An Elementary Method of Finding Circles of Curvature at Points, Multiple or Other, of a Plane Curve Whose Equation Is Given in Rectangular Coordinates

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More Advanced Stage.—Experiments involving the application of the Calculus.

Area of cross section of a vessel from observations of volume and depth.

Moments of Inertia. Oscillating lamina. Compound pendulum.

Kinetic energy of fly wheel.

Applications of coordinate geometry in three dimensions.

R. C. FAWDRY.

The President next called on Mr. A. Lodge for his Paper on :

AN ELEMENTARY METHOD OF FINDING CIRCLES OF CURVATURE AT POINTS, MULTIPLE OR OTHER, OF A PLANE CURVE WHOSE EQUATION IS GIVEN IN RECTANGULAR COORDINATES.

THE method of finding the centre of curvature to which I wish to draw attention is as follows :

If PT is the tangent at a point P of a curve, a circle which touches the curve at P will in general cut it at other points $Q, \dots$

Now, if P be taken as origin, and a homogeneous equation be formed by means of the equations of the curve and a tangent circle, this equation will represent all the chords $PQ, \dots$. The circle of curvature will be distinguished by the fact that one of these chords must coincide with PT . That is the whole spirit of the method : there is no calculus, and the method is available for all who can transform an equation to a new point as origin and can appreciate the meaning of a homogeneous equation. In many cases, comprehension of what terms are small enough to be omitted will facilitate matters, but it is not essential, and indeed the method can be made to develop such comprehension.

Ex. 1. *To find the circle of curvature at $(0, 0)$ of the curve*

$$(x+y)^2 = a(x-y). \dots\dots\dots(1)$$

Let its equation be $x^2 + y^2 = k(x-y). \dots\dots\dots(2)$

Then the homogeneous equation is

$$k(x+y)^2 = a(x^2 + y^2), \dots\dots\dots(3)$$

and this will be satisfied by $x-y=0$ if

$$k = \frac{1}{2}a;$$

$\therefore$ the circle of curvature is

$$x^2 + y^2 = \frac{a}{2}(x-y), \dots\dots\dots(4)$$

and the centre of curvature is at $\left(\frac{a}{4}, -\frac{a}{4}\right)$.

Moreover, if in (3) we put $k = \frac{1}{2}a$, it becomes

$$(x+y)^2 = 2(x^2 + y^2),$$

$$\text{i.e. } (x-y)^2 = 0.$$

Hence in this case the contact is of the 3rd order, as *all* the chords coincide with the tangent.

Ex. 2. *Case where the tangent at the origin is one of the axes.*

In this case the method leads to a quicker result.

Thus the circle of curvature at $(0, 0)$ of $y^2 = 4ax$ is

$$x^2 + y^2 = 4ax,$$

i.e. the presence or absence of x^2 in the presence of x has no effect on the curvature.

Similarly, in the case of $Ax^2 + Bxy + Cy^2 = x$, the circle of curvature at $(0, 0)$ is $Cx^2 + Cy^2 = x$.

Ex. 3. To find the curvature at $(at^2, 2at)$ of the parabola $y^2 = 4ax$.

We have $(y - 2at)^2 = 4a(x - ty + at^2)$,
 or, taking P as origin, $y^2 = 4a(x - ty)$.
 Tangent circle : $x^2 + y^2 = k(x - ty)$;
 chord equation : $ky^2 = 4a(x^2 + y^2)$; $\therefore k = 4a(1 + t^2)$;
 circle of curvature : $x^2 + y^2 = 4a(1 + t^2)(x - ty)$;
 revised chord equation : $(1 + t^2)y^2 = x^2 + y^2$,
 i.e. $(ty - x)(ty + x) = 0$;

showing that the remaining chord and the tangent are equally inclined to the axis.

The coordinates of the centre of curvature with the old origin are $(2a + 3at^2, -2at^3)$.

The method is equally efficacious in the case of an ellipse or hyperbola.

Ex. 4. (Double Point.) Find the curvature circle at $(0, 0)$ of the curve

$$x(x^2 + y^2) = a(x^2 - y^2).$$

Take (1) $x^2 + y^2 = k(x - y)$,
 the chord being $kx = a(x + y)$; $\therefore k = 2a$;
 (2) $x^2 + y^2 = k(x + y)$,
 the chord being $kx = a(x - y)$; $\therefore k = 2a$.

$\therefore$ the circles of curvature are

$$(i) \ x^2 + y^2 = 2a(x - y) ; \quad (ii) \ x^2 + y^2 = 2a(x + y).$$

For the curvature circle at $(a, 0)$, write $x + a$ for x , so as to move the origin to this point ;

$$\therefore x(x^2 + 2ax + a^2) + y^2(x + 2a) = 0, \text{ i.e. } (x + 2a)(x^2 + y^2) + a^2x = 0 ;$$

$$\therefore 2y^2 + ax = 0 \text{ is the parabola of curvature}$$

and $x^2 + y^2 + \frac{1}{2}ax = 0$ is the circle of curvature.

The parabola of curvature to one of the branches at the origin can be obtained by the same method, but although the parabola has the interest of lying closer to the curve for a longer distance, it is not so easily drawn, nor is it quite so easily obtained.

However, taking $(Ax + By)^2 = k(x - y)$ as a tangent parabola, we have

$$kx(x^2 + y^2) = a(x + y)(Ax + By)^2$$

for chords ;

$$\therefore \text{ if } x = y \text{ is one of them, } k = a(A + B)^2 ;$$

$$\therefore (A + B)^2 x(x^2 + y^2) = (x + y)(Ax + By)^2.$$

To obtain a second chord coinciding with $x = y$ is rather lengthy unless we use the fact that if $f(y) = 0$ has two equal roots, $f'(y) = 0$ has the same root.

Using this method, we obtain

$$(A + B)^2 \cdot 2xy = (Ax + By)^2 + 2B(x + y)(Ax + By) ;$$

whence, if $x = y$, $(A + B)^2 = 4B(A + B)$;

$$\therefore A + B = 0$$

or

$$A = 3B.$$

$A+B=0$ is useless, as it gives merely $(x-y)^2=0$.

∴ putting $A=3B$ in the equation of the parabola, we have

$$(3x+y)^2=16a(x-y)$$

as the equation of the parabola of curvature ;

and the corresponding chord equation is

$$16x(x^2+y^2)=(x+y)(3x+y)^2,$$

which reduces to

$$(x-y)^2(7x-y)=0.$$

∴ the curve cuts the parabola again on the line $y=7x$,

$$\text{i.e. where } (10x)^2 = -96ax,$$

$$\text{i.e. } x = -0.96a,$$

$$y = -6.72a.$$

Other Examples.

For the curve $(x^2+y^2)^2=a(3x^2y-y^3)$,

the circles of curvature at the origin are

$$x^2+y^2=3ay,$$

$$2(x^2+y^2)=3a(x\sqrt{3}-y),$$

$$2(x^2+y^2)+3a(x\sqrt{3}+y)=0;$$

the centres of curvature being equally spaced on the circumference of the circle, $4(x^2+y^2)=9a^2$.

For the curve

$$y^3=(2x-3y)(3x+2y),$$

the origin circles are

$$8(x^2+y^2)=169(2x-3y),$$

$$27(x^2+y^2)+169(3x+2y)=0.$$

For the curve

$$y=e^x, \quad (=1+x+\frac{1}{2}x^2+\dots),$$

the curvature circle at $(0, 1)$ is, when the origin is moved there,

$$x^2+y^2+4(x-y)=0.$$

For the curve

$$x^2(x^2+y^2)=a^2(x-y)^2,$$

the curvature circles at the origin are

$$x^2+y^2=\pm a\sqrt{2}(x-y);$$

and when the origin is moved to $(-a, 0)$, the circle there is

$$3(x^2+y^2)=4a(x+y).$$

By help of these circles and the fact that $x=\pm a$ are asymptotes, the curve can be very accurately drawn with very little trouble.

The cases where $k=0$ or ∞ correspond to cusps and points of inflexion. In such cases the same method could be used to find semi-cubical or cubical parabolas which lie close to the curve. But this is an extension into which I do not propose now to go, though I have done it successfully in several cases.

A. LODGE.

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