THE

MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; Prof. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.
E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON:

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,

AND BOMBAY.

Vol. II.	March, 1902.	No. 32.

THE TRIGONOMETRY OF THE TETRAHEDRON.

So far as I know the properties of a tetrahedron are not given in any mathematical text-book. In the late Dr. Wolstenholme's *Examples for Practice in the use of Seven-figure Logarithms*, the formulae which are necessary to determine the plane and dihedral angles and the volume of the tetrahedron, having given the six edges, are stated, but nothing beyond this. Dr. Wolstenholme worked at the subject for some years, and several of his results were sent as problems to the *Educational Times*, where they appeared for the most part after his death. In one set of questions he remarks that they are "the equations I have been looking for for years," and so I think we may fairly infer that these questions represent the more advanced results of his investigations. The solutions of these questions, so far as I have been able to find them, are practically all given in this paper. With regard to the rest of the matter, I hardly dare claim any originality. Wolstenholme must have known all or nearly all the results, and any one seriously attacking the subject must come upon them. All I can say is that I have never seen them in print, and that under any circumstances the bringing of them together must be helpful to students in many branches of mathematics.

The following notation will be used:

$OABC$ is the tetrahedron; V its volume.
$\Delta_0, \Delta_1, \Delta_2, \Delta_3$ the areas of the faces opposite to O, A, B, C respectively.
a, b, c, x, y, z the lengths of OA, OB, OC, BC, CA, AB.
p_0, p_1, p_2, p_3 the perpendiculars from O, A, B, C on the opposite faces.
d_1, d_2, d_3 the joins of the mid-points of a and x, b and y, c and z.
l, m, n the perpendicular distances between a and x, b and y, c and z.
$\alpha_1, \beta_2, \gamma_3$ the angles BOC, COA, AOB;
$\alpha_0, \beta_3, \gamma_2$ " BAC, OAB, CAO;
$\alpha_3, \beta_0, \gamma_1$ " OBA, ABC, CBO;
$\alpha_2, \beta_1, \gamma_0$ " OCA, BCO, ACB.

It should be noticed that

α is *not* adjacent to either a or x,
β " " b or y,
γ " " c or z,

and that the subscripts are the same as those of the faces Δ in which the angles lie.

A, B, C, X, Y, Z the dihedral angles whose edges are a, b, c, x, y, z.
λ, μ, ν, and their supplements, the angles between a and x, b and y, c and z.
θ_1, ϕ_2, ψ_3 the inclinations of a and Δ_1, b and Δ_2, c and Δ_3;
θ_0, ϕ_3, ψ_2 " a and Δ_0, y and Δ_3, z and Δ_2;
θ_3, ϕ_0, ψ_1 " x and Δ_3, b and Δ_0, z and Δ_1;
θ_2, ϕ_1, ψ_0 " x and Δ_2, y and Δ_1, c and Δ_0.
Again it should be noticed that the angles
θ are the inclinations of a and x to the faces opposite to them;
ϕ " " b and y " " "
ψ " " c and z " " "
and that the subscripts are the same as those of the faces Δ, to which the edges are respectively inclined.

The fraction $4\Delta_0\Delta_1\Delta_2\Delta_3/9V^2$ will be called h^2.

R, r, r_0, r_1, r_2, r_3 the radii of the circum-sphere, the in-sphere, and the four ex-spheres opposite to O, A, B, C respectively.

1. Speaking generally the tetrahedron is completely determined when we know six parts, one of which must be a length or an area; and so five angles are required in order to determine the other angles. We have however the four necessary relations

$$\alpha_0+\beta_0+\gamma_0=\alpha_1+\beta_1+\gamma_1=\alpha_2+\beta_2+\gamma_2=\alpha_3+\beta_3+\gamma_3=\pi. \quad \text{.........(1)}$$

$\alpha_1, \beta_2, \gamma_3$ are the sides; A, B, C the opposite angles; and θ_1, ϕ_2, ψ_3 the perpendiculars from the angles on the opposite sides of a spherical triangle, which we shall speak of as the spherical triangle at O.

The corresponding parts of the spherical triangle

at A are $\alpha_0, \beta_3, \gamma_2, A, Y, Z, \theta_0, \phi_3, \psi_2$;
at B are $\alpha_3, \beta_0, \gamma_1, X, B, Z, \theta_3, \phi_0, \psi_1$;
at C are $\alpha_2, \beta_1, \gamma_0, X, Y, C, \theta_2, \phi_1, \psi_0$.

The identities

$$1-\Sigma\cos^2\alpha+2\Pi\cos\alpha=\begin{vmatrix} 1, & \cos\beta, & \cos\gamma \\ \cos\beta, & 1, & \cos\alpha \\ \cos\gamma, & \cos\alpha, & 1 \end{vmatrix}$$

and

$$1-\Sigma\cos^2 A-2\Pi\cos A=-\begin{vmatrix} -1, & \cos B, & \cos C \\ \cos B, & -1, & \cos A \\ \cos C, & \cos A, & -1 \end{vmatrix}$$

will be found useful.

2. We commence by giving the relations between the different angles at a vertex. Neglecting the subscripts and calling the dihedral angles A, B, C, as if the vertex was O, we have from the ordinary formula of spherical trigonometry,

$$\cos A=(\cos\alpha-\cos\beta\cos\gamma)/(\sin\beta\sin\gamma), \text{ etc.}, \quad \text{.........(2)}$$

$$\sin A/\sin\alpha=\ldots=\ldots=\sqrt{1-\Sigma\cos^2\alpha+2\Pi\cos\alpha}/\Pi\sin\alpha$$
$$=2\sqrt{\sin\sigma\Pi\sin(\sigma-\alpha)}/\Pi\sin\alpha, \quad \text{.........(3)}$$

$$\tan\frac{A}{2}\sin(\sigma-\alpha)=\ldots=\ldots=\sqrt{\Pi\sin(\sigma-\alpha)/\sin\sigma}, \quad \text{.........(4)}$$

where $2\sigma=\Sigma\alpha$.

The last set are most useful for numerical calculations.

3. Similarly the plane angles are given in terms of the dihedral angles,

$$\cos\alpha=(\cos A+\cos B\cos C)/\sin B\sin C, \text{ etc.}, \quad \text{.........(5)}$$

$$\sin\alpha/\sin A=\ldots=\ldots=\sqrt{1-\Sigma\cos^2 A-2\Pi\cos A}/\Pi\sin A, \quad \text{.........(6)}$$

with corresponding equations for $\tan\frac{\alpha}{2}$, etc.

4. From the spherical triangle we have the equations

$$\sin\theta=\sin\beta\sin C=\sin\gamma\sin B,$$

with similar equations for $\sin\phi$ and $\sin\psi$.

From (3) and (6),

$$\sin\theta\sin\alpha=\sin\phi\sin\beta=\sin\psi\sin\gamma=\sqrt{1-\Sigma\cos^2\alpha+2\Pi\cos\alpha}, \quad\text{......(7)}$$

$$\sin\theta\sin A=\sin\phi\sin B=\sin\psi\sin C=\sqrt{1-\Sigma\cos^2 A-2\Pi\cos A}. \quad\text{......(8)}$$

Hence, given the three plane or dihedral angles at any vertex, the remaining angles are easily determined.

5. If the three angles θ, ϕ, ψ are given, the determination of the plane or dihedral angles is not so simple.

Multiplying (7) by $\sin\phi\sin\psi$, squaring and transposing, we have

$$\begin{aligned}2\Pi\cos\alpha\,.\,\sin^2\phi\sin^2\psi=2\sin^2\phi\sin^2\psi&-\sin^2\alpha\cos^2\theta\sin^2\phi\sin^2\psi\\&-\sin^2\beta\sin^2\phi\sin^2\psi-\sin^2\gamma\sin^2\phi\sin^2\psi,\end{aligned}$$

or

$$\begin{aligned}&2\cos\alpha\sin\phi\sin\psi\sqrt{(\sin^2\phi-\sin^2\alpha\sin^2\theta)(\sin^2\psi-\sin^2\alpha\sin^2\theta)}\\&\quad=2\sin^2\phi\sin^2\psi-\sin^2\alpha\cos^2\theta\sin^2\phi\sin^2\psi-\sin^2\alpha\sin^2\theta(\sin^2\phi+\sin^2\psi),\end{aligned}$$

which may be reduced to

$$4\sin^4\alpha\sin^2\theta-\sin^2\alpha\left\{2\Sigma\sin^2\theta-\left(1+\Sigma\frac{\cos 2\theta}{\sin^4\theta}\right)\Pi\sin^2\theta\right\}+4\sin^2\phi\sin^2\psi=0.\text{...(9)}$$

The determination of A in terms of θ, ϕ, ψ results in the same equation; consequently the two roots of this equation give both $\sin^2\alpha$ and $\sin^2 A$. This might have been inferred from the symmetry of the relations (7) and (8).

The different signs in the product of the three cosines of α, β, γ and of A, B, C in (7) and (8) show that when the three perpendiculars from the angular points of a spherical triangle on the opposite sides are given, there are two such triangles, each of which is the polar of the other.

6. The following are the most important expressions for the volume of the tetrahedron:

$$6V=2p_1\Delta_1=abc\sin\theta_1\sin\alpha_1,$$

which from (7)

$$\begin{aligned}&=abc\sqrt{1-\cos^2\alpha_1-\cos^2\beta_2-\cos^2\gamma_3+2\cos\alpha_1\cos\beta_2\cos\gamma_3} \quad\text{(10)}\\&=abc\sin\beta_2\sin\gamma_3\sin A,\ \text{from (3),} \quad\text{............................(11)}\\&=4\Delta_2\Delta_3\sin A/a=4\Delta_0\Delta_1\sin X/x\ \text{(by symmetry)}\\&=4\Delta_1\Delta_3\sin B/b=4\Delta_0\Delta_2\sin Y/y=\text{etc.} \quad\text{........................(12)}\end{aligned}$$

Hence we obtain

$$\frac{ax}{\sin A\sin X}=\frac{by}{\sin B\sin Y}=\frac{cz}{\sin C\sin Z}=\frac{4\Delta_0\Delta_1\Delta_2\Delta_3}{9V^2}=h^2. \quad\text{...........(13)}$$

From (10) we have

$$36V^2=a^2b^2c^2\begin{vmatrix}1, & \cos\gamma_3, & \cos\beta_2\\ \cos\gamma_3, & 1, & \cos\alpha_1\\ \cos\beta_2, & \cos\alpha_1, & 1\end{vmatrix};$$

$$\therefore\ 288V^2=\begin{vmatrix}2a^2, & a^2+b^2-z^2, & c^2+a^2-y^2\\ a^2+b^2-z^2, & 2b^2, & b^2+c^2-x^2\\ c^2+a^2-y^2, & b^2+c^2-x^2, & 2c^2\end{vmatrix}, \quad\text{.............(14)}$$

which, if expanded, gives

$$144V^2=\Sigma a^2x^2(b^2+y^2+c^2+z^2-a^2-x^2)-\Sigma x^2y^2z^2. \quad\text{...............(15)}$$

See Salmon's *Geometry of Three Dimensions*, Art. 53.

In the *Educational Times* for May, 1897, part of problem 13494 by Wolstenholme is

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ b^2+c^2-x^2, & y^2+z^2-x^2, & a^2-b^2-z^2, & a^2-y^2-c^2 \\ c^2+a^2-y^2, & b^2-a^2-z^2, & z^2+x^2-y^2, & b^2-c^2-x^2 \\ a^2+b^2-z^2, & c^2-a^2-y^2, & c^2-x^2-b^2, & x^2+y^2-z^2 \end{vmatrix} = -1152V^2,$$

which agrees with (14).

7. Besides the four identities (1) the twelve plane angles must be connected by three equations, which may be thus found:

$$x/\sin\alpha_0 = y/\sin\beta_0 = z/\sin\gamma_0 \text{ (from triangle } ABC),$$
$$x/\sin\alpha_1 = b/\sin\beta_1 = c/\sin\gamma_1 \text{ (from triangle } OBC),$$
$$a/\sin\alpha_2 = y/\sin\beta_2 = c/\sin\gamma_2 \text{ (from triangle } OCA),$$
$$a/\sin\alpha_3 = b/\sin\beta_3 = z/\sin\gamma_3 \text{ (from triangle } OAB);$$
$$\therefore\ a/\sin\alpha_3 = x\sin\beta_1/\sin\alpha_1\sin\beta_3 = x\sin\gamma_0/\sin\alpha_0\sin\gamma_3,$$

and
$$a/\sin\alpha_2 = x\sin\beta_0/\sin\alpha_0\sin\beta_2 = x\sin\gamma_1/\sin\alpha_1\sin\gamma_2;$$

$$\therefore\ \frac{\sin\alpha_2\sin\beta_0}{\sin\alpha_0\sin\beta_2} = \frac{\sin\alpha_3\sin\beta_1}{\sin\alpha_1\sin\beta_3} = \frac{\sin\alpha_3\sin\gamma_0}{\sin\alpha_0\sin\gamma_3} = \frac{\sin\alpha_2\sin\gamma_1}{\sin\alpha_1\sin\gamma_2} = \frac{a}{x}. \quad\ldots\ldots\ldots(16)$$

And of course there are similar expressions equal to b/y and c/z. If however five plane angles are given the rest cannot be found *directly* from these relations.

These relations (16) are equivalent to the statement that: Considering any three triangles having a common vertex, the product of the ratios of the sines of the two angles not at the common vertex in each triangle taken in order is equal to unity.

It should be noticed that from the above equations the ratio of the lengths of any pair of edges can be expressed in terms of the sines of some of the plane angles.

8. Relations between the θ, ϕ, ψ angles can be found by expressing the lengths of the perpendiculars from the vertices on the opposite faces, thus:

$$p_0 = a\sin\theta_0 = b\sin\phi_0 = c\sin\psi_0,$$
$$p_1 = a\sin\theta_1 = y\sin\phi_1 = z\sin\psi_1,$$
$$p_2 = x\sin\theta_2 = b\sin\phi_2 = z\sin\psi_2,$$
$$p_3 = x\sin\theta_3 = y\sin\phi_3 = c\sin\psi_3.$$

From which we may obtain, as in case of the plane angles, that

$$\frac{a}{x} = \frac{\sin\theta_2\sin\phi_0}{\sin\theta_0\sin\phi_2} = \frac{\sin\theta_3\sin\phi_1}{\sin\theta_1\sin\phi_3} = \frac{\sin\theta_3\sin\psi_0}{\sin\theta_0\sin\psi_3} = \frac{\sin\theta_2\sin\psi_1}{\sin\theta_1\sin\psi_2}. \quad\ldots\ldots\ldots(17)$$

These relations are identical in form to those just found (16) existing between the plane angles, and may be expressed in the statement that: Considering any three edges in the same plane, the product of the ratios of the sines of the angles which each edge makes with the two opposite faces taken in order is equal to unity.

As in the case of the plane angles, the above equations enable us to express the ratio of the lengths of any pair of edges in terms of the sines of the θ, ϕ, ψ angles.

The above give only three independent relations between these twelve angles; the remaining four may be found from equations corresponding to (9). As there mentioned, (9) is a quadratic in $\sin^2 A$ as well as in $\sin^2\alpha$; and if we form the corresponding quadratic in $\sin^2 A$ from the angles at the vertex A, the eliminant may be written down, but it is extremely cumbrous, and does not look as if it would reduce to anything reasonable. Of course six such equations could be found of which two must result from the other four by the aid of (17).

9. The relation between the six dihedral angles may be found by projecting in turn three of the faces on the fourth. Hence we obtain

$$\left.\begin{array}{l} -\Delta_0 \quad +\Delta_1 \cos X+\Delta_2 \cos Y+\Delta_3 \cos Z=0 \\ \Delta_0 \cos X-\Delta_1 \quad +\Delta_2 \cos C+\Delta_3 \cos B=0 \\ \Delta_0 \cos Y+\Delta_1 \cos C-\Delta_2 \quad +\Delta_3 \cos A=0 \\ \Delta_0 \cos Z+\Delta_1 \cos B+\Delta_2 \cos A-\Delta_3 \quad =0 \end{array}\right\} \quad \text{(A)}$$

and the eliminant is

$$\begin{vmatrix} -1, & \cos X, & \cos Y, & \cos Z \\ \cos X, & -1, & \cos C, & \cos B \\ \cos Y, & \cos C, & -1, & \cos A \\ \cos Z, & \cos B, & \cos A, & -1 \end{vmatrix}=0, \quad (18)$$

which expands into

$$1-\Sigma \cos^2 A-2\Sigma \cos A \cos B \cos C$$
$$-2\Sigma \cos B \cos Y \cos C \cos Z+\Sigma \cos^2 A \cos^2 X=0.$$

If five of these angles are given, this is a quadratic for determining the sixth. If this equation be solved for $\cos A$, we have

$$\sin^2 X \cos A=-(\cos B \cos C+\cos Y \cos Z+\cos B \cos X \cos Y+\cos C \cos Z \cos X)$$
$$\pm\sqrt{(1-\cos^2 B-\cos^2 Z-\cos^2 X-2 \cos B \cos Z \cos X)}$$
$$\times\sqrt{(1-\cos^2 C-\cos^2 X-\cos^2 Y-2 \cos C \cos X \cos Y)}. \quad (19)$$

This result might also be obtained from (5) and (6); and from these equations we can see that only one root in (19) is admissible: for from (5) at the vertex B, $\cos \gamma_1$, and consequently γ_1, has but one value; similarly at the vertex C, β_1 has but one value; therefore from (1) α_1 has but one value; and therefore from the first equation of (5) A has only one value. It thus follows that if five *specified* dihedral angles are given and one edge, only one tetrahedron is determined.

10. To find the ratios of the lengths of the edges in terms of the dihedral angles.

Denoting by $\delta(\cos A)$, $\delta(\cos X)$, etc., the co-factors of $\cos A$, $\cos X$, etc., in the development of the determinant (18), we obtain from equations (**A**)

$$\begin{array}{llll} & \Delta_1/\delta(\cos X) & =\Delta_2/\delta(\cos Y) & =\Delta_3/\delta(\cos Z), \\ \Delta_0/\delta(\cos X) & & =\Delta_2/\delta(\cos C) & =\Delta_3/\delta(\cos B), \\ \Delta_0/\delta(\cos Y) & =\Delta_1/\delta(\cos C) & & =\Delta_3/\delta(\cos A), \\ \Delta_0/\delta(\cos Z) & =\Delta_1/\delta(\cos B) & =\Delta_2/\delta(\cos A). & \end{array}$$

Noting that $\delta(\cos A) . \delta(\cos X)=\delta(\cos B) . \delta(\cos Y)=\delta(\cos C) . \delta(\cos Z)$, and that the co-factors are all negative quantities, we obtain

$$\Delta_0\Delta_1/\delta(\cos X)=\Delta_2\Delta_3/\delta(\cos A)=-\{\Delta_0\Delta_1\Delta_2\Delta_3/\delta(\cos A) . \delta(\cos X)\}^{\frac{1}{2}}$$
$$=\Delta_0\Delta_2/\delta(\cos Y)=\Delta_0\Delta_3/\delta(\cos Z)=\Delta_1\Delta_2/\delta(\cos C)=\Delta_1\Delta_3/\delta(\cos B).$$

Combining these equations with (12) we have

$${}^*a/\sin A\,\delta(\cos A)=b/\sin B\,\delta(\cos B)=\ldots=-h/\{\delta(\cos A)\delta(\cos X)\}^{\frac{1}{2}} \quad (20)$$

* For this result I have to thank Dr. F. S. Macaulay.

11. The areas of the faces may also be expressed in terms of the dihedral angles and the quantity h.

From (6) we have

$$\begin{aligned}(1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C)^3 \\ &=(\sin\alpha_1\sin\beta_2\sin\gamma_3\sin^2 A\sin^2 B\sin^2 C)^2 \\ &=\left\{\frac{8\Delta_1\Delta_2\Delta_3}{a^2b^2c^2}\times\left(\frac{3Va}{2\Delta_2\Delta_3}\cdot\frac{3Vb}{2\Delta_3\Delta_1}\cdot\frac{3Vc}{2\Delta_1\Delta_2}\right)^2\right\}^2, \text{ from (12)}, \\ &=(8\Delta_0{}^3/h^6)^2;\end{aligned}$$

$$\therefore\ ^*1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C=4\Delta_0{}^2/h^4;$$

$$\therefore\ \frac{\Delta_0{}^2}{\begin{vmatrix}-1, & \cos B, & \cos C\\ \cos B, & -1, & \cos A\\ \cos C, & \cos A, & -1\end{vmatrix}}=\frac{\Delta_1{}^2}{\begin{vmatrix}-1, & \cos Y, & \cos Z\\ \cos Y, & -1, & \cos A\\ \cos Z, & \cos A, & -1\end{vmatrix}}$$

$$=\frac{\Delta_2{}^2}{\begin{vmatrix}-1, & \cos X, & \cos Z\\ \cos X, & -1, & \cos B\\ \cos Z, & \cos B, & -1\end{vmatrix}}=\frac{\Delta_3{}^2}{\begin{vmatrix}-1, & \cos X, & \cos Y\\ \cos X, & -1, & \cos C\\ \cos Y, & \cos C, & -1\end{vmatrix}}=-\frac{h^4}{4}\text{.......(21)}$$

12. The following is due to Wolstenholme. Shortly before his death I happened to meet him, and he wrote down the result as a problem for me to work at. I sent it to the *Educational Times*, where it appeared June, 1897, No. 13521 :

$$\frac{(a+x)^2+u}{\sin^2\frac{1}{2}(A+X)}=\frac{(a-x)^2+u}{\sin^2\frac{1}{2}(A-X)}=\frac{(b+y)^2+u}{\sin^2\frac{1}{2}(B+Y)}=\ldots=4h^2, \text{ ....(22)}$$

where

$$u\equiv(\Delta_0+\Delta_1+\Delta_2+\Delta_3)(\Delta_0+\Delta_1-\Delta_2-\Delta_3)(\Delta_0+\Delta_2-\Delta_1-\Delta_3)(\Delta_0+\Delta_3-\Delta_1-\Delta_2)/9V^2,$$

$$\begin{aligned}9V^2u&=\{(\Delta_0+\Delta_1)^2-(\Delta_2+\Delta_3)^2\}\{(\Delta_0-\Delta_1)^2-(\Delta_2-\Delta_3)^2\}\\ &=(\Delta_0{}^2+\Delta_1{}^2-\Delta_2{}^2-\Delta_3{}^2)^2-4(\Delta_0\Delta_1-\Delta_2\Delta_3)^2.\end{aligned}$$

Also by multiplying equations (A) [§ 9] by Δ_0, Δ_1, $-\Delta_2$, $-\Delta_3$ in order, and adding we obtain

$$\Delta_0{}^2+\Delta_1{}^2-\Delta_2{}^2-\Delta_3{}^2=2\Delta_0\Delta_1\cos X-2\Delta_2\Delta_3\cos A.\text{..............(23)}$$

Hence

$$\begin{aligned}9V^2u&=4(\Delta_0\Delta_1\cos X-\Delta_2\Delta_3\cos A)^2-4(\Delta_0\Delta_1-\Delta_2\Delta_3)^2\\ &=8\Delta_0\Delta_1\Delta_2\Delta_3(1-\cos A\cos X)-4(\Delta_0{}^2\Delta_1{}^2\sin^2 X+\Delta_2{}^2\Delta_3{}^2\sin^2 A)\\ &=8\Delta_0\Delta_1\Delta_2\Delta_3(1-\cos A\cos X)-9(a^2+x^2)V^2, \text{ from (12)};\end{aligned}$$

$$\begin{aligned}\therefore\ (a+x)^2+u&=2h^2(1-\cos A\cos X)+2ax\\ &=2h^2\{1-\cos(A+X)\}, \text{ from (13)},\\ &=4h^2\sin^2\tfrac{1}{2}(A+X).\end{aligned}$$

Similarly $(a-x)^2+u=4h^2\sin^2\frac{1}{2}(A-X)$.

And so for $B\pm Y$, $C\pm Z$.

This investigation has been abbreviated from my own solution by a hint received from the solution of H. W. Curjel, M.A., in the reprint of the *Mathematical Times Solutions*, vol. LXVIII., p. 102.

13. The *Educational Times*, September, 1897.

13605. (The late Professor Wolstenholme, M.A., Sc.D.)—" *Tetrahedron, of course.* The equations I have been looking for for years :

$$[(b\pm y)^2-(c\pm z)^2]\sin^2\tfrac{1}{2}(A\pm X)+[(c\pm z)^2-(a\pm x)^2]\sin^2\tfrac{1}{2}(B\pm Y)$$
$$+[(a\pm x)^2-(b\pm y)^2]\sin^2\tfrac{1}{2}(C\pm Z)\equiv 0;\text{...(24)}$$

* For this result I have to thank Dr. F. S. Macaulay.

the ambiguities being always of the same sign in any term $(a\pm x)^2$ as in the corresponding term $\sin^2\frac{1}{2}(A\pm X)$. The relations between the lengths of edges and dihedral angles are

$$\frac{(a+x)^2}{k+\sin^2\frac{1}{2}(A+X)}=\frac{(b+y)^2}{k+\sin^2\frac{1}{2}(B+Y)}=\frac{(c+z)^2}{k+\sin^2\frac{1}{2}(C+Z)}$$
$$=\frac{(a-x)^2}{k+\sin^2\frac{1}{2}(A-X)}=\frac{(b-y)^2}{k+\sin^2\frac{1}{2}(B-Y)}=\frac{(c-z)^2}{k+\sin^2\frac{1}{2}(C-Z)}=4h^2.$$

.... Can any one supply the exact value of k? I cannot at present."

["I think I have now accomplished all I can in the theory of the tetrahedron, and propose to pretermit the study for some time. There is, however, still a great deal to be done in the goniometry of the tetrahedron, and I expect sometime the theory of elliptic functions will start from it. I am too old and broken to have any hope of accomplishing this myself, or I would not divulge the idea." J. W., Oct. 22, 1889.]

Starting with (22), namely,

$$\frac{(a+x)^2+u}{\sin^2\frac{1}{2}(A+X)}=\frac{(b+y)^2+u}{\sin^2\frac{1}{2}(B+Y)}=\frac{(c+z)^2+u}{\sin^2\frac{1}{2}(C+X)};$$

it at once follows that each of these fractions

$$=\frac{0}{[(b+y)^2-(c+z)^2]\sin^2\frac{1}{2}(A+X)+[(c+z)^2-(a+x)^2]\times\sin^2\frac{1}{2}(B+Y)+[(a+x)^2-(b+y)^2]\sin^2\frac{1}{2}(C+Z)},$$

which establishes the first identity when the upper signs are taken; and similarly when the lower signs are taken.

Again, since
$$\frac{(a+x)^2+u}{\sin^2\frac{1}{2}(A+X)}=4h^2;$$
$$\therefore\ (a+x)^2=4h^2\sin^2\tfrac{1}{2}(A+X)-u.$$

If we let $u=-4h^2k$ we obtain $\dfrac{(a+x)^2}{k+\sin^2\frac{1}{2}(A+X)}=4h^2$, where

$$k=-\frac{(\Delta_0+\Delta_1+\Delta_2+\Delta_3)(\Delta_0+\Delta_1-\Delta_2-\Delta_3)(\Delta_0+\Delta_2-\Delta_1-\Delta_3)(\Delta_0+\Delta_3-\Delta_1-\Delta_2)}{16\Delta_0\Delta_1\Delta_2\Delta_3}.$$

And similarly for the equality of the other fractions.

It is evident that when Wolstenholme set me (22) he had found the value of k.

14. The following is also by Wolstenholme. See *Educational Times*, July, 1897, No. 13551:

$$\text{“}\frac{\sin(B+Y)+\sin(C+Z)}{b+y+c+z}$$
$$=\frac{3V}{32\Delta_0\Delta_1\Delta_2\Delta_3}[\{a^2+(y-c)(b-z)\}\{x^2+(b-c)(y-z)\}-(by+cz)(b+y-c-z)^2], \quad (25)$$

and any equation derived from this by cyclical permutation or by changing throughout the signs of any dihedral angles and the corresponding (opposite) edges is also true."

From (2) $\sin\gamma_3\sin\alpha_1\cos B=\cos\beta_1-\cos\gamma_3\cos\alpha_1$;

$$\therefore\ 16\Delta_1\Delta_3\cos B=2b^2(c^2+a^2-y^2)-(a^2+b^2-z^2)(b^2+c^2-x^2)$$
$$=b^2(c^2+z^2+a^2+x^2-b^2-y^2)+a^2x^2-b^2y^2+c^2z^2-c^2a^2-z^2x^2.$$

Similarly,

$$16\Delta_0\Delta_2 \cos Y = y^2(c^2+z^2+a^2+x^2-b^2-y^2)+a^2x^2-b^2y^2+c^2z^2-a^2z^2-c^2x^2,$$
$$16\Delta_1\Delta_2 \cos C = c^2(a^2+x^2+b^2+y^2-c^2-z^2)+a^2x^2+b^2y^2-c^2z^2-a^2b^2-x^2y^2,$$
$$16\Delta_0\Delta_3 \cos Z = z^2(a^2+x^2+b^2+y^2-c^2-z^2)+a^2x^2+b^2y^2-c^2z^2-b^2x^2-a^2y^2. \quad (26)$$

And from (12),

$$3V = 2\Delta_1\Delta_3 \sin B/b = 2\Delta_0\Delta_2 \sin Y/y = 2\Delta_1\Delta_2 \sin C/c = 2\Delta_0\Delta_3 \sin Z/z.$$

Hence $32\Delta_0\Delta_1\Delta_2\Delta_3\{\sin(B+Y)+\sin(C+Z)\}/3V$

$$\begin{aligned}&= by(b+y)(c^2+z^2+a^2+x^2-b^2-y^2)+a^2x^2(b+y)\\&\quad -b^2y^2(b+y)+c^2z^2(b+y)-a^2(bz^2+c^2y)-x^2(bc^2+yz^2)\\&\quad +cz(c+z)(a^2+x^2+b^2+y^2-c^2-z^2)+a^2x^2(c+z)\\&\quad +b^2y^2(c+z)-c^2z^2(c+z)-a^2(cy^2+b^2z)-x^2(b^2c+y^2z),\end{aligned}$$

and on reduction

$$=(b+y+c+z)[\{a^2+(y-c)(b-z)\}\{x^2+(b-c)(y-z)\}-(b+y-c-z)^2(by+cz)].$$

And of course any equation formed by cyclical permutation is also true; but in changes of sign, it must be observed that it is not by simultaneous changes of B and y, but of B and b, that the resulting equation still holds; so that instead of 'opposite' in Wolstenholme's enunciation we should read 'adjacent.'

15. Let ABC be what we have called the spherical triangle at O, and suppose a line through O parallel to the side BC of the tetrahedron meet the sphere in D. OB, OC, and OD are in the same plane, and consequently D lies on the side BC of the spherical triangle ABC, and $CD=\beta_1$. AD is either λ or its supplement. Suppose $AD=\pi-\lambda$, then from the triangle ABC

$$\cos\gamma_3 = \cos\alpha_1 \cos\beta_2 + \sin\alpha_1 \sin\beta_2 \cos C,$$

and from ACD $\quad -\cos\lambda = \cos\beta_2\cos\beta_1 - \sin\beta_2 \sin\beta_1 \cos C;$

$$\therefore \quad \cos\gamma_3 \sin\beta_1 - \cos\lambda \sin\alpha_1 = \cos\beta_2(\sin\beta_1\cos\alpha_1+\sin\alpha_1\cos\beta_1)$$
$$= \cos\beta_2 \sin\gamma_1, \text{ from (1)};$$

$$\therefore \quad \cos\lambda = \cos\gamma_3 \,.\, \sin\beta_1/\sin\alpha_1 - \cos\beta_2 \,.\, \sin\gamma_1/\sin\alpha_1 \quad \ldots\ldots\ldots\ldots(27)$$
$$= \frac{a^2+b^2-z^2}{2ab}\cdot\frac{b}{x} - \frac{c^2+a^2-y^2}{2ca}\cdot\frac{c}{x}$$
$$= \frac{b^2+y^2-c^2-z^2}{2ax}. \quad \ldots\ldots\ldots\ldots(28)$$

By cyclic changes we obtain

$$\cos\mu = \cos\alpha_1 \sin\gamma_2/\sin\beta_2 - \cos\gamma_3\sin\alpha_2/\sin\beta_2 = (c^2+z^2-a^2-x^2)/2by, \ldots(29)$$
$$\cos\nu = \cos\beta_2\sin\alpha_3/\sin\gamma_3 - \cos\alpha_1\sin\beta_3/\sin\gamma_3 = (a^2+x^2-b^2-y^2)/2cz. \ldots(30)$$

Hence $$\Sigma ax\cos\lambda = 0. \quad \ldots\ldots\ldots\ldots(31)$$

With the sign we have adopted for $\cos\lambda$, the other values are

$$\begin{aligned}\cos\lambda &= \cos\gamma_2\sin\beta_0/\sin\alpha_0 - \cos\beta_3\sin\gamma_0/\sin\alpha_0\\&= \cos\gamma_1\sin\beta_3/\sin\alpha_3 - \cos\beta_0\sin\gamma_3/\sin\alpha_3\\&= \cos\gamma_0\sin\beta_2/\sin\alpha_2 - \cos\beta_1\sin\gamma_2/\sin\alpha_2.\end{aligned}$$

And similarly for $\cos\mu$ and $\cos\nu$.

16. To find the shortest distances (the perpendiculars l, m, n) between the three pairs of opposite edges.

Through O draw OD parallel and equal to CB: join AD, BD. Then $OCBD$ is a parallelogram, and consequently the tetrahedra $AOBD$, $AOBC$ are equal. Also the perpendicular from B on AOD is l, and the area of AOD is $\frac{1}{2}ax\sin\lambda$;

$$\therefore\quad lax\sin\lambda=6V;$$

$$\therefore\quad l=6V/ax\sin\lambda,\quad m=6V/by\sin\mu,\quad n=6V/cz\sin\nu.\qquad\text{(32)}$$

17. To find the distances between the mid-points of the opposite edges.

The mid-points of the six edges taken four by four are obviously the angular points of three parallelograms of which d_1, d_2, d_3 are, two by two, the diagonals, which are therefore concurrent and bisect each other.

Also d_1, $b/2$, $y/2$ are sides of a triangle, the angle opposite d_1 being μ or $\pi-\mu$. Hence

$$4d_1^2=b^2+y^2\pm 2by\cos\mu=b^2+y^2\pm(c^2+z^2-a^2-x^2),\ \text{from (29)}.$$

Similarly $\quad 4d_1^2=c^2+z^2\mp 2cz\cos\nu=c^2+z^2\mp(a^2+x^2-b^2-y^2),\quad$ from (30).

As these results must be identical, it is obvious that the upper signs must be taken throughout, so that

$$4d_1^2=b^2+y^2+c^2+z^2-a^2-x^2.\qquad\text{(33)}$$

Similarly for $4d_2^2$ and $4d_3^2$;

$$\therefore\quad 4(d_3^2-d_2^2)=2(b^2+y^2-c^2-z^2);\qquad\text{(34)}$$

$$\therefore\quad d_3^2-d_2^2=ax\cos\lambda,\ \text{from (28)}.$$

Also $$2(d_3^2+d_2^2)=a^2+x^2.\qquad\text{(35)}$$

And similarly for the other pairs.

18. To find the radius R of the circum-sphere.

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coordinates of the circum-centre referred to OA, OB, OC as coordinate axes; then we have (neglecting the subscripts of α_1, β_2, γ_3)

$$R^2=\Sigma\bar{x}^2+2\Sigma\bar{y}\bar{z}\cos\alpha.$$

Also $R^2=(\bar{x}-a)^2+\bar{y}^2+\bar{z}^2+2\bar{y}\bar{z}\cos\alpha+2\bar{z}(\bar{x}-a)\cos\beta+2(\bar{x}-a)\bar{y}\cos\gamma.$

Subtracting we get $\quad \bar{x}+\bar{y}\cos\gamma+\bar{z}\cos\beta=\frac{1}{2}a.$

Similarly
$$x\cos\gamma+\bar{y}+\bar{z}\cos\alpha=\tfrac{1}{2}b,$$
$$\bar{x}\cos\beta+\bar{y}\cos\alpha+\bar{z}=\tfrac{1}{2}c.$$

$$\therefore\quad \tfrac{1}{2}(a\bar{x}+b\bar{y}+c\bar{z})=\bar{x}^2+\bar{y}^2+\bar{z}^2+2\bar{y}\bar{z}\cos\alpha+2\bar{z}\bar{x}\cos\beta+2\bar{x}\bar{y}\cos\gamma=R^2.$$

Solving these three equations we have

$$2\bar{x}\begin{vmatrix}1, & \cos\beta, & \cos\gamma\\ \cos\beta, & 1, & \cos\alpha\\ \cos\gamma, & \cos\alpha, & 1\end{vmatrix}=a-b\cos\gamma-c\cos\beta-\cos\alpha(a\cos\alpha-b\cos\beta-c\cos\gamma);$$

$$\therefore\quad \frac{72V^2\bar{x}}{a^2b^2c^2}=a-b\frac{a^2+b^2-z^2}{2ab}-c\frac{c^2+a^2-y^2}{2ca}-\frac{b^2+c^2-x^2}{2bc}$$
$$\times\left(a\frac{b^2+c^2-x^2}{2bc}-b\frac{c^2+a^2-y^2}{2ca}-c\frac{a^2+b^2-z^2}{2ab}\right),$$

or $288V^2\bar{x}/a=2b^2c^2(y^2+z^2-b^2-c^2)-(b^2+c^2-x^2)(b^2y^2+c^2z^2-a^2x^2-2b^2c^2)$
$$=(c^2a^2+a^2b^2-2b^2c^2)x^2+(b^2c^2-b^4)y^2+(b^2c^2-c^4)z^2+x^2(b^2y^2+c^2z^2-a^2x^2).$$

Similarly $288V^2\bar{y}/b$ and $288V^2\bar{z}/c$ may be written down.

Hence $$288V^2(a\bar{x}+b\bar{y}+c\bar{z})=2\Sigma b^2c^2y^2z^2-\Sigma a^4x^4,$$

or $$576V^2R^2=\Sigma ax\,\Pi\,(by+cz-ax).\qquad\text{(36)}$$

See Salmon's *Geometry of Three Dimensions*, Art. 55; also Frost and Wolstenholme's *Solid Geometry*, Art. 437.

19. It is obvious that $r=3V/(\Delta_0+\Delta_1+\Delta_2+\Delta_3)$, ...(37)

and $r_0=3V/(\Delta_1+\Delta_2+\Delta_3-\Delta_0)$, ...(38)

with similar expressions for r_1, r_2, r_3.

Hence $1/r_0+1/r_1+1/r_2+1/r_3=2/r$. ....(39)

20. The following problems are suggested as applications of this paper:

(1) All tetrahedra which have a common edge and whose opposite vertices are on two given straight lines parallel to this edge are equal in volume.

(2) If a line is equally inclined to OA, OB, OC, then this inclination is $\cot^{-1}(\cos\frac{1}{2}\beta_2\cos\frac{1}{2}\gamma_3\sin A/\sin\frac{1}{2}\alpha_1)$; and the conditions that the four lines at the four vertices should be concurrent are $a+x=b+y=c+z$.

(3) The conditions that a sphere should meet the four faces in their nine-point circles are that the opposite edges should be at right angles.

(4) $\Delta_0^2=\Sigma\Delta_1^2-2\Sigma\Delta_2\Delta_3\cos A$.

(Wolstenholme's *Mathematical Problems*, Second edition, No. 2021.)

(5) If $b=y$ and $c=z$, then $B=Y$ and $C=Z$. Is the converse true?

(6) If $\Delta_0+\Delta_1=\Delta_2+\Delta_3$, then

$$(a-x)/\sin\tfrac{1}{2}(X-A)=(b-y)/\sin\tfrac{1}{2}(Y-B)=(c-z)/\sin\tfrac{1}{2}(Z-C)=2h.$$

(*Educational Times*, October, 1900, 14675, by Wolstenholme.)

(7) $\cos\beta_0\cos C\sin Z-\cos\beta_1\sin C\cos Z=\cos\gamma_0\cos B\sin Y-\cos\gamma_1\sin B\cos Y$.

(8) $24V(a\cot A-x\cot X)=4d_1^2(a^2-x^2)-(b^2-y^2)(c^2-z^2)$.

(9) $\Sigma\cot\lambda/l=0$.

(10) The sines of the angles between the three planes, each through four mid-points of the edges, are $d_1mn/3V$, $d_2nl/3V$, $d_3lm/3V$.

(11) The volume of the tetrahedron described on any face of $OABC$, its other edges being d_1, d_2, d_3 (d_1 being opposite to a or x, etc.), is $\frac{1}{2}V$.

(12) The diameter of the circum-sphere in terms of the angles and edges at the vertex O, neglecting subscripts, is

$$\left(\frac{\Sigma a^2\sin^2\alpha-2\Sigma bc\sin\beta\sin\gamma\cos A}{1-\Sigma\cos^2\alpha+2\Pi\cos\alpha}\right)^{\frac{1}{2}}.$$

(13) $(1/r-1/r_0)(1/r-1/r_1)(1/r-1/r_2)(1/r-1/r_3)=4h^2/9V^2$.

(14) $1/r_0r_1+1/r_2r_3=h^2/\Delta_0\Delta_1+h^2/\Delta_2\Delta_3$.

(15) The tetrahedral equation of the circum-sphere is

$$x^2\beta\gamma+y^2\gamma\alpha+z^2\alpha\beta+a^2\alpha o+b^2\beta o+c^2\gamma o=0,$$

where $o=0$, $\alpha=0$, $\beta=0$, $\gamma=0$ are the equations of the faces opposite to O, A, B, C respectively. (Frost and Wolstenholme's *Solid Geometry*, Art. 435; see also Salmon, Art. 229.)

(16) If the four perpendiculars from the vertices on the opposite faces are concurrent, then the opposite edges are at right angles;

$$a^2+x^2=b^2+y^2=c^2+z^2,$$

and $$\cos A\cos X=\cos B\cos Y=\cos C\cos Z.$$

The tetrahedral coordinates of the point of concurrence are proportional to

$$\Delta_0(\sec A\sec B\sec C)^{\frac{1}{3}},\quad \Delta_1(\sec Y\sec Z\sec A)^{\frac{1}{3}},\ \text{etc.*}$$

G. Richardson.

* For the extension of the modern Geometry of a triangle to special tetrahedra, see a memoir by Professor M. J. Neuberg in *Mémoires Couronnés, L'Académie Royale de Belgique*, tome xxxvii., Bruxelles, Janvier, 1886.