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XXIII. *Hydrodynamical Notes.*
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Potential and Kinetic Energies of Wave Motion.—Waves moving into Shallower Water.—Concentrated initial Disturbance with inclusion of Capillarity.—Periodic Waves in Deep Water advancing without change of Type.—Tide Races.—Rotational Fluid Motion in a Corner.—Steady Motion in a Corner of Viscous Fluid.

IN the problems here considered the fluid is regarded as incompressible, and the motion is supposed to take place in two dimensions.

Potential and Kinetic Energies of Wave Motion.

When there is no dispersion, the energy of a progressive wave of any form is half potential and half kinetic. Thus in the case of a long wave in shallow water, "if we suppose that initially the surface is displaced, but that the particles have no velocity, we shall evidently obtain (as in the case of sound) two equal waves travelling in opposite directions, whose total energies are equal, and together make up the potential energy of the original displacement. Now the elevation of the derived waves must be half of that of the original displacement, and accordingly the potential energies less in the ratio of 4:1. Since therefore the potential energy of each derived wave is one quarter, and the total energy one half that of the original displacement,

* Communicated by the Author.

it follows that in the derived wave the potential and kinetic energies are equal”*.

The assumption that the displacement in each derived wave, when separated, is similar to the original displacement fails when the medium is dispersive. The equality of the two kinds of energy in an infinite progressive train of simple waves may, however, be established as follows.

Consider first an infinite series of simple stationary waves, of which the energy is at one moment wholly potential and half a period later wholly kinetic. If t denote the time and E the total energy, we may write

$$\text{K.E.} = E \sin^2 nt, \quad \text{P.E.} = E \cos^2 nt.$$

Upon this superpose a similar system, displaced through a quarter wave-length in space and through a quarter period in time. For this, taken by itself, we should have

$$\text{K.E.} = E \cos^2 nt, \quad \text{P.E.} = E \sin^2 nt.$$

And, the vibrations being *conjugate*, the potential and kinetic energies of the combined motion may be found by simple addition of the components, and are accordingly independent of the time, and each equal to E . Now the resultant motion is a simple progressive train, of which the potential and kinetic energies are thus seen to be equal.

A similar argument is applicable to prove the equality of energies in the motion of a simple conical pendulum.

It is to be observed that the conclusion is in general limited to vibrations which are infinitely small.

Waves moving into Shallower Water.

The problem proposed is the passage of an infinite train of simple infinitesimal waves from deep water into water which shallows gradually in such a manner that there is no loss of energy by reflexion or otherwise. At any stage the whole energy, being the double of the potential energy, is proportional per unit length to the square of the height; and for motion in two dimensions the only remaining question for our purpose is what are to be regarded as corresponding lengths along the direction of propagation.

In the case of long waves, where the wave-length (λ) is long in comparison with the depth (l) of the water, corresponding parts are as the velocities of propagation (V), or

* “On Waves,” *Phil. Mag.* i. p. 257 (1876); ‘*Scientific Papers*,’ i. p. 254.

since the periodic time (τ) is constant, as λ . Conservation of energy then requires that

$$(\text{height})^2 \times V = \text{constant}; \quad . \quad . \quad . \quad . \quad (1)$$

or since V varies as $l^{\frac{1}{2}}$, height varies as $l^{-\frac{1}{4}}$.*

But for a dispersive medium corresponding parts are not proportional to V , and the argument requires modification. A uniform regime being established, what we are to equate at two separated places where the waves are of different character is the *rate of propagation of energy* through these places. It is a general proposition that in any kind of waves the ratio of the energy propagated past a fixed point in unit time to that resident in unit length is U , where U is the *group-velocity*, equal to $d\sigma/dk$, where $\sigma = 2\pi/\tau$, $k = 2\pi/\lambda$ †. Hence in our problem we must take

$$\text{height varies as } U^{-\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (2)$$

which includes the former result, since in a non-dispersive medium $U = V$.

For waves in water of depth l ,

$$\sigma^2 = gk \tanh kl, \quad . \quad . \quad . \quad . \quad (3)$$

whence

$$2\sigma U/g = \tanh kl + kl(1 - \tanh^2 kl). \quad . \quad . \quad . \quad (4)$$

As the wave progresses, σ remains constant, (3) determines k in terms of l , and U follows from (4). If we write

$$\sigma^2 l/g = l', \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(3) becomes

$$kl \cdot \tanh kl = l', \quad . \quad . \quad . \quad . \quad (6)$$

and (4) may be written

$$2\sigma U/g = kl + (l' - l'^2)/kl. \quad . \quad . \quad . \quad (7)$$

By (6), (7) U is determined as a function of l' or by (5) of l .

If kl , and therefore l' , is very great, $kl = l'$, and then by (7) if U_0 be the corresponding value of U ,

$$2\sigma U_0/g = 1, \quad . \quad . \quad . \quad . \quad (8)$$

and in general

$$U/U_0 = kl + (l' - l'^2)/kl. \quad . \quad . \quad . \quad (9)$$

* *Loc. cit.* p. 255.

† Proc. Lond. Math. Soc. ix. 1877; 'Scientific Papers,' i. p. 326.

Equations (2), (5), (6), (9) may be regarded as giving the solution of the problem in terms of a known σ . It is perhaps more practical to replace σ in (5) by λ_0 , the corresponding wave-length in a great depth. The relation between σ and λ_0 being $\sigma^2 = 2\pi g/\lambda_0$, we find in place of (5)

$$l' = 2\pi l/\lambda_0 = k_0 l \dots \dots (10)$$

Starting in (16) from λ_0 and l we may obtain l' , whence (6) gives kl , and (9) gives U/U_0 . But in calculating results by means of tables of the hyperbolic functions it is more convenient to start from kl . We find

kl .	l' .	U/U_0 .	kl .	l' .	U/U_0 .
∞	kl	1.000	.6	.322	.964
10	kl	1.000	.5	.231	.855
5	4.999	1.001	.4	.152	.722
2	1.928	1.105	.3	.087	.566
1.5	1.358	1.176	.2	.039	.390
1.0	.762	1.182	.1	.010	.200
.8	.531	1.110	kl	$(kl)^2$	$2kl$
.7	.423	1.048	—	—	—

It appears that U/U_0 does not differ much from unity between $l' = .23$ and $l' = \infty$, so that the shallowing of the water does not at first produce much effect upon the height of the waves. It must be remembered, however, that the wave-length is diminishing, so that waves, even though they do no more than maintain their height, grow *steeper*.

*Concentrated Initial Disturbance with inclusion of
Capillarity.*

A simple approximate treatment of the general problem of initial linear disturbance is due to Kelvin*. We have for the elevation η at any point x and at any time t

$$\begin{aligned} \eta &= \frac{1}{\pi} \int_0^\infty \cos kx \cos \sigma t \, dk \\ &= \frac{1}{2\pi} \int_0^\infty \cos (kx - \sigma t) \, dk + \frac{1}{2\pi} \int_0^\infty \cos (kx + \sigma t) \, dk, \quad (1) \end{aligned}$$

in which σ is a function of k , determined by the character

* Proc. Roy. Soc. vol. xlii. p. 80 (1887); 'Math. and Phys. Papers,' iv. p. 303.

of the dispersive medium—expressing that the initial elevation ($t=0$) is concentrated at the origin of x . When t is great, the angles whose cosines are to be integrated will in general vary rapidly with k , and the corresponding parts of the integral contribute little to the total result. The most important part of the range of integration is the neighbourhood of places where $kx \pm \sigma t$ is stationary with respect to k , i. e. where

$$x \pm t \frac{d\sigma}{dk} = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the vast majority of practical applications $d\sigma/dk$ is positive, so that if x and t are also positive the second integral in (1) makes no sensible contribution. The result then depends upon the first integral, and only upon such parts of that as lie in the neighbourhood of the value, or values, of k which satisfy (2) taken with the lower sign. If k_1 be such a value, Kelvin shows that the corresponding term in η has an expression equivalent to

$$\eta = \frac{\cos(\sigma_1 t - k_1 x - \frac{1}{4}\pi)}{\sqrt{\{-2\pi t d^2\sigma/dk_1^2\}}}, \quad . \quad . \quad . \quad . \quad (3)$$

σ_1 being the value of σ corresponding to k_1 .

In the case of deep-water waves where $\sigma = \sqrt{gk}$, there is only one predominant value of k for given values of x and t , and (2) gives

$$k_1 = gt^2/4x^2, \quad \sigma_1 = gt/2x, \quad . \quad . \quad . \quad . \quad (4)$$

making

$$\sigma_1 t - k_1 x - \frac{1}{4}\pi = gt^2/4x - \frac{1}{4}\pi, \quad . \quad . \quad . \quad . \quad (5)$$

and finally

$$\eta = \frac{g^{\frac{1}{2}}t}{2\pi^{\frac{1}{2}}x^{\frac{3}{2}}} \cos \left\{ \frac{gt^2}{4x} - \frac{\pi}{4} \right\}, \quad . \quad . \quad . \quad . \quad (6)$$

the well-known formula of Cauchy and Poisson.

In the numerator of (3) σ_1 and k_1 are functions of x and t . If we inquire what change (Λ) in x with t constant alters the angle by 2π , we find

$$\Lambda \left\{ k_1 + \left(x - t \frac{d\sigma}{dk_1} \right) \frac{dk_1}{dx} \right\} = 2\pi,$$

so that by (2) $\Lambda = 2\pi/k_1$, i. e. the effective wave-length Λ coincides with that of the predominant component in the original integral (1), and a like result holds for the periodic

time *. Again, it follows from (2) that $k_1x - \sigma_1t$ in (3) may be replaced by $\int k_1 dx$, as is exemplified in (4) and (6).

When the waves move under the influence of a capillary tension T in addition to gravity,

$$\sigma^2 = gk + Tk^3/\rho, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

ρ being the density, and for the wave-velocity (V)

$$V^2 = \sigma^2/k^2 = g/k + Tk/\rho, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

as first found by Kelvin. Under these circumstances V has a minimum value when

$$k^2 = g\rho/T. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The group-velocity U is equal to $d\sigma/dk$, or to $d(kV)/dk$; so that when V has a minimum value, U and V coincide. Referring to this, Kelvin towards the close of his paper remarks "The working out of our present problem for this case, or any case in which there are either minimums or maximums, or both maximums and minimums, of wave-velocity, is particularly interesting, but time does not permit of its being included in the present communication."

A glance at the simplified form (3) shows, however, that the special case arises, not when V is a minimum (or maximum), but when U is so, since then $d^2\sigma/dk^2$ vanishes. As given by (3), η would become infinite—an indication that the approximation must be pursued. If $k = k_1 + \xi$, we have in general in the neighbourhood of k_1 ,

$$\begin{aligned} kx - \sigma t &= k_1x - \sigma_1t + \left(x - t \frac{d\sigma}{dk_1}\right)\xi \\ &\quad - \frac{t}{1.2} \frac{d^2\sigma}{dk_1^2} \xi^2 - \frac{t}{1.2.3} \frac{d^3\sigma}{dk_1^3} \xi^3. \quad . \quad . \quad . \quad (10) \end{aligned}$$

In the present case where the term in ξ^2 disappears, as well as that in ξ , we get in place of (3) when t is great

$$\eta = \frac{\cos(k_1x - \sigma_1t)}{2\pi \left\{ \frac{1}{6} t \frac{d^3\sigma}{dk_1^3} \right\}^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \cos \alpha^3. d\alpha, \quad . \quad . \quad . \quad (11)$$

varying as $t^{-\frac{1}{2}}$ instead of as $t^{-\frac{1}{3}}$.

The definite integral is included in the general form

$$\int_{-\infty}^{+\infty} \cos \alpha^m d\alpha = \frac{2}{m} \Gamma\left(\frac{1}{m}\right) \cos \frac{\pi}{2m}, \quad . \quad . \quad . \quad . \quad (12)$$

* Cf. Green, Proc. Roy. Soc. Ed. xxix. p. 445 (1909).

giving

$$\int_{-\infty}^{+\infty} \cos \alpha^2. d\alpha = \sqrt{\left(\frac{\pi}{2}\right)}; \quad \int_{-\infty}^{+\infty} \cos \alpha^3. d\alpha = \frac{1}{\sqrt{3}} \Gamma\left(\frac{1}{3}\right). \quad (13)$$

The former is employed in the derivation of (3).

The occurrence of stationary values of U is determined from (7) by means of a quadratic. There is but one such value (U_0), easily seen to be a minimum, and it occurs when

$$k^2 = \left\{ \sqrt{\frac{4}{3}} - 1 \right\} \frac{g\rho}{T} = .1547 \frac{g\rho}{T}. \quad . \quad . \quad (14)$$

On the other hand, the minimum of V occurs when $k^2 = g\rho/T$ simply.

When t is great, there is no important effect so long as x (positive) is less than $U_0 t$. For this value of x the Kelvin formula requires the modification expressed by (11). When x is decidedly greater than $U_0 t$, there arise two terms of the Kelvin form, indicating that there are now two systems of waves of different wave-lengths, effective at the same place.

It will be seen that the introduction of capillarity greatly alters the character of the solution. The quiescent region inside the annular waves is easily recognized a few seconds after a very small stone is dropped into smooth water*, but I have not observed the duplicity of the annular waves themselves. Probably the capillary waves of short wave-length are rapidly damped, especially when the water-surface is not quite clean. It would be interesting to experiment upon truly linear waves, such as might be generated by the sudden electrical charge or discharge of a wire stretched just above the surface. But the full development of the peculiar features to be expected on the inside of the wave-system seems to require a space larger than is conveniently available in a laboratory.

Periodic Waves in Deep Water advancing without change of Type.

The solution of this problem when the height of the waves is infinitesimal has been familiar for more than a century, and the pursuance of the approximation to cover the case of moderate height is to be found in a well-known paper by

* A checkered background, *e. g.* the sky seen through foliage, shows the waves best.

Stokes *. In a supplement published in 1880 † the same author treated the problem by another method in which the space coordinates x, y are regarded as functions of ϕ, ψ the velocity and stream functions, and carried the approximation a stage further.

In an early publication ‡ I showed that some of the results of Stokes's first memoir could be very simply derived from the expression for the stream-function in terms of x and y , and lately I have found that this method may be extended to give, as readily if perhaps less elegantly, all the results of Stokes' supplement.

Supposing for brevity that the wave-length is 2π and the velocity of propagation unity, we take as the expression for the stream-function of the waves, reduced to rest,

$$\psi = y - \alpha e^{-y} \cos x - \beta e^{-2y} \cos 2x - \gamma e^{-3y} \cos 3x, \quad (1)$$

in which x is measured horizontally and y vertically downwards. This expression evidently satisfies the differential equation to which ψ is subject, whatever may be the values of the constants α, β, γ . From (1) we find

$$\begin{aligned} U^2 - 2gy &= (d\psi/dx)^2 + (d\psi/dy)^2 - 2gy \\ &= 1 - 2\psi + 2(1-g)y + 2\beta e^{-2y} \cos 2x + 4\gamma e^{-3y} \cos 3x \\ &\quad + \alpha^2 e^{-2y} + 4\beta^2 e^{-4y} + 9\gamma^2 e^{-6y} + 4\alpha\beta e^{-3y} \cos x \\ &\quad + 6\alpha\gamma e^{-4y} \cos 2x + 12\beta\gamma e^{-5y} \cos x. \quad (2) \end{aligned}$$

The condition to be satisfied at a free surface is the constancy of (2).

The solution to a moderate degree of approximation (as already referred to) may be obtained with omission of β and γ in (1), (2). Thus from (1) we get, determining ψ so that the mean value of y is zero,

$$y = \alpha(1 + \frac{5}{8}\alpha^2) \cos x - \frac{1}{2}\alpha^2 \cos 2x + \frac{3}{8}\alpha^3 \cos 3x, \quad (3)$$

which is correct as far as α^3 inclusive.

If we call the coefficient of $\cos x$ in (3) a , we may write with the same approximation

$$y = a \cos x - \frac{1}{2}a^2 \cos 2x + \frac{3}{8}a^3 \cos 3x. \quad (4)$$

* Camb. Phil. Soc. Trans. viii. p. 441 (1847); 'Math. and Phys. Papers,' i. p. 197.

† *Z. c. i.* p. 314.

‡ Phil. Mag. i. p. 257 (1876); Sci. Papers, i. p. 262. See also Lamb's 'Hydrodynamics,' § 280.

Again from (2) with omission of β , γ ,

$$U^2 - 2gy = \text{const.} + 2(1 - g - \alpha^2 - \alpha^4)y + \alpha^4 \cos 2x - \frac{4}{3}\alpha^5 \cos 3x. \quad (5)$$

It appears from (5) that the surface condition can be satisfied with α only, provided that α^4 is neglected and that

$$1 - g - \alpha^2 = 0. \quad (6)$$

In (6) α may be replaced by a , and the equation determines the velocity of propagation. To exhibit this we must restore generality by introduction of k ($= 2\pi/\lambda$) and c the velocity of propagation, hitherto treated as unity. Consideration of "dimensions" shows that (6) becomes

$$kc^2 - g - a^2 c^2 k^3 = 0, \quad (7)$$

or

$$c^2 = g/k \cdot (1 + k^2 a^2). \quad (8)$$

Formulæ (4) and (8) are those given by Stokes in his first memoir.

By means of β and γ the surface condition (2) can be satisfied with inclusion of α^4 and α^5 , and from (5) we see that β is of the order α^4 and γ of the order α^5 . The terms to be retained in (2), in addition to those given in (5), are

$$\begin{aligned} & 2\beta(1 - 2y) \cos 2x + 4\gamma \cos 3x + 4\alpha\beta \cos x \\ & = 2\beta \cos 2x + 2\alpha\beta (\cos x + \cos 3x) + 4\gamma \cos 3x + 4\alpha\beta \cos x. \end{aligned}$$

Expressing the terms in $\cos x$ by means of y , we get finally

$$\begin{aligned} U^2 - 2gy &= \text{const.} + 2y(1 - g - \alpha^2 - \alpha^4 + \beta) \\ &+ (\alpha^4 + 2\beta) \cos 2x + (4\gamma - \frac{4}{3}\alpha^5 - 2\alpha\beta) \cos 3x. \quad (9) \end{aligned}$$

In order to satisfy the surface condition of constant pressure, we must take

$$\beta = -\frac{1}{2}\alpha^4, \quad \gamma = \frac{1}{12}\alpha^5, \quad (10)$$

and in addition

$$1 - g - \alpha^2 - \frac{3}{2}\alpha^4 = 0, \quad (11)$$

correct to α^5 inclusive. The expression (1) for ψ thus assumes the form

$$\psi = y - \alpha e^{-y} \cos x + \frac{1}{2}\alpha^4 e^{-2y} \cos 2x - \frac{1}{12}\alpha^5 e^{-3y} \cos 3x, \quad (12)$$

from which y may be calculated in terms of x as far as α^5 inclusive.

By successive approximation, determining ψ so as to make the mean value of y equal to zero, we find as far as α^4

$$y = (\alpha + \frac{5}{8}\alpha^3) \cos x - (\frac{1}{2}\alpha^2 + \frac{4}{3}\alpha^4) \cos 2x + \frac{3}{8}\alpha^3 \cos 3x - \frac{1}{3}\alpha^4 \cos 4x, \quad \dots \quad (13)$$

or, if we write as before a for the coefficient of $\cos x$,

$$y = a \cos x - (\frac{1}{2}a^2 + \frac{1}{2}\frac{7}{4}a^4) \cos 2x + \frac{3}{8}a^3 \cos 3x - \frac{1}{3}a^4 \cos 4x, \quad \dots \quad (14)$$

in agreement with equation (20) of Stokes' Supplement.

Expressed in terms of a , (11) becomes

$$g = 1 - a^2 - \frac{1}{4}a^4, \quad \dots \quad (15)$$

or on restoration of k, c ,

$$g = kc^2 - k^3a^2c^2 - \frac{1}{4}k^5a^4c^2, \quad \dots \quad (16)$$

Thus the extension of (8) is

$$c^2 = g/k \cdot (1 + k^2a^2 + \frac{4}{5}k^4a^4), \quad \dots \quad (17)$$

which also agrees with Stokes' Supplement.

If we pursue the approximation one stage further, we find from (12) terms in α^5 , additional to those expressed in (13). These are

$$y = \alpha^5 \left\{ \frac{373}{6.32} \cos x + \frac{243}{128} \cos 3x + \frac{125}{12.32} \cos 5x \right\} \quad \dots \quad (18)$$

It is of interest to compare the potential and kinetic energies of waves that are not infinitely small. For the stream-function of the waves regarded as progressive, we have, as in (1),

$$\psi = -\alpha e^{-y} \cos(x - ct) + \text{terms in } \alpha^4,$$

so that

$$(d\psi/dx)^2 + (d\psi/dy)^2 = \alpha^2 e^{-2y} + \text{terms in } \alpha^5.$$

Thus the mean kinetic energy per length x measured in the direction of propagation is

$$\begin{aligned} \frac{\alpha^2}{2} \int dx \int_y^\infty e^{-2y} dy &= \frac{\alpha^2}{4} \int dx e^{-2y} = \frac{\alpha^2}{4} \int dx (1 - 2y + 2y^2) \\ &= \frac{\alpha^2}{4} \left\{ x + 2 \int y^2 dx \right\}, \end{aligned}$$

where y is the ordinate of the surface. And by (3)

$$\int y^2 dx = \left\{ \frac{1}{2}(\alpha^2 + \frac{5}{4}\alpha^4) + \frac{1}{8}\alpha^4 \right\} x.$$

Hence correct to α^4 ,

$$\text{K.E.} = \frac{1}{4}\alpha^2(1 + \alpha^2)x. \quad . \quad . \quad . \quad . \quad (19)$$

Again, for the potential energy

$$\text{P.E.} = \frac{1}{2}g \int y^2 dx = \frac{1}{2}gx(\frac{1}{2}\alpha^2 + \frac{3}{4}\alpha^4);$$

or since $g = 1 - \alpha^2$,

$$\text{P.E.} = \frac{1}{4}\alpha^2(1 + \frac{1}{2}\alpha^2)x. \quad . \quad . \quad . \quad . \quad (20)$$

The kinetic energy thus exceeds the potential energy, when α^4 is retained.

Tide Races.

It is, I believe, generally recognised that seas are apt to be exceptionally heavy when the tide runs against the wind. An obvious explanation may be founded upon the fact that the relative motion of air and water is then greater than if the latter were not running, but it seems doubtful whether this explanation is adequate.

It has occurred to me that the cause may be rather in the motion of the stream relatively to itself, *e.g.* in the more rapid movement of the upper strata. Stokes's theory of the highest possible wave shows that in non-rotating water the angle at the crest is 120° and the height only moderate. In such waves the surface strata have a mean motion forwards. On the other hand, in Gerstner and Rankine's waves the fluid particles retain a mean position, but here there is *rotation* of such a character that (in the absence of waves) the surface strata have a relative motion backwards, *i.e.* against the direction of propagation*. It seems possible that waves moving against the tide may approximate more or less to the Gerstner type and thus be capable of acquiring a greater height and a sharper angle than would otherwise be expected. Needless to say, it is the steepness of waves, rather than their mere height, which is a source of inconvenience and even danger to small craft.

The above is nothing more than a suggestion. I do not know of any detailed account of the special character of these waves, on which perhaps a better opinion might be formed.

Rotational Fluid Motion in a Corner.

The motion of incompressible inviscid fluid is here supposed to take place in two dimensions and to be bounded by two fixed planes meeting at an angle α . If there is no rotation,

* Lamb's *Hydrodynamics*, § 247.

the stream-function ψ , satisfying $\nabla^2\psi=0$, may be expressed by a series of terms

$$r^{\pi/\alpha} \sin \pi\theta/\alpha, \quad r^{2\pi/\alpha} \sin 2\pi\theta/\alpha, \quad \dots \quad r^{n\pi/\alpha} \sin n\pi\theta/\alpha,$$

where n is an integer, making $\psi=0$ when $\theta=0$ or $\theta=\alpha$. In the immediate vicinity of the origin the first term predominates. For example, if the angle be a right angle,

$$\psi=r^2 \sin 2\theta=2xy, \quad . \quad . \quad . \quad . \quad (1)$$

if we introduce rectangular coordinates.

The possibility of irrotational motion depends upon the fixed boundary not being closed. If $\alpha<\pi$, the motion near the origin is finite; but if $\alpha>\pi$, the velocities deduced from ψ become infinite.

If there be rotation, motion may take place even though the boundary be closed. For example, the circuit may be completed by the arc of the circle $r=1$. In the case which it is proposed to consider the rotation ω is *uniform*, and the motion may be regarded as steady. The stream-function then satisfies the general equation

$$\nabla^2\psi=d^2\psi/dx^2+d^2\psi/dy^2=2\omega, \quad . \quad . \quad . \quad (2)$$

or in polar coordinates

$$\frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} + \frac{1}{r^2} \frac{d^2\psi}{d\theta^2} = 2\omega. \quad . \quad . \quad . \quad (3)$$

When the angle is a right angle, it might perhaps be expected that there should be a simple expression for ψ in powers of x and y , analogous to (1) and applicable to the immediate vicinity of the origin; but we may easily satisfy ourselves that no such expression exists*. In order to express the motion we must find solutions of (3) subject to the conditions that $\psi=0$ when $\theta=0$ and when $\theta=\alpha$.

For this purpose we assume, as we may do, that

$$\psi=\Sigma R_n \sin n\pi\theta/\alpha, \quad . \quad . \quad . \quad . \quad (4)$$

where n is integral and R_n a function of r only; and in deducing $\nabla^2\psi$ we may perform the differentiations with respect to θ (as well as with respect to r) under the sign of summation, since $\psi=0$ at the limits. Thus

$$\nabla^2\psi=\Sigma\left(\frac{d^2R_n}{dr^2} + \frac{1}{r} \frac{dR_n}{dr} - \frac{n^2\pi^2}{\alpha^2 r^2} R_n\right) \sin \frac{n\pi\theta}{\alpha}. \quad . \quad (5)$$

* In strictness the satisfaction of (2) at the origin is inconsistent with the evanescence of ψ on the rectangular axes.

The right-hand member of (3) may also be expressed in a series of sines of the form

$$2\omega = \delta\omega/\pi \cdot \sum n^{-1} \sin n\pi\theta/\alpha, \quad . \quad . \quad . \quad (6)$$

where n is an *odd* integer; and thus for all values of n we have

$$r^2 \frac{d^2 R_n}{dr^2} + r \frac{dR_n}{dr} - \frac{n^2 \pi^2 R_n}{\alpha^2} = \frac{4\omega}{n\pi} \{1 - (-1)^n\}. \quad . \quad (7)$$

The general solution of (7) is

$$R_n = A_n r^{n\pi/\alpha} + B_n r^{-n\pi/\alpha} + \frac{4\omega\alpha^2 r^2 \{1 - (-1)^n\}}{n\pi(4\alpha^2 - n^2 \pi^2)}, \quad . \quad (8)$$

the introduction of which into (4) gives ψ .

In (8) A_n and B_n are arbitrary constants to be determined by the other conditions of the problem. For example, we might make R_n , and therefore ψ , vanish when $r=r_1$ and when $r=r_2$, so that the fixed boundary enclosing the fluid would consist of two radii vectores and two circular arcs. If the fluid extend to the origin, we must make $B_n=0$; and if the boundary be completed by the circular arc $r=1$, we have $A_n=0$ when n is even, and when n is odd

$$A_n + \frac{8\omega\alpha^2}{n\pi(4\alpha^2 - n^2 \pi^2)} = 0. \quad . \quad . \quad . \quad (9)$$

Thus for the fluid enclosed in a circular sector of angle α and radius unity

$$\psi = 8\omega\alpha^2 \sum \frac{r^{n\pi/\alpha} - r^2}{n\pi(n^2 \pi^2 - 4\alpha^2)} \sin \frac{n\pi\theta}{\alpha}, \quad . \quad . \quad (10)$$

the summation extending to all odd integral values of n .

The above formula (10) relates to the motion of uniformly *rotating* fluid bounded by *stationary* radii vectores at $\theta=0$, $\theta=\alpha$. We may suppose the containing vessel to have been rotating for a long time and that the fluid (under the influence of a very small viscosity) has acquired this rotation so that the whole revolves like a solid body. The motion expressed by (10) is that which would ensue if the rotation of the vessel were suddenly stopped. A related problem was solved a long time since by Stokes*, who considered the *irrotational* motion of fluid in a *revolving* sector. The solution of Stokes's problem is derivable from (10) by mere

* Camb. Phil. Trans. vol. viii. p. 533 (1847); Math. and Phys. Papers, vol. i. p. 305.

addition to the latter of $\psi_0 = -\frac{1}{2}\omega r^2$, for then $\psi + \psi_0$ satisfies $\nabla^2(\psi + \psi_0) = 0$; and this is perhaps the simplest method of obtaining it. The results are in harmony; but the fact is not immediately apparent, inasmuch as Stokes expresses the motion by means of the velocity-potential, whereas here we have employed the stream-function.

That the subtraction of $\frac{1}{2}\omega r^2$ makes (10) an harmonic function shows that the series multiplying r^2 can be summed. In fact

$$8\alpha^2 \sum \frac{\sin(n\pi\theta/\alpha)}{n\pi(n^2\pi^2 - 4\alpha^2)} = \frac{\cos(2\theta - \alpha)}{2\cos\alpha} - \frac{1}{2},$$

so that

$$\psi/\omega = \frac{1}{2}r^2 - \frac{r^2 \cos(2\theta - \alpha)}{2\cos\alpha} + 8\alpha^2 \sum \frac{r^{n\pi/\alpha} \sin n\pi\theta/\alpha}{n\pi(n^2\pi^2 - 4\alpha^2)}. \quad (11)$$

In considering the character of the motion defined by (11) in the immediate vicinity of the origin we see that if $\alpha < \frac{1}{2}\pi$, the term in r^2 preponderates even when $n=1$. When $\alpha = \frac{1}{2}\pi$ exactly, the second term in (11) and the first term under Σ corresponding to $n=1$ become infinite, and the expression demands transformation. We find in this case

$$\begin{aligned} \psi/\omega = & \frac{1}{2}r^2 + \frac{2r^2}{\pi}(\theta - \frac{1}{4}\pi) \cos 2\theta + r^2 \sin 2\theta \left(\frac{2}{\pi} \log r - \frac{3}{2\pi} \right) \\ & + \frac{2}{\pi} \sum \frac{r^{2n} \sin 2n\theta}{n(n^2 - 1)}, \quad \dots \dots \dots (12) \end{aligned}$$

the summation commencing at $n=3$. On the middle line $\theta = \frac{1}{4}\pi$, we have

$$\psi/\omega = \frac{1}{2}r^2 - \frac{3r^2}{2\pi} + \frac{2}{\pi} \left\{ r^2 \log r - \frac{r^6}{3.8} + \frac{r^{10}}{5.24} - \dots \right\}. \quad (13)$$

The following are derived from (13):—

$r.$	$-\frac{1}{2}\pi\psi.$	$r.$	$-\frac{1}{2}\pi\psi.$	$r.$	$-\frac{1}{2}\pi\psi.$
0.0	·00000	0.4	·14112	0.8	·13030
0.1	·02267	0.5	·16507	0.9	·07641
0.2	·06296	0.6	·17306	1.0	·00000
0.3	·10521	0.7	·16210		

The maximum value occurs when $r = .592$. At the point $r = .592$, $\theta = \frac{1}{4}\pi$, the fluid is stationary.

A similar transformation is required when $\alpha=3\pi/2$.

When $\alpha=\pi$, the boundary becomes a semicircle, and the leading term ($n=1$) is

$$\psi/\omega = -\frac{8}{3\pi}r \sin \theta = -\frac{3}{8\pi}y \dots \dots (14)$$

which of itself represents an irrotational motion.

When $\alpha=2\pi$, the two bounding radii vectores coincide and the containing vessel becomes a circle with a single partition wall at $\theta=0$. In this case again the leading term is irrotational, being

$$\psi/\omega = -\frac{32}{15\pi}r^{\frac{3}{2}} \sin \frac{1}{2}\theta. \dots \dots (15)$$

Steady Motion in a Corner of a Viscous Fluid.

Here again we suppose the fluid to be incompressible and to move in two dimensions free from external forces, or at any rate from such as cannot be derived from a potential. If in the same notation as before ψ represents the stream-function, the general equation to be satisfied by ψ is

$$\nabla^4 \psi = 0; \dots \dots (1)$$

with the conditions that when $\theta=0$ and $\theta=\alpha$,

$$\psi=0, \quad d\psi/d\theta=0. \dots \dots (2)$$

It is worthy of remark that the problem is analytically the same as that of a plane elastic plate clamped at $\theta=0$ and $\theta=\alpha$, upon which (in the region considered) no external forces act.

The general problem thus presented is one of great difficulty, and all that will be attempted here is the consideration of one or two particular cases. We inquire what solutions are possible such that ψ , as a function of r (the radius vector), is proportional to r^m . Introducing this supposition into (1), we get

$$\left\{ m^2 + \frac{d^2}{d\theta^2} \right\} \left\{ (m-2)^2 + \frac{d^2}{d\theta^2} \right\} \psi = 0, \dots (3)$$

as the equation determining the dependence on θ . The most general value of ψ consistent with our suppositions is thus

$$\psi = r^m \{ A \cos m\theta + B \sin m\theta + C \cos (m-2)\theta + D \sin (m-2)\theta \}, \dots \dots (4)$$

where A, B, C, D are constants.

Equation (4) may be adapted to our purpose by taking
 $m = n\pi/\alpha, \dots \dots \dots (5)$

where n is an integer. Conditions (2) then give

$$A + C = 0, \quad A + C \cos 2\alpha - D \sin 2\alpha = 0,$$

$$\frac{n\pi}{\alpha} B + \left(\frac{n\pi}{\alpha} - 2\right) D = 0,$$

$$\frac{n\pi}{\alpha} B + \left(\frac{n\pi}{\alpha} - 2\right) C \sin 2\alpha + \left(\frac{n\pi}{\alpha} - 2\right) D \cos 2\alpha = 0.$$

When we substitute in the second and fourth of these equations the values of A and B , derived from the first and third, there results

$$C(1 - \cos 2\alpha) + D \sin 2\alpha = 0,$$

$$C \sin 2\alpha - D(1 - \cos 2\alpha) = 0;$$

and these can only be harmonized when $\cos 2\alpha = 1$, or $\alpha = s\pi$, where s is an integer. In physical problems, α is thus limited to the values π and 2π . To these cases (4) is applicable with C and D arbitrary, provided that we make

$$A + C = 0, \quad B + \left(1 - \frac{2s}{n}\right) D = 0. \quad \dots \quad (5)$$

Thus

$$\begin{aligned} \psi = & C r^{n/s} \left\{ \cos\left(\frac{n\theta}{s} - 2\theta\right) - \cos\frac{n\theta}{s} \right\} \\ & + D r^{n/s} \left\{ \sin\left(\frac{n\theta}{s} - 2\theta\right) - \left(1 - \frac{2s}{n}\right) \sin\frac{n\theta}{s} \right\}, \quad (6) \end{aligned}$$

making

$$\nabla^2 \psi = 4 \left(\frac{n}{s} - 1\right) r^{-2+n/s} \left\{ C \cos\left(\frac{n\theta}{s} - 2\theta\right) + D \sin\left(\frac{n\theta}{s} - 2\theta\right) \right\}. \quad \dots \dots \dots (7)$$

When $s=1$, $\alpha=\pi$, the corner disappears and we have simply a straight boundary (fig. 1). In this case $n=1$ gives a nugatory result. When $n=2$, we have

Fig. 1.

$$S = 1$$

$$\psi = C r^2 (1 - \cos 2\theta) = 2C y^2, \quad \dots \dots \dots (8)$$

and $\nabla^2 \psi = 4C$. When $n=3$,

$$\psi = Cr^3(\cos \theta - \cos 3\theta) + Dr^3(\sin \theta - \frac{1}{3} \sin 3\theta), \quad (9)$$

$$\nabla^2 \psi = 8r(C \cos \theta + D \sin \theta) = 8(Cx + Dy). \quad (10)$$

In rectangular coordinates

$$\psi = 4Cxy^2 + \frac{4}{3}Dy^3, \quad (11)$$

solutions which obviously satisfy the required conditions.

When $s=2$, $\alpha=2\pi$, the boundary consists of a straight wall extending from the origin in *one* direction (fig. 2). In

Fig. 2.

this case (6) and (7) give

$$\begin{aligned} \psi = Cr^{\frac{1}{2}n} \{ \cos (\tfrac{1}{2}n\theta - 2\theta) - \cos \tfrac{1}{2}n\theta \} \\ + Dr^{\frac{1}{2}n} \{ \sin (\tfrac{1}{2}n\theta - 2\theta) - \left(1 - \tfrac{1}{n}\right) \sin \tfrac{1}{2}n\theta \}, \end{aligned} \quad (12)$$

$$\begin{aligned} \nabla^2 \psi = (2n-4)r^{\frac{1}{2}n-2} \{ C \cos (\tfrac{1}{2}n\theta - 2\theta) \\ + D \sin (\tfrac{1}{2}n\theta - 2\theta) \}. \end{aligned} \quad (13)$$

Solutions of interest are afforded in the case $n=1$. The C-solution is ($C=\frac{1}{4}$)

$$\psi = \frac{1}{4}r^{\frac{1}{2}}(\cos \tfrac{3}{2}\theta - \cos \tfrac{1}{2}\theta) = -r^{\frac{1}{2}} \cos \tfrac{1}{2}\theta \sin^2 \tfrac{1}{2}\theta, \quad (14)$$

vanishing when $\theta=\pi$, as well as when $\theta=0$, $\theta=2\pi$, and for no other admissible value of θ . The values of ψ are reversed when we write $2\pi-\theta$ for θ . As expressed, this value is negative from 0 to π and positive from π to 2π . The minimum occurs when $\theta=109^\circ 28'$. Every streamline which enters the circle ($r=1$) on the left of this radius leaves it on the right.

The velocities, represented by $d\psi/dr$ and $r^{-1}d\psi/d\theta$, are infinite at the origin.

For the D-solution we may take

$$\psi = r^{\frac{1}{2}} \sin^3 \frac{1}{2} \theta. \quad . \quad . \quad . \quad (15)$$

Here ψ retains its value unaltered when $2\pi - \theta$ is substituted for θ . When r is given, ψ increases continuously from $\theta=0$ to $\theta=\pi$. On the line $\theta=\pi$ the motion is entirely transverse to it. This is an interesting example of the flow of viscous fluid round a sharp corner. In the application to an elastic plate ψ represents the displacement at any point of the plate, supposed to be clamped along $\theta=0$, and otherwise free from force within the region considered. The following table exhibits corresponding values of r and θ such as to make $\psi=1$ in (15) :—

θ .	r .	θ .	r .
180°	1.00	60°	64.0
150°	1.23	20°	$10^4 \times 3.65$
120°	2.37	10°	$10^6 \times 2.28$
90°	8.00	0°	∞

When $n=2$, (12) appears to have no significance.

When $n=3$, the dependence on θ is the same as when $n=1$. Thus (14) and (15) may be generalized :

$$\psi = (Ar^{\frac{1}{2}} + Br^{\frac{3}{2}}) \cos \frac{1}{2} \theta \sin^2 \frac{1}{2} \theta, \quad . \quad . \quad . \quad (16)$$

$$\psi = (A'r^{\frac{1}{2}} + B'r^{\frac{3}{2}}) \sin^3 \frac{1}{2} \theta. \quad . \quad . \quad . \quad (17)$$

For example, we could satisfy either of the conditions $\psi=0$, or $d\psi/dr=0$, on the circle $r=1$.

For $n=4$ the D-solution becomes nugatory ; but for the C-solution we have

$$\psi = Cr^2(1 - \cos 2\theta) = 2Cr^2 \sin^2 \theta = 2Cy^2. \quad . \quad (18)$$

The wall (or in the elastic plate problem the clamping) along $\theta=0$ is now without effect.

It will be seen that along these lines nothing can be done in the apparently simple problem of a horizontal plate clamped along the rectangular axes of x and y , if it be supposed free from force*. Ritz † has shown that the

* If indeed gravity act, $w=x^2y^2$ is a very simple solution.

† *Ann. d. Phys.* Bd. 28, p. 760, 1909.

solution is not developable in powers of x and y , and it may be worth while to extend the proposition to the more general case when the axes, still regarded as lines of clamping, are inclined at any angle α . In terms of the now oblique coordinates x, y the general equation takes the form

$$(d^2/dx^2 + d^2/dy^2 - 2 \cos \alpha d^2/dx dy)^2 w = 0, \quad . \quad . \quad (19)$$

which may be differentiated any number of times with respect to x and y , with the conditions

$$w = 0, \quad dw/dy = 0, \quad \text{when } y = 0, \quad . \quad . \quad (20)$$

$$w = 0, \quad dw/dx = 0, \quad \text{when } x = 0. \quad . \quad . \quad (21)$$

We may differentiate, as often as we please, (20) with respect to x and (21) with respect to y .

From these data it may be shown that at the origin *all* differential coefficients of w with respect to x and y vanish. The evanescence of those of zero and first order is expressed in (20), (21). As regards those of the second order we have from (20) $d^2w/dx^2 = 0$, $d^2w/dx dy = 0$, and from (21) $d^2w/dy^2 = 0$. Similarly for the third order from (20)

$$d^3w/dx^3 = 0, \quad d^3w/dx^2 dy = 0,$$

and from (21)

$$d^3w/dy^3 = 0, \quad d^3w/dx dy^2 = 0.$$

For the fourth order (20) gives

$$d^4w/dx^4 = 0, \quad d^4w/dx^3 dy = 0,$$

and (21) gives

$$d^4w/dy^4 = 0, \quad d^4w/dx dy^3 = 0.$$

So far $d^4w/dx^2 dy^2$ might be finite, but (19) requires that it also vanish. This process may be continued. For the $m+1$ coefficients of the m th order we obtain four equations from (20), (21) and $m-3$ by differentiations of (19), so that all the differential coefficients of the m th order vanish. It follows that every differential coefficient of w with respect to x and y vanishes at the origin. I apprehend that the conclusion is valid for all angles α less than 2π . That the displacement at a distance r from the corner should diminish rapidly with r is easily intelligible, but that it should diminish more rapidly than any power of r , however high, would, I think, not have been expected without analytical proof.