs equal to m and c and let the altitude on the hypotenuse of this triangle divide it into the segments r and s , r being adjacent to m .

$$\text{Then } \frac{r}{s} = \frac{m^2}{c^2} = \frac{a^2 + b^2 + c^2}{c^2} = \frac{h_3}{p_3}. \quad [\text{by (3)}]$$

$\therefore p_3$ is the fourth proportional to r, s , and h_3 .

Similarly p_2 may be constructed and the point O located.

(b) From the similar triangles ABC and OGI,

$$\frac{p_3}{h_3} = \frac{GI}{c} = \frac{c^3}{a^2 + b^2 + c^2}. \quad [\text{by (3), part (a)}]$$

$$\therefore GI = \frac{c^3}{a^2 + b^2 + c^2} \quad (6)$$

$$\text{Similarly} \quad EK = \frac{b^3}{a^2+b^2+c^2}, \quad (7)$$

$$\text{and } HF = \frac{a^3}{a^2+b^2+c^2}. \quad (8)$$

$$\text{Also } \frac{GO}{GI} = \frac{b}{c},$$

from similar triangles OGI and ABC. (9)

$$\text{From (9) and (6) } GO = \frac{c^2b}{a^2+b^2+c^2}.$$

$$\text{Similarly } EO = \frac{b^2c}{a^2+b^2+c^2}.$$

$$\therefore \frac{EO}{GO} = \frac{b}{c}, \text{ also } \angle EOG = \angle A.$$

$\therefore \triangle EOG$ is similar to $\triangle ABC$.

$\therefore \angle OEG = \angle C$ and $\angle EGO = \angle B$.

$\therefore \angle KEO + \angle OEG + \angle OIG = \angle A + \angle B + \angle C,$

or $\angle KEG + \angle OIG = 2 \text{ rt. } \angle s.$

$\therefore$ IG EK is inscriptible.

Similarly GEKH is inscriptible, and therefore the circumference through the points I, G, E, K passes through H. In the same manner it may be shown this circumference passes through F.

$\therefore$ IG EKHF is inscriptible.

(c) from (6), (7) and (8), part (b),

$$HF : EK : GI = a^3 : b^3 : c^3.$$

CREDIT FOR SOLUTIONS.

318. Julia M. Bligh. (1)
319. Julia M. Bligh. (1)
320. Julia M. Bligh, H. C. McMillin. (2)
321. Julia M. Bligh. (1)
323. A. Babbitt, T. M. Blakslee, Julia M. Bligh, Walter C. Eells, M. Fendel, John M. Gallagher (2 solutions), A. M. Harding, A. H. Hughey, A. MacNeish, R. M. Mathews, A. L. McCarty, M. E. McGregor, H. C. McMillin, Effie Morse, Otto J. Ramler, Nelson L. Roray, Elizabeth Sargent, Elmer Schuyler, Levi S. Shively, H. E. Trefethen, I. L. Winckler. (22)
324. T. M. Blakslee (2 solutions), Julia M. Bligh, M. Gertrude Buck, Walter C. Eells, John M. Gallagher (2 solutions), A. M. Harding, A. H. Hughey, A. L. McCarty, M. E. MacGregor, Agnes MacNeish, R. M. Mathews, H. C. McMillin, Otto J. Ramler, D. H. Richert, W. J. Risley, Nelson L. Roray, Elizabeth Sargent, Levi S. Shively, Elmer Schuyler, H. E. Trefethen, I. L. Winckler. (23)
325. A. M. Harding, A. H. Hughey, Frank A. Logan, H. C. McMillin, Agnes MacNeish, R. M. Mathews, Nelson L. Roray, Elmer Schuyler, Levi S. Shively, H. E. Trefethen, I. L. Winckler. (11)
326. A. Babbitt, A. H. Hughey, Nelson L. Roray, Elmer Schuyler, H. E. Trefethen, I. L. Winckler. (6)
327. A. H. Hughey, Agnes MacNeish, H. C. McMillin, Nelson L. Roray, Elmer Schuyler, H. E. Trefethen, I. L. Winckler. (7)

Total number of solutions. 74.

PROBLEMS FOR SOLUTION.

Algebra.

339. *Proposed by A. Babbitt, State College, Pa.*

Solve: $4x^4 - 16x^3 + 27x^2 - 21x + 9 = 0$.

340. *Proposed by H. E. Trefethen, Waterville, Maine.*

$(\lfloor n-1 \rfloor)^2 > n^{n-2}$. Prove.

Geometry.

341. *Proposed by L. M. S., New York City.*

From the vertex of a triangle draw a line to the base so that it may be a mean proportional between the segments of the base.

342. *Proposed by Ernest B. Lytle, Urbana, Ill.*

To find a point in a plane such that the sum of its distances from two given points on the same side of the plane shall be a minimum.

343. *Proposed by R. L. Clayton, Weatherford, Oklahoma.*

Given, any triangle, ABC, and a point, P, without the triangle. To draw a line through the given point so that it shall bisect the triangle. Prove by elementary geometry.

Trigonometry.

344. *Proposed by Nelson L. Roray, Metuchen, N. J.*

Solve the equation:

$$\sin^{-1} \frac{2a}{1+a^2} + \tan^{-1} \frac{2x}{1-x^2} = \cos^{-1} \frac{1-b^2}{1+b^2}$$

(From Hall & Knight's Elementary Trigonometry.)

SCHOOL SOCIAL CENTERS.

Three hundred and thirty-eight schools in 101 cities of the United States were used as social centers during the past season, according to a report compiled by Clarence Arthur Perry for the Sage Foundation. Officials of the United States Bureau of Education, who have examined the report, declare that it is bound to stimulate interest in this rapidly developing phase of the movement for wider use of the school plant.

Mr. Perry finds that in 44 of the 101 cities social centers were directed by paid workers. New York had 48 such centers and Chicago 16, while Philadelphia, Boston, Columbus, Detroit, Jersey City, Louisville, Rochester, and Trenton are also among the cities included in this list. There is wide variation in the length of the season, from five or six weeks in some localities to the full school term in others. In fact, little uniformity prevails as to what constitutes a social center. Mr. Perry presents in the report a tentative definition of a social center as follows: "A community may be said to have a schoolhouse social center if one of its school buildings is thrown open to the public on one or more fixed nights a week for at least 12 weeks a year, for activities of a social, recreational, or civic character, regularly directed by one or more trained leaders."

The report also presents data on the growing use of school buildings for political meetings. In Cleveland, Ohio, meetings were held