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Contribution to the Theory of Substitution

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SUGGESTIONS FOR THE PRACTICAL TREATMENT OF THE STANDARD CUBIC EQUATION, AND A CON- TRIBUTION TO THE THEORY OF SUBSTITUTION.

BY PROF. R. W. GENESE, M.A.

THE graphic methods in use are open to the objection of requiring an improvised graph, depending on the constants of the equation. To obtain even a 5 % approximation, the construction of such a graph demands calculations far in excess of those involved in Horner's method.

The object of the present paper is to advocate the desirability of either *stereotyped graphs*, or *special tables*. The former would also be most useful for didactic purposes.

If, in the equation $x^3 + 3px + q = 0$, we put $x = |p|^{\frac{1}{3}} X$,
then $X^3 \pm 3X = -qp^{-\frac{2}{3}} = c$, say,(1)
(the sign of the coefficient of X being that of p), and c can be quickly calculated by tables, or slide-rule.

The real value, or values, of X are, then, the abscissae of the intersections of $Y = X^3 \pm 3X$ and $Y = c$.

1°. For $Y = X^3 + 3X$, the graph shows at once that there is only one real value of X , say X_1 .

Since the sum of the roots of (1) is zero, if the imaginary roots be $\alpha \pm \beta i$, we have

$$2\alpha + X_1 = 0, \text{ giving } \alpha.$$

And from the coefficient of X ,

$$3 = \alpha^2 + \beta^2 + 2\alpha(-2\alpha),$$

$$\text{or } \frac{\beta^2}{3} - \alpha^2 = 1.$$

Hence, if the graph be accompanied by the auxiliary curve

$$\frac{y^2}{3} - x^2 = 1,$$

and the line $2x = -X_1$ meet this curve in Q, Q' , the imaginary roots of the X cubic are represented by OQ, OQ' (O origin).

In figure, $MK = -\frac{1}{2}MP$.

Unit of $x = 5$ small intervals.
 „ $y = \frac{1}{2}$ a small interval.
 Solution of $X^3 + 3X = 21$.
 $X = 2.4$ app. and $-1.2 \pm 2.7i$.

Instead of the cubic graph, four-figure tables might be prepared of the values of $X^3 + 3X$ (for the range likely to be required), from which X_1 might be read off; the conic graph would then give the imaginary roots of X .

Or, with loss of time, if tables of $\sinh$ and $\cosh$ be available, the substitution $X = 2 \sinh v$ gives $\sinh 3v = c/2$; v being found, the imaginary roots are $-\sinh v \pm \sqrt{-3} \cosh v$.

2°.

$$X^3 - 3X = c.$$

Graph for $c = 2.96$.

The graph shows at once that for imaginary roots $|c|$ must be greater than 2: and they are given by a construction similar to the previous one, the auxiliary conic being changed to

$$x^2 - \frac{y^2}{3} = 1.$$

In studying the relation between the roots (x_1, x_2, x_3) of the cubic equation, the form $x^3 - 3x + c = 0$ is to be preferred, as including the "irreducible case." (The slight modifications of the formulae obtained for this form, consequent on a change to $x^3 - 3px + c = 0$, will be given at the end of this note.)

If we use the transformation

$$y = \alpha x^2 + \beta x + \gamma \equiv S(x),$$

the roots of the resulting cubic in y are $S(x_1)$, $S(x_2)$, $S(x_3)$; and, theoretically, we can choose α , β , γ so that the cubic in y is identical with that in x .

Then $S(x_1) = x_1$, x_2 or x_3 ; and $S(x_2)$, $S(x_3)$ supply the other roots.

There are three possibilities.

$$1^\circ. \quad S(x_1) = x_1; \quad S(x_2) = x_2; \quad S(x_3) = x_3; \dots\dots\dots (A)$$

but, since S is quadratic, this can only arise when two of the roots are equal, or $|c| = 2$.

$$2^\circ. \quad S(x_1) = x_1; \quad S(x_2) = x_3; \quad S(x_3) = x_2. \dots\dots\dots (B)$$

$$3^\circ. \quad S(x_1) = x_2; \quad S(x_2) = x_3; \quad S(x_3) = x_1. \dots\dots\dots (C)$$

Distinct formulae will be obtained for each case.

The relation necessary for the vanishing of y^2 gives

$$\gamma = -2\alpha.$$

Thus

$$S(x) \equiv \alpha(x^2 - 2) + \beta x.$$

The next relation, from comparing terms of first degree, is

$$c = \frac{\alpha^2 + \beta^2 - 1}{\alpha\beta}.$$

$$\text{And the third, } \alpha^3 c^2 - (3\alpha^2 \beta + \beta^3 - 1)c + \alpha(6\beta^2 - 2\alpha^2) = 0.$$

Eliminating c , in order to obtain the relation between α and β for all cubics, there results a cubic equation in α^2 , which, however, admits of the solutions

$$\alpha^2 = (\beta - 1)^2, \quad \beta^2 + \beta \quad \text{and} \quad \beta^2 + \beta + 1.$$

1°. $\alpha^2 = (\beta - 1)^2$ gives $c = \pm 2$, that is, applies only to the case of equal roots. Taking $\alpha = \beta - 1$, $c = 2$,

$$\begin{aligned} S(x) &\equiv \alpha(x^2 - 2) + \beta x \\ &= x \text{ when } x = +1 \text{ or } -2. \end{aligned}$$

With $\alpha = 1 - \beta$, $c = -2$, and

$$\begin{aligned} S(x) &\equiv \alpha(x^2 - 2) + \beta x \\ &= x \text{ when } x = -1 \text{ or } +2. \end{aligned} \quad \text{This is the case (A).}$$

2°. $\alpha^2 = \beta^2 + \beta$, and then

$$\begin{aligned} c &= \frac{2\beta^2 + \beta - 1}{\beta^{\frac{3}{2}}(\beta + 1)^{\frac{1}{2}}} = -\frac{1 - 2\beta}{\beta} \frac{(1 + \beta)^{\frac{1}{2}}}{\beta^{\frac{1}{2}}} \\ &= -\left(\frac{1}{\beta} + 1 - 3\right)\left(\frac{1}{\beta} + 1\right)^{\frac{1}{2}} \end{aligned}$$

and, therefore, one root x_1 of the cubic is $\left(\frac{1}{\beta} + 1\right)^{\frac{1}{2}} = \frac{\alpha}{\beta}$.

$$\begin{aligned} S(x_1) &= \alpha(x_1^2 - 2) + \beta x_1 = \beta x_1(x_1^2 - 1) \\ &= \beta x_1 \left(\frac{1}{\beta}\right) \\ &= x_1. \end{aligned}$$

Hence, since the equation has not equal roots, we must have

$$S(x_2) = x_3 \text{ and } S(x_3) = x_2; \text{ which is case (B).}$$

Ex. gr. For $x^3 - 3x = \frac{9}{8}$, $x_1 = \frac{3}{2}$.

$$S(x_1) = \frac{6}{5}(x_1^2 - 2) + \frac{4}{5}x_1 = \frac{1}{5}x_1 + \frac{4}{5}x_1 = x_1,$$

$$S(x_2) = \frac{6}{5}(x_2^2 + \frac{3}{2}x_2 - \frac{3}{4}) - x_2 - \frac{3}{2} = x_3.$$

Conversely, if we begin with the last equations,

$$\alpha(x_2^2 - 2) + \beta x_2 = x_3,$$

$$\alpha(x_3^2 - 2) + \beta x_3 = x_2,$$

we get, subtracting and dividing by $x_2 - x_3$,

$$\alpha(x_2 + x_3) + \beta = -1,$$

$$\text{i.e. } -\alpha x_1 + \beta = -1,$$

$$x_1 = \frac{1 + \beta}{\alpha}$$

$$= \frac{\alpha}{\beta},$$

and, thus, return to the condition $\alpha^2 = \beta^2 + \beta$.

$$3^*. \quad \alpha^2 = \beta^2 + \beta + 1.$$

There is nothing left for this but case (C).

For the equation $x^3 - 3px + c = 0$, the relations become

$$\gamma = -2p\alpha,$$

$$c = \frac{p^2\alpha^2 + p\beta^2 - p}{\alpha\beta},$$

and

$$p\alpha^2 = (\beta - 1)^2, \quad \beta^2 + \beta \quad \text{and} \quad \beta^2 + \beta + 1$$

in the cases (A), (B), (C). (The reproduced root in (B) is $\frac{p\alpha}{\beta}$).

It may be of interest to add that to every transformation of the form

$$y = \alpha(x^2 - 2p) + \beta x,$$

there corresponds *only one cubic equation*. Ex. gr. $y = x^2 - 14 + 2x$ gives

$$p = 7 (= 2^2 + 2 + 1), \quad c = \frac{49 + 28 - 7}{2} = 35,$$

and the equation is

$$x^3 - 21x + 35 = 0.$$

All the above results may also be obtained from the sets of S equations (A), (B), (C).

Any values may be assigned to α and β . Then the last relation gives p and the preceding c .

The theory may be restated thus:

The condition that $y = \alpha x^2 + \beta x + \gamma$ should be a reproducing transformation, for some cubic $x^3 + 3px + q = 0$ is

$$-\frac{1}{2}\alpha\gamma = \text{either } (\beta - 1)^2 \quad \text{or} \quad \beta^2 + \beta \quad \text{or} \quad \beta^2 + \beta + 1.$$

In the 1st case, each root is untransformed.

„ 2nd „ one root $\left(= \frac{p\alpha}{\beta} \right)$ is untransformed,

the other two are interchanged.

„ 3rd „ the roots are transformed cyclically.

To each transformation corresponds *only one cubic equation*, given by

$$p = -\frac{\gamma}{2\alpha},$$

$$q = \frac{p^2\alpha^2 + p\beta^2 - p}{\alpha\beta}.$$

Example of the 2nd case (α, β are at our disposal).

Let $\alpha = 1, \beta = 2$, then $\gamma = -12, p = 6, q = 27$.

Thus, to $y = x^2 + 2x - 12$ there corresponds

$$x^3 - 18x + 27 = 0 = (x - 3)(x^2 + 3x - 9),$$

$$S(3) = 9 + 6 - 12 = 3.$$

If x_2, x_3 be the other roots,

$$S(x_2) = (9 - 3x_2) + 2x_2 - 12 = -3 - x_2 = x_3, \quad \therefore x_2 + x_3 = -3.$$

Aberystwyth, August 1916.

8-POINT AND 8-TANGENT CONICS, AND AN ASSOCIATED PENCIL AND RANGE OF CONICS.

THE following notes are a development of subject-matter which is treated in the writer's book on *Cross-Ratio Geometry*, chiefly in Chap. XIV., and the reader is referred to articles in that work for the demonstration of any property that is given here without proof.

1. In the figure, $T_1 \dots T_6$ is a complete quadrilateral, $P_1 P_2 P_3$ being its diagonal triangle. Then $P_1 P_2 P_3$ is self-conjugate for the quadrilateral.

2. Suppose a conic A is inscribed in the quadrilateral, and let $a_1 \dots a_4$ be its points of contact with the sides T_2T_4 , T_4T_1 , T_1T_3 , T_3T_2 . Then if we