SECT. II.—OTHER SELECTED PAPERS.

(*Paper No. 3592.*)

"A Problem Relating to Railway-bridge Piers of Masonry or Brickwork."

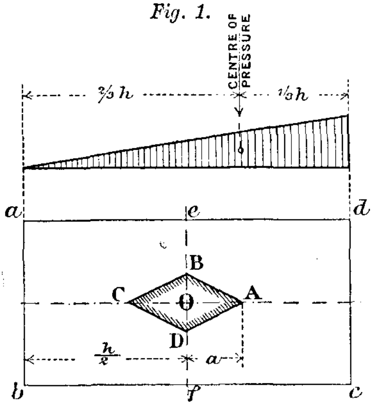
By FREDERICK KARL ESLING.

IN the design of railway-bridge piers of masonry or brickwork, a puzzling problem is met with when the horizontal forces (due to application of the brake, till sliding takes place, to a train exposed to wind-pressure), acting at an angle to each other, cause the centre of pressure of the combined vertical and horizontal forces to fall beyond a certain distance from the centre of gravity of the horizontal section of the pier. This section is usually square or rectangular in shape, and the following investigation will be confined to piers of these two sections.

The difficulty in calculating the exact stresses resulting in certain cases from the combined action of the horizontal and vertical forces, is due to the fact that masonry and brickwork, even when built in cement-mortar, cannot be relied upon to resist tension. Brickwork in cement is capable of withstanding considerable tension, and cases have occurred of partial damage to ordinary brick-in-cement culverts, in which the portions remaining intact must have been subjected to a tension of 80 lbs. per square inch. But both masonry and brickwork usually contain some joints at which the parts can be separated easily, and in most railway-piers the joint between the pier and its support is quite unable to resist tension.

In *Fig. 1* the horizontal forces are assumed to be acting with such intensity that the compression along $a\ b$ is reduced to *nil;* it will be seen at once that the compression along $c\ d$ must be double the compression due to vertical loads only, and that the centre of pressure will be at the point A. Supposing the line of no-stress to occupy successively every position possible around the rectangle $a\ b\ c\ d$, whilst always remaining in touch with it, then the corresponding centres of pressure describe the diamond-shaped figure A B C D. This rhombus is called the "kern" (kernel) by German scientists, and the distance of the point A from the centre O is—

Fig. 1.



$$a = \frac{I}{A.\frac{h}{2}},$$

A being the area of the rectangle and I its moment of inertia with $e\,f$ as axis.

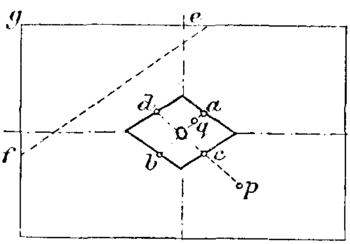
This gives for the rectangle:—

$$a = \frac{b.\,h^3}{12} \div \frac{b.\,h^2}{2} = \frac{h}{6}.$$

The kern therefore occupies the middle third of the two main axes, and with its assistance many of the problems relating to the combined action of the horizontal and vertical forces may be solved.

If the centre of pressure be within the limit of the kern, as at q in *Fig. 2*, it follows that there must be compression only, although of varying intensity, over the whole section of the pier. Let a straight line be drawn through q and O, intersecting the kern in a and b. Then the maximum compression is given by $\frac{W}{A} \times \frac{q\,b}{O\,b}$, and the minimum compression by $\frac{W}{A} \times \frac{q\,a}{O\,a}$, W being the total vertical load on the pier, and A the area of the latter. If the material of the pier be capable of resisting tension, the stresses can be found just as easily if the centre of pressure falls outside the kern, say at p, *Fig. 2.* Let the straight line through p and O intersect the kern in c and d. Then the maximum compression will be $\frac{W}{A} \times \frac{p\,d}{O\,d}$, and the maximum tension $\frac{W}{A} \times \frac{p\,c}{O\,c}$. But if the pier be unable to resist tension, a line of cleavage will be formed, say at $e\,f$, so that the area $e\,f\,g$ will be lost, and the compression will be increased on

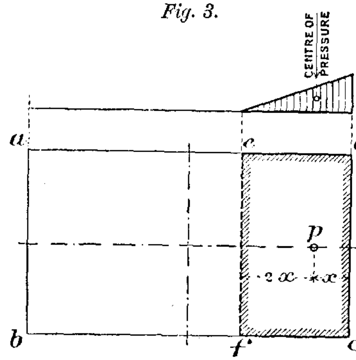
Fig. 2.



the remaining part of the original rectangle. Unfortunately there is no direct method of finding this line of cleavage (or no-stress); and the cases investigated in the text-books seem to be confined to those in which the centre of pressure falls on one of the two main axes of the rectangle.

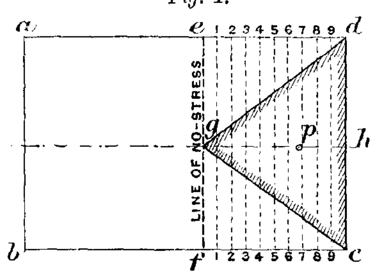
Supposing *a b c d*, *Fig. 3*, to be a horizontal section of a rectangular pier, with the centre of pressure, *p*, outside the kern; then it is known that *e f*, the line of no-stress, is twice as far from *p* as *p* is from *c d*, and that the maximum compression is along *c d*, and equal to $\frac{2\text{ W}}{\text{Area } c\,d\,e\,f}$, or that it is greater than the compression due to vertical loads only, in the ratio of $\frac{2\,b\,c}{f\,c}$. It will be shown that these data are sufficient for solution of the problem under consideration, namely, the determination of the maximum compression on the pier when the centre of pressure falls outside the kern, and away from the two main axes.

Fig. 3.



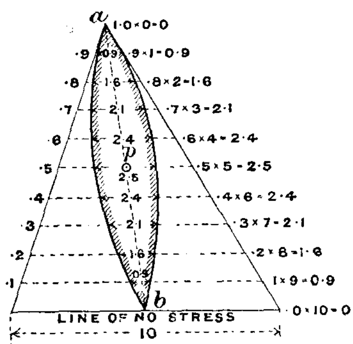
Examining the area *e f c d*, *Fig. 4*, at *e f* there is no pressure, whilst at *c d* there is double the average compression, and the centre of pressure, *p*, is $\frac{1}{3}\,g\,h$ from *c d*. Dividing *d e* into ten equal parts, as shown, and assuming the compression along *c d* to be 10, then along 9 9 it will be 9, along 8 8 it will be 8 . . . along 1 1 it will be 1, and along *e f nil*. Instead of taking the whole area *c d e f* as being active, an "equivalent area" of the maximum compression can now be constructed by marking these reduced lengths 9, 8, ... 1, 0, on the corresponding lines 9 9, 8 8 . . . 1 1 and *e f*. The area *c d g* is thus obtained, and this is a triangle, with the centre of pressure, *p*, on its centre of gravity. In *Fig. 5* is shown the "equivalent area" of a triangular section, with the base as the line of no-stress.

Fig. 4.



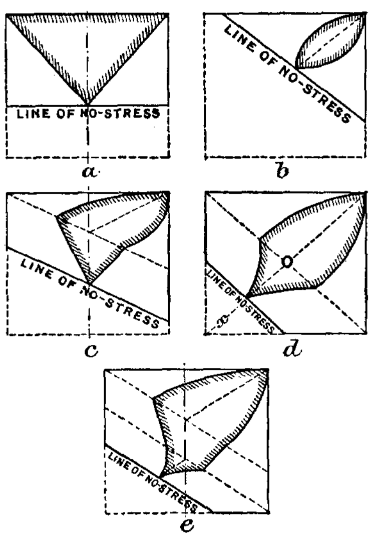
The equivalent area is bounded by two parabolas, is one-third the area of the triangle, and has its centre of gravity on the middle of $a\,b$. The "equivalent area" provides a simple method of finding the maximum compression, as it is only necessary to divide the total vertical loads by this area. In the case illustrated in *Fig. 4*, the maximum compression is $\frac{W}{\text{Area } c\ d\ g}$. The same reasoning applies to any position of the line of no-stress. Dividing the distance between this line and the farthest part of the rectangle into any number of equal parts by lines drawn parallel to the line of no-stress, and

Fig. 5.



marking off on these parallel lines the reduced lengths, 9 9, 8 8, . . . 1 1, as in *Figs. 4* and *5*, the "equivalent area" can be readily drawn and the maximum compression obtained; the centre of pressure will be on the centre of gravity of the "equivalent area." All the different "equivalent areas" come under one of the types shown in *Figs. 6*.

Figs. 6.



If a great number of these "equivalent areas" were drawn and their areas and centres of gravity fixed, lines of equal pressures could be plotted, which would serve to indicate at once the maximum compression caused by the action of the horizontal and vertical forces combined; but any attempt to solve the problem in this manner would soon prove to be an arduous task. What is wanted is a simple method of determining a few lines of pressure, and of deriving others from them.

The Author was first led to investigate this problem by finding that the "equivalent area" for a triangle having its base on the diagonal of the rectangle was one-sixth the area of the latter, and its centre of gravity, coinciding with the centre of pressure, was the middle point of the line joining the centre of the rectangle with the apex of the triangle under consideration. It seemed likely that there would be a curve between the centre of pressure of this triangle and that of the rectangle *a g h d* shown in *Fig. 7* (also giving an "equivalent area" equal to one-sixth the area of the pier), on which would be located the centres of pressure of all intermediate "equivalent areas" equal to one-sixth the area of the pier. For convenience, the terms "6-line," "3·85-line," &c., will be used to denote such curves for equivalent areas of $\frac{1}{6}$, $\frac{1}{\cdot 85}$, &c., of the area of the original rectangle. The term "6-line" will therefore denote the locus of the centre of pressure for all cases in which the resulting maximum compression due to the action of horizontal and vertical forces is six times that due to the vertical forces alone. The calculations were made for a rectangle 8 feet by 6 feet, but any other dimensions could have been chosen. For the 6-line the equivalent area must be $\frac{8 \times 6}{6} = 8$ square feet. Thus in *Fig. 7*, for the rectangle, *g h* is 2 feet from *a d*, with the centre of pressure, at *e*, 0·667 foot from *a d*. For the triangle *a c d* the 6-line centre of pressure is at *f*, the middle point of O *d*.

Fig. 7.

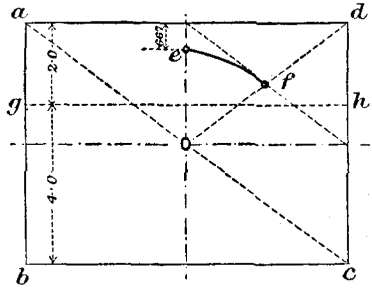
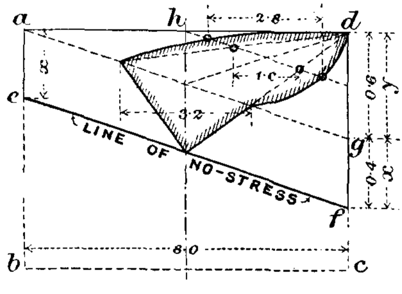


Fig. 8.



It also seemed very likely that all other areas represented by the 6-line would have one end of their fourth side on *a g* and the other end on *h c*. This assumption proved to be correct, after the following calculations had been made.

Taking a line of no-stress at *e f* (*Fig. 8*), so that *a e* and *f g* are both 0·4 foot and $d\,g$ 0·6 foot, the equivalent area is readily calculated from the dimensions given on *Fig. 8*. It consists of a triangle below $a\,g$, and a triangle plus two areas bounded by parabolas over $a\,g$. The former is found to be $3 \cdot 2 \times \frac{0 \cdot 4}{2} = 0 \cdot 64$ in area, and the latter $3 \cdot 2 \times \frac{0 \cdot 6}{2} + \frac{2}{3}(2 \cdot 8 - 1 \cdot 6) \times 0 \cdot 6 = 0 \cdot 96 + 0 \cdot 48$. Adding these, the total equivalent area is found to be 2·08 for $d\,f = 1 \cdot 0$. The co-ordinates of the centre of gravity of this area are 1·077 along $h\,d$, and 0·2603 at right-angles to $h\,d$. The calculation shows that the abscissa is a simple function of the length of the pier (in this case always 8·0), and the ordinate a simple function of $(x + y)$. The centre of pressure of any other area with the ratio of x to y equal to $\frac{0 \cdot 4}{0 \cdot 6}$ can therefore be easily determined. For the 6-line (area $= 8 \cdot 0$) x and y are multiplied by $\frac{8}{2 \cdot 08}$, giving $x = 1 \cdot 538$, and $y = 2 \cdot 308$, whilst the centre of pressure will have an abscissa of 1·077 on $h\,d$, and an ordinate of $0 \cdot 2603 \times \frac{8}{2 \cdot 08} = 1 \cdot 001$ under $h\,d$.

Evidently similar calculations can be performed for any position of $e\,f$ (so long as f is on the line $c\,d$), and a few trials will show that when x is to y as 0·4 is to 0·6, if the centre of gravity of the "equivalent area" is on any "z-line," it will have an abscissa of 1·077 on $h\ d$, and an ordinate of $\frac{6}{z} \times 1 \cdot 001$ below $a\,d$. Thus for the 4-line the ordinate is $\frac{6}{4} \times 1 \cdot 001 = 1 \cdot 501$. For the 12-line it is $\frac{6}{12} \times 1 \cdot 001 = 0 \cdot 50$, &c. An important case is that in which f falls on c, as it is clear that the proportionality to the 6-line will cease when f is anywhere on $b\,c$ except at the point c. The line $e\,f$ for this case will be termed the "limit-line." For $d\,f = 1$ the equivalent area has been found to be 2·08 and for $d\,f = d\,c = 6 \cdot 0$, it is clear that the "equivalent area" will be $6 \times 2 \cdot 08 = 12 \cdot 48$, and the limit-line of pressure $= \frac{8 \times 6}{12 \cdot 48} = 3 \cdot 85$. With the ratio of x to y that of 0·4 to 0·6, all lines of pressure between 3·85 and infinity can now be drawn by making the abscissa on $h\,d$ equal to 1·077, and the ordinate equal to $\frac{6}{z} \times 1 \cdot 001$, z representing any pressure-line between 3·85 and infinity. Similar calculations were made for various ratios of x to y, and the results obtained are given in the following Table:—

TABLE I.—ABSCISSAE AND ORDINATES OF THE 6-LINE.

$\frac{x}{y} =$	∞	$\frac{0\cdot9}{0\cdot1}$	$\frac{0\cdot8}{0\cdot2}$	$\frac{0\cdot7}{0\cdot3}$	$\frac{0\cdot6}{0\cdot4}$	$\frac{0\cdot5}{0\cdot5}$	$\frac{0\cdot4}{0\cdot6}$	$\frac{0\cdot3}{0\cdot7}$	$\frac{0\cdot2}{0\cdot8}$	$\frac{0\cdot1}{0\cdot9}$	0
$x =$	2·00	1·993	1·967	1·918	1·837	1·714	1·538	1·295	0·968	0·541	0
$y =$	0	0·221	0·492	0·822	1·225	1·714	2·308	3·021	3·871	4·865	6
Abscissa on $h\,d$ . .	0	0·140	0·295	0·466	0·653	0·857	1·077	1·309	1·548	1·784	2·0
Ordinate under $a\,d$.	0·667	0·702	0·744	0·792	0·850	0·918	1·001	1·100	1·217	1·352	1·5
Limit-line .	2·00	2·21	2·46	2·74	3·06	3·43	3·85	4·32	4·84	5·41	6·0

The foregoing Table gives eleven points of the 6-line of pressure, and by setting out these points the curve can be traced as shown at $e\,f$ in *Figs. 9* and *11*.

Calculating the ordinates below $a\,d$ for the "limit-lines" of Table I, another curve, namely, $i\,f$, *Figs. 9* and *11*, is obtained. The problem is now nearly determined. It will be seen (*Fig. 9*) that four sharply-defined areas, $O\,i\,m$, $i\,m\,f$, $i\,h\,n\,f$, and $n\,f\,d$, can be marked off, and that these four areas include all positions of

Fig. 9.

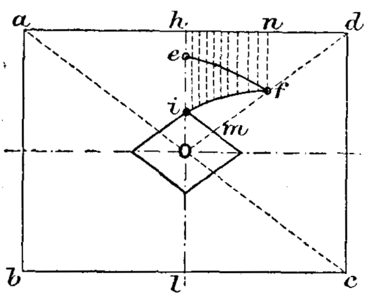
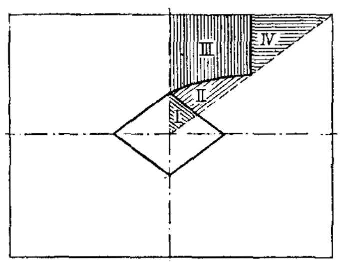


Fig. 10.

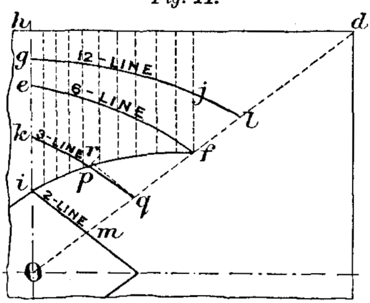


the centre of pressure. In *Fig. 10* these areas are shown shaded. Area I contains the centres of pressure for all pressure-lines between the 1-line and the 2-line, inclusive, *i.e.*, those due to the line of no-stress keeping outside, or just touching, the original rectangle; area II includes all centres of pressure due to the original rectangle being reduced by the line of no-stress to five-sided figures of types d and e, *Figs. 6*, the pressure-lines ranging from the 2-line to the 6-line; area III contains the centres of pressure for types a and c, *Figs. 6* (the

2-line to the infinity-line); whilst area IV includes the centres of pressure for type *b*, *Figs. 6*, ranging from the 6-line to the infinity-line.

It is now necessary to find some additional centres of pressure on the diagonal, O *d*, *Fig. 11*. For those between *f* and *d* this operation is simple, since here only triangles have to be dealt with, and these have "equivalent areas" equal to one-third the area of the triangle (with the line of no-stress on the base of the triangle), whilst the centre of gravity of each "equivalent area" is in the centre of the line drawn from the middle of the base-line to the apex opposite. This means that the centre of pressure for any z-line, where z is greater than 6, will be at a distance from d equal to $fd\sqrt{\frac{6}{z}}$. Supposing it be desired to draw the 12-line complete, the ordinates between g and j are obtained by halving those of the 6-line, and since $dl = 2 \cdot 5 \sqrt{\frac{6}{12}} = 1 \cdot 768$, the point l is fixed. To connect j with l a number of triangles are constructed having an "equivalent area" of $\frac{8 \times 6}{12} = 4 \cdot 0$, or an actual area of $12 \cdot 0$, and those for which the centres of pressure fall between l and j are selected. In this manner as many points as may be desired can be obtained, thus completing the 12-line from g to l.

Fig. 11.



To find the centres of pressure between O and f, a number of lines of no-stress between O and x (*Figs. 6, d*) are assumed, and the "equivalent areas" and distances of their centres of gravity along Of from O are calculated. The results for a number of cases are shown in Table II.

TABLE II.—CENTRES OF PRESSURE ON O f, FIGS. 6 (d) AND 11.

Pressure-lines .	2·0	2·22	2·477	2·773	3·0	3·33	3·645	4·0	4·364	5·0	5·533	6·0
Distance from O along Of	0·833	1·012	1·20	1·389	1·516	1·678	1·815	1·952	2·167	2·263	2·389	2·5

The pressure-lines referred to in Table II can be readily plotted. Thus the points on h O are obtained by dividing twice h i by each pressure and laying off the result obtained along $h\,i$ from h. (This gives for the 6-line, $h\,e = \frac{2 \times 2}{6} = 0 \cdot 667$; for the 3-line, $h\,k = \frac{2 \times 2}{3} = 1 \cdot 333$, &c.) To draw the 3-line complete, the ordinates between k and p are obtained by doubling those of the 6-line; the point q is obtained from Table II; and between p and q a flat curve is plotted by drawing $q\,r$ parallel to $i\,m$, and $p\,r$ parallel to $O\,f$, the curve $p\,q$ to be a parabola with ordinates parallel to $p\,r$. A few trials with equivalent areas of type e, *Figs. 6*, will show this curve to be practically correct.

It is now possible to determine any pressure-line between the 1-line and the infinity-line. In *Fig. 11* the 2-, 3-, 6-, and 12-lines are shown; but for the purposes of Table III many more were drawn. The quarter of the original rectangle is divided into smaller rectangles (*Fig. 12*), each one-fortieth the dimensions of the section of the pier. The values of the centres of pressure at the intersections of these lines are determined by scaling between the two nearest pressure-lines; the results obtained are recorded in Table III.

Fig. 12.

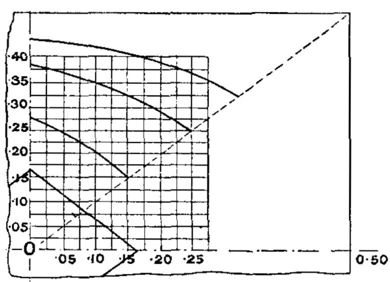


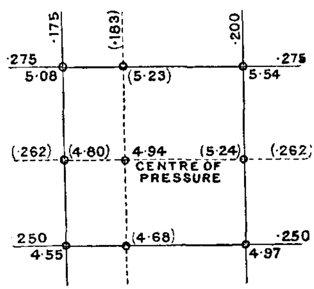
Table III can be used for the solution of most problems involving the action of horizontal and vertical forces on rectangular piers. It is only necessary to find the component of wind-pressure and vertical loads in decimals of the length of the pier, and the component of the brake-force and vertical loads in decimals of the breadth of the pier, and to take from Table III the value of the resultant centre of pressure. As an example, let it be desired to ascertain the maximum compression on a pier 100 feet in height and 25 feet by 12 feet in cross-section, the wind-pressure being 50 tons, the brake-force 50 tons, and the total vertical loads 1,850 tons. The component in decimals of the length of the pier will be $\frac{50 \times 100}{1{,}850 \times 25} = 0 \cdot 108$; and the component in decimals of the breadth of the pier will be $\frac{50 \times 100}{1{,}850 \times 12} = 0 \cdot 225$. In Table III the pressure-line at the intersection of $0 \cdot 225$ with $0 \cdot 100$ is found to be $3 \cdot 27$ and that at

TABLE III.—LINES OF PRESSURE AT INTERSECTIONS OF COMPONENTS OF FORCES.

Component = Breadth of Pier multiplied by	Component = Length of Pier multiplied by 0	0·025	0·050	0·075	0·100	0·125	0·150	0·175	0·200	0·225	0·250	0·275	0·300	0·325	0·350	0·375	0·400
0	1·0	1·15	1·30	1·45	1·60	1·75	1·90	2·05	2·22	2·43	2·67	2·96	3·33	3·87	4·44	5·33	6·67
0·025	1·15	1·30	1·45	1·60	1·75	1·90	2·05	2·21	2·39	2·61	2·87	3·18	3·58	4·08	4·77	5·73	7·18
0·50	1·30	1·45	1·60	1·75	1·90	2·05	2·21	2·38	2·58	2·81	3·09	3·43	3·87	4·41	5·16	6·17	7·73
0·075	1·45	1·60	1·75	1·90	2·05	2·20	2·37	2·56	2·78	3·03	3·33	3·70	4·16	4·74	5·56	6·66	8·30
0·100	1·60	1·75	1·90	2·05	2·20	2·37	2·56	2·76	2·99	3·27	3·60	3·99	4·48	5·14	5·99	7·16	9·00
0·125	1·75	1·90	2·05	2·20	2·37	2·56	2·75	2·97	3·24	3·54	3·90	4·32	4·81	5·57	6·47	7·78	9·74
0·150	1·90	2·05	2·21	2·37	2·56	2·75	2·96	3·22	3·51	3·84	4·22	4·66	5·28	6·03	7·04	8·45	10·6
0·175	2·05	2·21	2·38	2·56	2·76	2·97	3·22	3·50	3·81	4·16	4·55	5·08	5·73	6·55	7·66	9·17	11·5
0·200	2·22	2·39	2·58	2·78	2·99	3·24	3·51	3·81	4·13	4·50	4·97	5·54	6·24	7·12	8·33	9·98	..
0·225	2·43	2·61	2·81	3·03	3·27	3·54	3·84	4·16	4·50	4·93	5·48	6·05	6·83	7·82	9·13	10·9	..
0·250	2·67	2·87	3·09	3·33	3·60	3·90	4·22	4·55	4·97	5·48	6·00	6·67	7·50	8·57	10·0	12·0	..
0·275	2·96	3·18	3·43	3·70	3·99	4·32	4·66	5·08	5·54	6·05	6·67	7·41	8·37	9·55	11·1	..	..
0·300	3·33	3·58	3·87	4·16	4·48	4·81	5·28	5·73	6·24	6·83	7·50	8·37	9·37	10·8	..	..	..
0·325	3·87	4·08	4·41	4·74	5·14	5·57	6·03	6·55	7·12	7·82	8·57	9·55	10·8	..	..	..	..
0·350	4·44	4·77	5·16	5·56	5·99	6·47	7·04	7·66	8·33	9·13	10·0	11·1	..	..	..	..	..
0·375	5·33	5·73	6·17	6·66	7·16	7·78	8·45	9·17	9·98	10·9	12·0	..	..	..	..	..	..
0·400	6·67	7·18	7·73	8·30	9·00	9·74	10·6	11·5	..	..	..	..	..	..	..	..	..

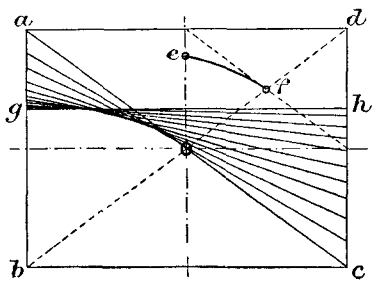
its intersection with 0·125 is 3·54. Dividing proportionately for 0·108, the resulting pressure-line is found to be 3·36, or, in other words, the maximum compression on the pier will be $\frac{1{,}850}{25 \times 12} \times 3 \cdot 36 = 20 \cdot 72$ tons per square foot. Supposing, in another case, the components of the pressure to be 0·262 and 0·183 of the length and breadth of the pier respectively, the line of pressure is found, on reference to Table III, to be in the square shown in *Fig. 13.* Dividing proportionately, vertically and horizontally, the pressure-line is found to be, in both cases, 4·94. If the two results should differ slightly, the mean value is taken.

Fig. 13.



The only method of dealing with this problem previously known to the Author is one which was described by Mr. J. H. Fraser in a Paper on "The Design of Bridges," read before the Engineering Students' Society of Melbourne University in 1892. In this method the centre of pressure is determined in the usual manner, and a trial line of cleavage (or no-stress) is assumed. Stiff paper is used for the drawing, and after the "equivalent area" is marked, it is cut out and balanced on a pin passed through the centre of pressure. Probably the piece cut out will overbalance to one side, and a new line of cleavage is assumed which will counterbalance the area shown to be in excess. The problem is solved when the "equivalent area" cut out balances perfectly on the centre of pressure determined in the first instance. Care and patience are necessary, as several trials are usually required to obtain a satisfactory result. By this method, however, great accuracy is attainable. The example already quoted, having reference to a pier 25 feet by 12 feet, was worked out by this method, and the maximum compression was found to be 20·7 tons per square foot, or the same as that obtained by means of Table III,

Fig. 14.



Referring again to *Fig. 7* and Table I, the following problem presents itself for solution by mathematics or graphical statics. If all lines of no-stress (for the 6-line) given by Table I, be drawn, it is seen that they gradually shift from $g\,h$ to $a\,c$, and during the whole of this operation they appear to be tangents to a curve O g, *Fig. 14*. It would be interesting to find the equation to this curve, and also that of the 6-line derived from it. The latter is, of course, always on the outside of the "kerns" of the different trapezoids formed by the movement of $g\,h$ round the curve O g.

The Paper is accompanied by a tracing, from which the Figures in the text have been prepared.