15.

Theoriae logarithmi integralis lineamenta nova.

(Auct. *Car. Ant. Bretschneider*, math. in Gymn. ill. Gothano praec. secundo.)

Ad difficiliora calculi integralis problemata theoria est referenda functionis illius, quae *logarithmus integralis* dicitur, in qua accuratius exstruenda jam plures analystae versati sunt. Imprimis huc referas nomina virorum cll. *Mascheroni, Soldner, Bessel, Buzengeiger* *), quorum diligentia atque sedulitas jam difficultates quasdam, easque non minimas, ab illa functione oblatas superavere. Tamen accuratior hujus rei disquisitio nullo modo superabundans censenda fuerit; nam non modo determinatio quantitatis constantis simplicior et rectior est constituenda, quam apud *Soldnerum* est **), sed formulae etiam gravissimae in hac theoria repertae vinculo et nexu, quo nunc omnino carent, angustiori inter se sunt jungendae. Huc accedit, quod si valorem functionis pro magnis valoribus variabilis x evolvere volueris, series adhuc inventae, adhibitis ipsis illis viri cl. *Bessel,* non satis convergere videntur. Quare hanc rem denuo tractabo atque ea, quae resultarunt, cum aliqua gaudeant utilitate, his quae sequuntur paragraphis proponam.

§. 1.

Denotato logarithmo integrali variabilis x per li x, efficiuntur ex evolutione quantitatum $\frac{d.e^{\pm lx}}{\pm lx}$ et $\frac{\partial x}{l(1\pm x)}$ et ex integratione subsequente aequationes fundamentales:

$$1.\qquad \mathrm{li}\, x^{\pm 1} = c + l(\pm lx) \pm \frac{lx}{1.1!} + \frac{(lx)^2}{2.2!} \pm \frac{(lx)^3}{3.3!} + \frac{(lx)^4}{4.4!} \pm \dots.$$

$$2.\qquad \mathrm{li}\,(1\pm x) = c + lx \pm \mathfrak{A}_1 x - \tfrac{1}{2}\mathfrak{A}_2 x^2 \pm \tfrac{1}{3}\mathfrak{A}_3 x^3 - \tfrac{1}{4}\mathfrak{A}_4 x^4 \pm \dots.$$

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*) *Mascheroni* in s. adnotation. ad calculum integr. Euleri. — *Soldner* in théorie et tables d'une nouvelle fonction transcendante; à Munic, 1809. — *Bessel* in Königsberger Archiv für Mathem. und Naturwiss. ann. 1811, fasc. 1. — *Buzengeiger* in de Zach, Monatl. Corr. Vol. XXVI. pag. 285. — Disquisitiones a cl. Mascheroni institutas non cognovi nisi ex illis, quae cl. Bessel in disputatione sua citavit. Operis ipsius copia mihi non erat; investigatio autem constantis libri caput esse dicitur.

**) Conf. Hallische Litteraturzeitung. 1811. No. 104.

Si $x > 1$, in aequatione (1.) signo superiori, si $x < 1$, inferiori utaris. Quantitas c est constans integrationis utrique seriei communis; $n! = 1.2.3.4 \ldots (n-1)n$; denique coëfficientes seriei (2.) sunt:

$$\mathfrak{A}_1 = \tfrac{1}{2} = 0{,}5$$

$$\tfrac{1}{2}\mathfrak{A}_2 = \tfrac{1}{24} = 0{,}041666\ 666666\ 666666\ 666666\ 666666\ 666666\ 6\ldots$$

$$\tfrac{1}{3}\mathfrak{A}_3 = \tfrac{1}{72} = 0{,}013888\ 888888\ 888888\ 888888\ 888888\ 888888\ 8\ldots$$

$$\tfrac{1}{4}\mathfrak{A}_4 = \tfrac{19}{2880} = 0{,}006597\ 222222\ 222222\ 222222\ 222222\ 222222\ 2\ldots$$

$$\tfrac{1}{5}\mathfrak{A}_5 = \tfrac{3}{800} = 0{,}00375$$

$$\tfrac{1}{6}\mathfrak{A}_6 = \tfrac{863}{362880} = 0{,}002378\ 196649\ 029982\ 363315\ 696649\ 029982\ 3\ldots$$

$$\tfrac{1}{7}\mathfrak{A}_7 = \tfrac{275}{169344} = 0{,}001623\ 913454\ 270597\ 127739\ 984882\ 842025\ 6\ldots$$

$$\tfrac{1}{8}\mathfrak{A}_8 = \tfrac{33953}{29030400}$$
$$= 0{,}001169\ 567074\ 514991\ 181657\ 848324\ 514991\ 1\ldots$$

$$\tfrac{1}{9}\mathfrak{A}_9 = \tfrac{8183}{9331200}$$
$$= 0{,}000876\ 950445\ 816186\ 556927\ 297668\ 038408\ 7\ldots$$

$$\tfrac{1}{10}\mathfrak{A}_{10} = \tfrac{3250433}{4790016000}$$
$$= 0{,}000678\ 584998\ 463470\ 685692\ 907915\ 130137\ 3\ldots$$

$$\tfrac{1}{11}\mathfrak{A}_{11} = \tfrac{4671}{8673280}$$
$$= 0{,}000538\ 550582\ 939787\ 485242\ 030696\ 576151\ 1\ldots$$

$$\tfrac{1}{12}\mathfrak{A}_{12} = \tfrac{13695779093}{31384184832000}$$
$$= 0{,}000436\ 391104\ 829190\ 422223\ 931924\ 108290\ 9\ldots$$

$$\tfrac{1}{13}\mathfrak{A}_{13} = \tfrac{24466579093}{67999067136000}$$
$$= 0{,}000359\ 807569\ 772481\ 885831\ 499181\ 112530\ 7\ldots$$

$$\tfrac{1}{14}\mathfrak{A}_{14} = \tfrac{132282840127}{439378587648000}$$
$$= 0{,}000301\ 068017\ 071819\ 489777\ 413687\ 718550\ 4\ldots$$

etc. etc.

Ad investigandam constantem, posito li $0 = 0$, fiat in aequatione (2.), adhibitis signis inferioribus, $x = 1$, quo prodit valor

$$3.\quad c = \mathfrak{A}_1 + \tfrac{1}{2}\mathfrak{A}_2 + \tfrac{1}{3}\mathfrak{A}_3 + \tfrac{1}{4}\mathfrak{A}_4 + \tfrac{1}{5}\mathfrak{A}_5 + \ldots$$

ex quo videre licet, c inter valores 0,5 et 0,6 contineri. Porro cum sit

$$\mathfrak{A}_1 = \frac{1}{1.2}$$

$$\tfrac{1}{2}\mathfrak{A}_2 = \frac{1}{2.3} - \frac{1}{2.2}\mathfrak{A}_1$$

$$\tfrac{1}{3}\mathfrak{A}_3 = \frac{1}{3.4} - \frac{1}{3.3}\mathfrak{A}_1 + \frac{1}{2.3}\mathfrak{A}_2$$

$$\tfrac{1}{4}\mathfrak{A}_4 = \frac{1}{4.5} - \frac{1}{4.4}\mathfrak{A}_1 - \frac{1}{3.4}\mathfrak{A}_2 - \frac{1}{2.4}\mathfrak{A}_3$$

etc. etc.

additis quae in linea verticali sese excipiunt membris, series efficitur:

$$4.\quad c = 1 - \mathfrak{A}_1\left(\frac{1}{2.2}+\frac{1}{3.3}+\frac{1}{4.4}+\dots\right) - \mathfrak{A}_2\left(\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\dots\right)$$
$$-\mathfrak{A}_3\left(\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\dots\right) - \text{etc.}$$

quae citius convergit quam (3.) et, deficiente alia via, ad determinationem constantis c adhiberi possit. Quamquam non sufficit, ut valorem solum numericum constantis enuclees, sed necesse est, ut veram cognoscas hujus quantitatis indolem, quem finem modo in sequentibus proposito assequamur.

Constat enim esse

$$5.\quad \mathfrak{A}_p = (-1)^{p+1}\int_0^1 \frac{\nu(\nu-1)(\nu-2)\dots(\nu-p+1)}{1.2.3.4\dots p}\,\partial\nu$$
$$= \frac{(-1)^{p+1}}{p!}\left(\frac{{}^p\mathfrak{F}_0}{p+1} - \frac{{}^p\mathfrak{F}_1}{p+2} + \frac{{}^p\mathfrak{F}_2}{p+3} - \frac{{}^p\mathfrak{F}_3}{p+4} + \dots \pm \frac{{}^p\mathfrak{F}_{p-2}}{3} \pm \frac{{}^p\mathfrak{F}_{p-1}}{2}\right)$$

(denotante ${}^p\mathfrak{F}_n$ coëfficientem $n$tum evolutionis facultatis $(1;+x)^p$, si p est numerus integer positivus). Quos valores si substituas in (2.), solutis parenthesibus et membris illis, quae eadem utuntur differentia $p-n$ indicum coëfficientis ${}^p\mathfrak{F}_n$, contractis, haec prodit aequatio:

$$\text{li}(1\pm x) = c + lx \pm \tfrac{1}{2}\left(\frac{{}^1\mathfrak{F}_0}{1}\cdot\frac{x}{1!} \mp \frac{{}^2\mathfrak{F}_1}{2}\cdot\frac{x^2}{2!} + \frac{{}^3\mathfrak{F}_2}{3}\cdot\frac{x^3}{3!} \mp \frac{{}^4\mathfrak{F}_3}{4}\cdot\frac{x^4}{4!} + \dots\right)$$
$$+ \tfrac{1}{3}\left(\frac{{}^2\mathfrak{F}_0}{2}\cdot\frac{x^2}{2!} \mp \frac{{}^3\mathfrak{F}_1}{3}\cdot\frac{x^3}{3!} + \frac{{}^4\mathfrak{F}_2}{4}\cdot\frac{x^4}{4!} \mp \frac{{}^5\mathfrak{F}_3}{5}\cdot\frac{x^5}{5!} + \dots\right)$$
$$\pm \text{ etc.}$$

Ad summandas series in parenthesibus inclusas habes, denotante ${}^a\mathfrak{B}_n$ coëfficientem binomialem $n$tum exponentis a,

$$\pm\frac{{}^a\mathfrak{B}_1 x}{1} + \frac{{}^a\mathfrak{B}_2 x^2}{2} \pm \frac{{}^a\mathfrak{B}_3 x^3}{3} + \dots = \pm a\left(\frac{{}^1\mathfrak{F}_0}{1}\cdot\frac{x}{1!} \mp \frac{{}^2\mathfrak{F}_1}{2}\cdot\frac{x^2}{2!} + \dots\right)$$
$$+ a^2\left(\frac{{}^2\mathfrak{F}_0}{2}\cdot\frac{x^2}{1!} \mp \frac{{}^3\mathfrak{F}_1}{3}\cdot\frac{x^3}{3!} + \dots\right)$$
$$\pm \text{ etc.}$$

Pars autem hujus aequationis ad sinistram scripta est aequalis quantitati

$$\int\frac{(1\pm x)^a - 1}{x}\,\partial x = \int\frac{\partial x}{x}\left[\frac{a}{1!}\,l(1\pm x) + \frac{a^2}{2!}[l(\pm x)]^2 + \frac{a^3}{3!}[l(1\pm x)]^3 + \dots\right),$$

quo efficitur

$$6.\quad (\pm 1)^n\left(\frac{{}^n\mathfrak{F}_0}{n}\cdot\frac{x^n}{n!} \mp \frac{{}^{n+1}\mathfrak{F}_1}{n+1}\cdot\frac{x^{n+1}}{(n+1)!} + \frac{{}^{n+2}\mathfrak{F}_2}{n+2}\cdot\frac{x^{n+2}}{(n+2)!} \mp \dots\right)$$
$$= \int\frac{[l(1\pm x)]^n}{n!}\cdot\frac{\partial x}{x},$$

ideoque

$$7.\quad \mathrm{li}(1\pm x)=c+\int\frac{[l(1\pm x)]^0}{1!}\cdot\frac{\partial x}{x}+\int\frac{[l(1\pm x)]^1}{2!}\cdot\frac{\partial x}{x}+\int\frac{[l(1\pm x)]^2}{3!}\cdot\frac{\partial x}{x}+\ldots.$$

Jam vero a cl. *Eulero* est demonstratum, esse

$$8.\quad (-1)^n\int_0^1\frac{[l(1-x)]^n}{n!}\cdot\frac{\partial x}{x}=\frac{{}^n\mathfrak{F}_0}{n.n!}+\frac{{}^{n+1}\mathfrak{F}_1}{(n+1)(n+1)!}+\frac{{}^{n+2}\mathfrak{F}_2}{(n+2)(n+2)!}+\ldots.$$

$$=\sigma_{n+1}=\frac{1}{1^{n+1}}+\frac{1}{2^{n+1}}+\frac{1}{3^{n+1}}+\frac{1}{4^{n+1}}+\ldots.$$

Adhibitis igitur in aequatione (7.) signis inferioribus et posito $x=1$, invenitur

$$9.\quad c=\tfrac{1}{2}\sigma_2-\tfrac{1}{3}\sigma_3+\tfrac{1}{4}\sigma_4-\tfrac{1}{5}\sigma_5+-\ldots.,$$

qui est valor quantitatis illius constantis, quam vir cl. *Kramp* in summatione seriei harmonicae investigavit et in numeros sequentes $\gamma=c=$ 0,577215 664901 532860 618112 090082 3.. computavit.

§. 2.

Constante ita determinata nunc ad ipsam logarithmorum integralium computationem progredi licet, quae ex serie (1.) pro quovis valore variabilis x, ex serie (2.) autem solummodo pro $x<1$ proficit. Ubi autem x vel in aequatione (1.) aliquanto accrescit vel in (2.) prope ad limitem tendit, utraque series vix convergit et laborem computationis tam immensum reddit, ut necessario ad alia subsidia aliasque methodos sit confugiendum. Primum quod occurrit subsidium est illud, ut vel in formulas plane novas inquiras vel jamjam repertas tali modo transformes, ut multo citius convergant. Expressionem omnino novam ex integratione pro parte instituta accipis, quae iterata vice perfecta ad aequationem perducit:

$$10.\quad \mathrm{li}\,x^{\pm1}=\frac{x^{\pm1}}{\pm lx}\left[1\pm\frac{1!}{lx}+\frac{2!}{(lx)^2}\pm\frac{3!}{(lx)^3}+\frac{4!}{(lx)^4}\pm\frac{5!}{(lx)^5}+\ldots\right]+R.$$

Seriem autem hanc divergentem tum demum pro vero valore functionis $\mathrm{li}\,x$ habeas, cum evolutum residuum $R=n!\int\frac{\partial x}{(lx)^{n+1}}$, posito $n=\infty$, plane evanescere intellexeris. Qua quidem evolutione facta, aequatio illa viri cl. *Mascheroni* prodit:

$$11.\quad \mathrm{li}\,x^{\pm1}$$

$$=\frac{x^{\pm1}}{\pm lx}\left[1\pm\frac{1!}{lx}+\frac{2!}{(lx)^2}\pm\frac{3!}{(lx)^3}+\ldots\pm\frac{(n-1)!}{(lx)^{n-1}}\right]$$

$$\mp\frac{n}{lx}\left[1\pm\frac{n-1}{2\;lx}+\frac{(n-1)(n-2)}{3(lx)^2}\pm\ldots\pm\frac{(n-1)(n-2)\ldots4.3.2}{(n-1)(lx)^{n-2}}+\frac{(n-1)\ldots4.3.2.1}{n(lx)^{n-1}}\right]$$

$$+\gamma-\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}\right)$$

$$+l(\pm lx)\pm\frac{lx}{1(n+1)}+\frac{(lx)^2}{2(n+1)(n+2)}\pm\frac{(lx)^3}{3(n+1)(n+2)(n+3)}+\ldots$$

quae pro magnis quoque valoribus variabilis x satis convergit, at ipsum computationis laborem minime imminuit.

Posito nunc $n = \infty$, tota aequatio, cum sit

$$\gamma = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots + \frac{1}{\infty} - l\infty,$$

reducitur ad

$$\text{li}\, x^{\pm 1} = \frac{x^{\pm 1}}{\pm\, l x}\left[1 \pm \frac{1!}{lx} + \frac{2!}{(lx)^2} + \frac{3!}{(lx)^3} + \ldots \text{ in inf.}\right]$$
$$\pm\left[\frac{\infty}{lx} + \tfrac{1}{2}\left(\frac{\infty}{lx}\right)^2 + \tfrac{1}{3}\left(\frac{\infty}{lx}\right)^2 \pm \ldots\right] + l(\pm\, lx) - l\infty.$$

Serierum harum inferior evolutionem praebet expressionis $+ l\left(1 \mp \frac{\infty}{lx}\right) = l(lx \mp \infty) - Ux = l(\mp\infty) - llx$, quae in antecedente substituta, quia $l(-1) = \pi\sqrt{-1}$ invenitur, pro signorum diversitate aequationes efficit, quae sequuntur:

$$12. \quad \text{li}\,\frac{1}{x} = -\frac{1}{x\,lx}\left[1 - \frac{1!}{lx} + \frac{2!}{(lx)^2} - \frac{3!}{(lx)^3} + \frac{4!}{(lx)^4} - + \ldots\right],$$

$$13. \quad \text{li}\, x = \frac{x}{lx}\left[1 + \frac{1!}{lx} + \frac{2!}{(lx)^2} + \frac{3!}{(lx)^4} + \frac{4!}{(lx)^4} + \ldots\right] + \pi\sqrt{-1}.$$

Quod novum affert documentum rei in analysi satis jam compertae, *seriei quamvis divergentis summam reperiri posse finitam atque realem, si singulorum terminorum opposita sese excipiant signa; sin vero termini seriei divergentis iisdem omnino signis utantur, seriem aut infinitam habere summam aut imaginariae quantitatis praebere symbolum.* Quae series, satis quidem memorabiles, cum ad computationem logarithmi integralis omnino nihil faciant, necesse est alias investigemus.

Novam hic commemoremus seriem a viro cl. *Bessel* ex aequatione

$$\frac{\partial x}{lx} = \frac{\partial x}{n\, l x^{\frac{1}{n}}} = \frac{1}{n}\left[(x^{\frac{1}{n}} - 1) - \tfrac{1}{2}(x^{\frac{1}{n}} - 1)^2 + \tfrac{1}{3}(x^{\frac{1}{n}} - 1)^3 - + \ldots\right]^{-1} \partial x$$

derivatam, puta

$$14. \quad \text{li}\, x = l[\pm(x^{\frac{1}{n}} - 1)] + x^{\frac{1}{n}} + \tfrac{1}{2} x^{\frac{2}{n}} + \tfrac{1}{3} x^{\frac{3}{n}} + \ldots$$
$$\ldots + \frac{1}{n-1} x^{\frac{n-1}{n}} + \frac{x^n}{n}\left[\mathfrak{A}_1 - \mathfrak{A}_2 \Delta^{(1)} + \mathfrak{A}_3 \Delta^{(2)} - \mathfrak{A}_4 \Delta^{(3)} + \ldots\right],$$

in qua $\mathfrak{A}_1$, $\mathfrak{A}_2$ etc. sunt coëfficientes seriei (2.), et quantitates $\Delta^{(1)}$, $\Delta^{(2)}$ etc. primos denotant terminos serierum differentiarum, quae ex singulis terminis 1, $\dfrac{x^{\frac{1}{n}}}{1+\frac{1}{n}}$, $\dfrac{x^{\frac{2}{n}}}{1+\frac{2}{n}}$, $\dfrac{x^{\frac{3}{n}}}{1+\frac{3}{n}}$ etc. successive descendunt. Valor quantitatis arbitrariae n ita est determinandus, ut sit $x^{\frac{1}{n}} < 2$, qua una tantum conditione series convergit. Si modicus variabilis x datur valor, computatio functionis $\mathrm{li}\, x$ satis commode efficitur; at magni argumenti x valores computum tam operosum reddunt, ut calculatoris quamvis indefessi sedulitatem reprimant.

Alias etiam nanciscimur formulas discerptis ratione modo dicenda coëfficientibus $\mathfrak{A}_1$ $\mathfrak{A}_2$ etc. Constat nimirum, coëfficientes ex evolutione quantitatis $\frac{1}{l(1 \pm x)}$ progredientes ita quoque repraesentari posse, ut sit:

$$\begin{aligned}
\mathfrak{A}_1 &= \tfrac{1}{2}, \\
\tfrac{1}{2}\mathfrak{A}_2 &= \frac{\beta_1 \cdot {}^1\mathfrak{F}_0}{2.2!}, \\
\tfrac{1}{3}\mathfrak{A}_3 &= \frac{\beta_1 \cdot {}^2\mathfrak{F}_1}{2.3!}, \\
\tfrac{1}{4}\mathfrak{A}_4 &= \frac{\beta_1 \cdot {}^3\mathfrak{F}_2}{2.4!} - \frac{\beta_2 \cdot {}^3\mathfrak{F}_0}{4.4!}, \\
\tfrac{1}{5}\mathfrak{A}_5 &= \frac{\beta_1 \cdot {}^4\mathfrak{F}_3}{2.5!} - \frac{\beta_2 \cdot {}^4\mathfrak{F}_1}{4.5!}, \\
\tfrac{1}{6}\mathfrak{A}_6 &= \frac{\beta_1 \cdot {}^5\mathfrak{F}_4}{2.6!} - \frac{\beta_2 \cdot {}^5\mathfrak{F}_2}{4.6!} + \frac{\beta_3 \cdot {}^5\mathfrak{F}_0}{6.6!}, \\
\tfrac{1}{7}\mathfrak{A}_7 &= \frac{\beta_1 \cdot {}^6\mathfrak{F}_5}{2.7!} - \frac{\beta_2 \cdot {}^6\mathfrak{F}_3}{4.7!} + \frac{\beta_3 \cdot {}^6\mathfrak{F}_1}{6.7!} \\
&\text{etc.,}
\end{aligned}$$

ubi β_1, β_2, β_3 etc. numeros designant Bernouillianos. Quibus valoribus in serie (2.) substitutis nova prodit aequatio:

$$15. \quad \begin{aligned}
\mathrm{li}(1 \pm x) = \gamma + lx \pm \tfrac{1}{2}x &- \frac{\beta_1}{2}\left(\frac{{}^1\mathfrak{F}_0}{2!}x^2 \mp \frac{{}^2\mathfrak{F}_1}{3!}x^3 + \frac{{}^3\mathfrak{F}_2}{4!}x^4 \mp \ldots\right) \\
&+ \frac{\beta_2}{4}\left(\frac{{}^3\mathfrak{F}_0}{4!}x^4 \mp \frac{{}^4\mathfrak{F}_1}{5!}x^5 + \frac{{}^5\mathfrak{F}_2}{6!}x^6 \mp \ldots\right) \\
&- \text{etc.,}
\end{aligned}$$

in qua ut serierum ad dextram scriptarum summas faciamus est necesse. Quem ad finem commemoremus formulam illam notam

$$\frac{{}^a\mathfrak{B}_n}{n+1}x^{n+1}(\pm 1)^{n+1}$$

$$= (\pm 1)^{n+1}\frac{x^{n+1}}{(n+1)!}({}^n\mathfrak{F}_0 \cdot a^n - {}^n\mathfrak{F}_1 \cdot a^{n-1} + - \ldots \mp {}^n\mathfrak{F}_{n-2} \cdot a^2 \pm {}^n\mathfrak{F}_{n-1} \cdot a),$$

ex qua, posito n successive $= 1, 2, 3,$ etc. et additis qui inde proveniunt terminis, aequatio prodit:

16. $$\frac{^a\mathfrak{B}_1}{2}x^2 \pm \frac{^a\mathfrak{B}_2}{3}x^3 + \frac{^a\mathfrak{B}_3}{4}x^4 \pm \dots = a\left(\frac{^1\mathfrak{F}_0}{2!}x^2 \mp \frac{^2\mathfrak{F}_1}{3!}x^3 + \frac{^3\mathfrak{F}_2}{4!}x^4 \mp \dots\right) \pm a^2\left(\frac{^2\mathfrak{F}_0}{3!}x^3 \mp \frac{^3\mathfrak{F}_1}{4!}x^4 + \frac{^4\mathfrak{F}_2}{5!}x^5 \mp \dots\right) + \text{etc.},$$

cujus pars ad sinistram scripta formam quoque induit sequentem:

$$\frac{(1\pm x)^{a+1} - 1 \mp (a+1)x}{a+1}$$
$$= \mp x + \frac{l(1\pm x)}{1!} + (a+1)\frac{[l(1\pm x)]^2}{2!} + (a+1)^2\frac{[l(1\pm x)]^3}{3!} + \dots,$$

vel evolutione facta et posito $1\pm x = u$, in aequationem

$$\frac{u^{a+1} - 1 + (a+1)(1-u)}{a+1} = a\frac{(lu)^2}{1!}\left(\frac{1}{2} + \frac{lu}{1!3} + \frac{(lu)^2}{2!4} + \frac{(lu)^3}{3!5} + \dots\right) + a^2\frac{(lu)^3}{2!}\left(\frac{1}{3} + \frac{lu}{1!4} + \frac{(lu)^2}{2!5} + \frac{(lu)^3}{3!6} + \dots\right) + \text{etc.}$$

abit. Brevitatis causa nunc designemus series modo evolutas symbolo G^{lu}, ita ut sit

17. $$\frac{1}{n} \pm \frac{lu}{1!(n+1)} + \frac{(lu)^2}{2!(n+2)} \pm \frac{(lu)^3}{3!(n+3)} + \frac{(lu)^4}{4!(n+4)} \pm \dots = G_n^{\pm lu},$$

quo nanciscimur

18. $$\frac{^n\mathfrak{F}_0}{(n+1)!}x^{n+1} \mp \frac{^{n+1}\mathfrak{F}_1}{(n+2)!}x^{n+2} + \frac{^{n+2}\mathfrak{F}_2}{(n+3)!}x^{n+3} \mp \dots = (\pm 1)^{n+1}\frac{[l(1\pm x)]^{n+1}}{n!}G_{n+1}^{l(1\pm x)};$$

tunc ex aequatione (15.) descendit aequatio haec nova et memorabilis

19. $$\text{li}\,u = \gamma + l[\mp(1-u)] - \frac{1-u}{2} - \frac{\beta_1}{2}\cdot\frac{(lu)^2}{1!}G_2^{lu} + \frac{\beta^2}{4}\cdot\frac{(lu)^4}{3!}G_4^{lu} - \frac{\beta^3}{6}\cdot\frac{(lu)^6}{5!}G_6^{lu} + - \dots$$

in quo numeri Bernouilliani coëfficientes constituunt singulorum terminorum. Quae quidem evolutio subsidium adeo nobis suppeditat, quo constantis valorem alia prorsus et nova, quam qua supra usi sumus, methodo evolvamus. Habemus enim in aequatione (16.), adhibitis signis inferioribus et posito $x = 1$,

$$\frac{^a\mathfrak{B}_1}{2} - \frac{^a\mathfrak{B}_2}{3} + \frac{^a\mathfrak{B}_3}{4} - \frac{^a\mathfrak{B}_4}{5} + - \dots = \frac{a}{1+a} = a - a^2 + a^3 - a^4 + \dots,$$

ideoque

20. $$1 = \frac{^n\mathfrak{F}_0}{(n+1)!} + \frac{^{n+1}\mathfrak{F}_1}{(n+2)!} + \frac{^{n+2}\mathfrak{F}_2}{(n+3)!} + \frac{^{n+3}\mathfrak{F}_3}{(n+4)!} + \dots,$$

cujus ope ex formula (15.) valorem constantis

$$21.\quad c = \tfrac{1}{2} + \frac{\beta_1}{2} - \frac{\beta_2}{4} + \frac{\beta_3}{6} - \frac{\beta_4}{8} + \frac{\beta_5}{10} - + \ldots = \gamma$$

eruimus eundem, quem supra jam invenimus.

Coëfficientes symbole G denotatos, quorum in tota hac theoria et maximus usus et maxima est utilitas, alias etiam per formas repraesentare licet memoratu dignas. Differentiata aequatione (17.) secundum lu accipimus:

$$\frac{\partial \,.\, G_n^{\pm lu}}{\partial(\pm lu)} = \frac{1}{n+1} \pm \frac{lu}{1!(n+2)} + \frac{(lu)^2}{2!(n+3)} \pm \frac{(lu)^3}{3!(n+4)} + \ldots = G_{n+1}^{\pm lu},$$

ex quo concludimus, coëfficientes G enucleari posse posteriorem nimirum quemque ex antecedenti sola adhibita differentiatione. Itaque cum sit $G_1^{\pm lu} = \frac{u^{\pm 1} - 1}{\pm lu}$, habemus etiam

$$22.\quad G_{n+1}^{\pm lu} = \frac{\partial^n \left(\frac{u^{\pm 1} - 1}{\pm lu}\right)}{\partial(\pm lu)^n}$$

sive, differentiatione perpetrata,

$$23.\quad G_{n+1}^{lu} = \frac{u[(lu)^n - n(lu)^{n-1} + n(n-1)(lu)^{n-2} - \ldots \pm n(n-1)\ldots 3.2\,lu \pm n!] \mp n!}{(lu)^{n+1}},$$

$$24.\quad G_{n+1}^{-lu} = \frac{u^{-1}[(lu)^n + n(lu)^{n-1} + n(n-1)(lu)^{n-2} + \ldots + n(n-1)\ldots 3.2\,lu + n!] - n!}{(-lu)^{n+1}},$$

quarum aequationum beneficio etiam licet, coëfficientes G per formulam hanc recurrentem

$$25.\quad G_{n+1}^{\pm lu} = \frac{n\,.\,G_n^{\pm lu} - u^{\pm 1}}{\mp lu}$$

inter se conjungere. Invenimus autem eadem via:

$$G_n^{\pm lu} = \frac{u^{\pm 1}}{n} + \frac{\mp lu}{n} G_{n+1}^{\pm u}, \qquad G_{n+1}^{\pm lu} = \frac{u^{\pm 1}}{n+1} + \frac{\mp lu}{n+1} G_{n-2}^{\pm lu} \text{ etc.}$$

qui valores, perpetua adhibita substitutione, cum residuum $\frac{(\mp lu)^\nu}{n(n+1)(n+2)\ldots(n+\nu-1)} G_{n+\nu}^{\pm lu}$ pro $\nu = \infty$ ad nihilum decrescat, novam praebent aequationem:

$$26.\quad G_n^{\pm lu} = \frac{u^{\pm 1}}{1}\left[\frac{1}{n} \mp \frac{lu}{n(n+1)} + \frac{(lu)^2}{n(n+1)(n+2)} \mp \frac{(lu)^3}{n(n+1)(n+2)(n+3)} + \ldots\right]$$
$$= u^{\pm 1} C_n^{\mp lu},$$

quae multo citius convergit, quam series in (17.) inventa, ideoque ad coëfficientium G seu C computum numericum magnopere praestat.

Alia quoque exstat nec non rectior via, qua eundem valorem (26.) assequi licet. Multiplicata enim serie (17.) per

$$u^{\mp 1} = e^{\mp lu} = 1 \mp \frac{lu}{1!} + \frac{(lu)^2}{2!} \mp \frac{(lu)^3}{3!} + \dots$$

efficitur, si termini easdem quantitatis lu dignitates continentes conjunguntur,

27. $$u^{\mp 1} G^{\pm lu} = \frac{1}{n} \mp lu \left(\frac{1}{n} - \frac{1}{1!(n+1)}\right) + (lu)^2 \left(\frac{1}{2!n} - \frac{1}{1!1!(n+1)} + \frac{1}{2!(n+2)}\right) \mp (lu)^3 \left(\frac{1}{3!n} - \frac{1}{2!1!(n+1)} + \frac{1}{1!2!(n+2)} - \frac{1}{3!(n+3)}\right) + \text{etc.}$$

Coëfficiens termini $(lu)^p$, nimirum

$$\left(\frac{1}{p!n} - \frac{1}{(p-1)!1!(n+1)} + \frac{1}{(p-2)!2!(n+2)} - \dots \pm \frac{1}{1!(p-1)!(n+p-1)} \mp \frac{1}{p!(n+p)}\right),$$

formam potest induere

$$\frac{1}{p!}\left[\frac{1}{n} - \frac{{}^p\mathfrak{B}_1}{n+1} + \frac{{}^p\mathfrak{B}_2}{n+2} - \frac{{}^p\mathfrak{B}_3}{n+3} + \dots \pm \frac{{}^p\mathfrak{B}_2}{n+p-2} \mp \frac{{}^p\mathfrak{B}_1}{n+p-1} \pm \frac{1}{n+p}\right] = \frac{1}{p!}(n.^{n+p}\mathfrak{B}_p)^{-1};$$

cujus ope, cum sit

$$\frac{1}{n.p!} \cdot \frac{1}{{}^{n+p}\mathfrak{B}_p} = \frac{1}{n(n+1)(n+2)\dots(n+p)},$$

series (27.) extemplo in priorem (26.) transmutatur.

Denique, ut omnia ad coëfficientes G spectantia hic colligamus, nunc integrale hoc definitum

28. $$G_{n+1}^{\pm lu} = \int_0^1 u^{\pm v} . v^n \, \partial v$$

commemoremus, quod rite evolutum omnes fere, quos adhuc invenimus, nobis praebet valores coëfficientis G.

§. 3.

Series adhuc repertae cum ad computum numericum functionis nostrae pro magnis argumenti x valoribus non admodum prosint, experiamur, an discerpando et transformando effici possit, ut citius convergant. Quem ad finem meminisse juvabit, esse

$$-l(1-x) = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots = X_0,$$

$$\frac{1}{1.1!} - \frac{1-x}{1.x} X_0 = \frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \frac{x^4}{4.5} + \dots = X_1,$$

$$\frac{1}{2\,2!} - \frac{1-x}{2.x} X_1 = \frac{x}{1.2.3} + \frac{x^3}{2.3.4} + \frac{x^4}{3.4.5} + \frac{x^5}{4.5.6} + \dots = X_2,$$

etc. etc.

$$\frac{1}{n.n!} - \frac{1-x}{n.x} X_{n-1} = \frac{x}{1.2\dots(n+1)} + \frac{x^2}{2.3.4\dots(n+2)} + \frac{x^3}{3.4.5\dots(n+3)} + \dots = X_n,$$

unde posito $x=1$, cum sit $(1-1)\,l(1-1)=0$, descendunt valores

$$\frac{1}{1.1!}=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\dots,$$
$$\frac{1}{2.2!}=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\dots,$$
$$\dots\dots\dots\dots\dots\dots$$
$$\frac{1}{n.n!}=\frac{1}{1.2\dots(n+1)}+\frac{1}{2.3\dots(n+2)}+\frac{1}{3.4\dots(n+3)}+\dots,$$

qui in formula fundamentali (1.) substituti, singulis terminis rite dispositis et ratione habita aequationis (26.), novam efficiunt seriem

$$29.\quad \mathrm{li}\,x^{\pm 1}=\gamma+l(\pm lx)\pm lx\left[\tfrac{1}{1}C_2^{\pm lx}+\tfrac{1}{2}C_3^{\pm lx}+\tfrac{1}{3}C_4^{\pm lx}+\dots\right]$$
$$=\gamma+l(\pm lx)\pm lx.x^{\pm 1}\left[\tfrac{1}{1}G_2^{\mp lx}+\tfrac{1}{2}G_3^{\mp lx}+\tfrac{1}{3}G_4^{\mp lx}+\dots\right].$$

Itaque si valores coëfficientium C aliunde jam computatos habes, logarithmum integralem facillime computes; si non, valores illos e formula recurrenti (25.) commode evolvas, evolutosque in functionis computatione adhibeas. Crescente x seriei convergentia imminuitur, ita ut pro magnis argumenti valoribus computus nimis reddatur operosus, quo quidem casu alia necesse est utaris seriei (1.) transformatione. Retentis nimirum in formula fundamentali terminis omnibus, quorum valor unitatem superat, discerptionem supra doctam in illo demum seriei termino adhibeas, quod unitatis limitem haud attingit, unde aequationem novam:

$$30.\quad \mathrm{li}\,x^{\pm 1}=\gamma+l(\pm lx)\pm lx+\frac{(lx)^2}{2.2!}\pm\frac{(lx)^3}{3.3!}+\dots\pm\frac{(lx)^{n-1}}{(n-1)(n-1)!}$$
$$+(lx)^n\left[\frac{C_{n+1}^{\pm lx}}{1.2.3\dots n}+\frac{C_{n+2}^{\pm lx}}{2.3.4\dots(n+1)}+\frac{C_{n+3}^{\pm lx}}{3.4.5\dots(n+2)}+\dots\right]$$

accipias pro omni arbitrariae x valore satis convergentem.

Ultimo loco duarum quoque aequationum mentionem faciamus, quae logarithmum integralem per differentialia successiva exhibent, et quarum altera ex (29.) et (22.) statim invenitur:

$$31.\quad \mathrm{li}\,x^{\pm 1}$$
$$=\gamma+l(\pm lx)\pm x^{\pm 1}lx\left(\frac{\partial\left(\frac{x^{\mp 1}-1}{\mp lx}\right)}{\partial(\mp lx)}+\tfrac{1}{2}\cdot\frac{\partial^2\left(\frac{x^{\mp 1}-1}{\mp lx}\right)}{\partial(\mp lx)^2}+\tfrac{1}{3}\cdot\frac{\partial^3\left(\frac{x^{\mp 1}-1}{\mp lx}\right)}{\partial(\mp lx)^3}+\dots\right),$$

altera vero ex formula (19.) sequenti modo eruitur. Habes enim ex aequatione (23.) valores:

$$\frac{(lu)^2}{1!}G_2^{lu}=u\left(\frac{lu}{1!}-1\right)+1,\quad \frac{(lu)^4}{3!}G_4^{lu}=u\left(\frac{(lu)^3}{3!}-\frac{(lu)^2}{2!}+\frac{lu}{1!}-1\right)+1 \text{ etc.}$$

qui in (19.) substituti, adhibita aequatione (21.), formulam exhibent:

$$\left.\begin{aligned}\mathrm{li}\, u = l[\mp(1-u)] + u\Big\{\tfrac{1}{2} &- \frac{\beta_1 . lu}{2.1!} + \frac{\beta_2 . (lu)^3}{4.3!} - \frac{\beta_3 . (lu)^5}{6.5!} + \dots \\ &+ \frac{\beta_1}{2} - \frac{\beta_2 . (lu)^2}{4.2!} + \frac{\beta_3 . (lu)^4}{6.4!} - \dots \\ &+ \frac{\beta_2 . lu}{4.1!} - \frac{\beta_3 . (lu)^3}{6.3!} + \dots \\ &- \frac{\beta_2}{4} + \frac{\beta_3 (lu)^2}{6.2!} - \dots \\ &\text{etc.}\end{aligned}\right\}.$$

Facillimo nunc intelligis negotio, serierum horizontalium inferiorem quamque esse differentiale superioris, ita ut dato valore supremae seriei tota aequatio satis sit determinata. Constat autem esse

$$\frac{e^u+1}{e^u-1} = \frac{2}{u}\left(1 + \frac{\beta_1 u^2}{2!} - \frac{\beta_2 u^4}{4!} + \frac{\beta_3 u^6}{6!} - \frac{\beta_4 u^8}{8!} + \dots\right),$$

cujus ope seriei illius supremae summam invenimus aequalem $-\left(\frac{1}{u-1} - \frac{1}{lu}\right)$, quod formulam nobis praebet gravissimam

$$32. \quad l(\mp 1 \pm u) - \mathrm{li}\, u$$

$$= u\left\{\left(\frac{1}{u-1} - \frac{1}{lu} - \frac{\partial\left(\frac{1}{u-1} - \frac{1}{lu}\right)}{\partial(lu)} + \frac{\partial^2\left(\frac{1}{1-u} - \frac{1}{lu}\right)}{\partial(lu)^2} - + \dots\right\},$$

sive

$$33. \quad l[\pm(u-1)] - \mathrm{li}\, u$$

$$= u\left\{\frac{\partial[l(u-1) - \mathrm{li}\, u]}{u\,\partial lu} - \frac{\partial^2[l(u-1) - \mathrm{li}\, u]}{\partial(lu)^2} + \frac{\partial^3[l(u-1) - \mathrm{li}\, u]}{\partial(lu)^3} - + \dots\right\}.$$

§. 4.

Aliud idemque prorsus novum subsidium, quo adhibito termini seriei cujusdam magis convergant, theoria suppeditat fractionum continuarum, cujus alias quoque in analysi utilitas est atque usus. Ut li x fractione continua exhibeatur non nisi adhibita aequatione (12.) effici potest, cum numeratores et denominatores particulares ex seriebus (1.) et (2.) orientes tantopere sint impediti, ut lex qua utuntur reperiri vix possit. Itaque si data est series

$$fx = 1 - (n+1)x + (n+1)(n+2)x^2 - (n+1)(n+2)(n+3)x^3 + \dots$$

in fractionem continuam transformanda, habes:

$$a.\quad fx = \cfrac{1}{1+\cfrac{(n+1)x}{1+\cfrac{1.x}{1+\cfrac{(n+2)x}{1+\cfrac{2x}{1+\cfrac{(n+3)x}{1+\cfrac{3x}{1+\text{etc.}}}}}}}}$$

Denotemus nunc singulos numeratores particulares suo ordine per $a_0\, a_1\, a_2\, a_3 \ldots.$ nec non denominatores fractionum approximatarum, successive evolutarum, per $M_0\, M_1\, M_2\, M_3 \ldots.$, nemo est qui dubitet, quin sit

$$fx = \frac{a_0}{M_0} - \frac{a_0 a_1}{M_0 M_1} + \frac{a_0 a_1 a_2}{M_1 M_2} - \frac{a_0 a_1 a_2 a_3}{M_2 M_3} + - \ldots.$$

Invenimus autem $M_p = a_p . M_{p-2} + M_{p-1}$, qua de causa, subtractis terminis seriei negativis ab antecedentibus positivis, nanciscimur

$$\frac{a_0 a_1 \ldots. a_{2p}}{M_{2p}}\left(\frac{M_{2p+1} - a_{2p+1} . M_{2p-1}}{M_{2p-1} M_{2p+1}}\right) = \frac{a_0 a_1 a_2 \ldots. a_{2p}}{M_{2p-1} . M_{2p+1}},$$

quod suo loco substitutum seriem supra evolutam exhibet sub forma sequenti

$$b.\quad fx = \frac{a_0}{M_1} + \frac{a_0 a_1 a_2}{M_1 M_3} + \frac{a_0 a_1 a_2 a_3 a_4}{M_3 M_5} + \frac{a_0 a_1 \ldots. a_5 a_6}{M_5 M_7} + \ldots.$$

Itaque cum quivis denominator $M_1 M_3 M_5 \ldots.$ hoc schemate universali

$$c.\quad M_{2p+1} =$$

$$1 + (p+1)\,{}^{n+p+1}\mathfrak{B}_1 x + (p+1)p\,{}^{n+p+1}\mathfrak{B}_2 x^2 + (p+1)p(p-1)\,{}^{n+p+1}\mathfrak{B}_3 x^3 + \ldots$$
$$\ldots + (p+1)p(p-1)\ldots.2.1\,{}^{n+p+1}\mathfrak{B}_{p+1} x^{p+1}$$

contineatur, series nostra ($b.$) hanc induit formam:

$$d.\quad fx = \frac{1}{1+(n+1)x} + \frac{1(n+1)x^2}{[1+(n+1)x]\,[1+2(n+2)x+(n+2)(n+1)x^2]}$$
$$+ \frac{1.2.(n+1)(n+2)x^4}{[1+2(n+2)x+(n+2)(n+1)x^2]\,[1+3(n+3)x+3(n+3)(n+2)x^2+(n+3)(n+2)(n+1)x^3]}$$
$$+ \text{etc.}$$

ad computationem aptissimam. Et facile intelligitur, hanc seriem transformatam eo citius convergere, quo magis series supra proposita diverserit.

Jam vero licet, loco fractionis continuae ($a.$) in seriem transformandae valores enucleare fractionum approximatarum, quae functionem fx usque ad certum limitem satis exacte repraesentant. Verum enim vero cum fractiones approximatae ex ($a.$) evolutae nimis tarde convergant, praestat ut fractionem continuam ($a.$) in aliam transformemus, quae non nisi dimidiam terminorum partem contineat. Constat nimirum esse

$$e.\quad fx = 1 - \cfrac{(n+1)x}{1+(n+2)x - \cfrac{1(n+2)x^2}{1+(n+4)x} - \cfrac{2(n+3)x^2}{1+(n+6)x - \cfrac{3(n+4)x^2}{1+(n+8)x - \text{etc.}}}};$$

quod nobis fractiones approximatas sequentes

$$f.\quad fx_1 = 1 - \frac{(n+1)x}{1+(n+2)x},\quad fx_2 = 1 - \frac{(n+1)x + 1(n+1)(n+2)x^2}{1+2(n+3)x+(n+3)(n+2)x^2}$$

$$fx_3 = 1 - \frac{(n+1)x+2(n+1)(n+5)x^2+[1(n+1)(n+5)(n+4)-1(n+1)(n+2)]x^3}{1+3(n+4)x+3(n+4)(n+3)x^2+(n+4)(n+3)(n+2)x^3} \text{ etc.}$$

suppeditat, quarum schema universale

$$g.\quad fx_p =$$

$$1 - \frac{N_1^p x + N_2^p x^2 + N_3^p x^3 + \dots + N_r^p x^r + \dots + N_{p-1}^p x^{p-1} + N_p^p x^p}{1+p.{}^{n+p+1}\mathfrak{B}_1 x + p(p-1){}^{n+p+1}\mathfrak{B}_2 x^2 + \dots + p(p-1)\dots 3.2\,{}^{n+p+1}\mathfrak{B}_{p-1} + p!\,{}^{n+p+1}\mathfrak{B}_p x^p}$$

invenitur. Numeratoris coëfficientes N_r^p ex illis numeratoris classis antecedentis $(p-1)$tae et $(p-2)$tae per formulas recurrentes derivantur

$$\begin{aligned}
N_1^p &= N_1^{p-1} = n+1,\\
N_2^p &= N_2^{p-1} + (n+2p)N_1^{p-1},\\
N_3^p &= N_3^{p-1} + (n+2p)N_2^{p-1} - (p-1)(n+p)N_1^{p-2},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
N_r^p &= N_r^{p-1} + (n+2p)N_{r-1}^{p-1} - (p-1)(n+p)N_{r-2}^{p-2},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
N_{p-1}^p &= N_{p-1}^{p-1} + (n+2p)N_{p-2}^{p-1} - (p-1)(n+p)N_{p-3}^{p-2},\\
N_p^p &= \qquad (n+2p)N_{p-1}^{p-1} - (p-1)(n+p)N_{p-2}^{p-2},
\end{aligned}$$

quae, substitutione rite perpetrata, aequationem nobis praebent:

$$h.\quad N_r^p = (r-1)!(n+1)\,{}^{p-1}\mathfrak{B}_{r-1}.{}^{n+p+2}\mathfrak{B}_{r-1} - (r-2)!(n+1)(n+2)\,{}^{p-2}\mathfrak{B}_{r-2}.{}^{n+p+2}\mathfrak{B}_{r-3}$$
$$+(r-3)!(n+1)(n+2)(n+3)\,{}^{p-3}\mathfrak{B}_{r-3}.{}^{n+p+2}\mathfrak{B}_{r-5} - + \dots,$$

quae pro impari valore indicis r termino illo finitur, quod ${}^{n+p+2}\mathfrak{B}_{r-(r-1)}$ continet, pro pari autem indicis valore, termino per ${}^{n+p+2}\mathfrak{B}_{r-r}$ multiplicato. Facili negotio nunc demonstrari potest, aequationem (*h.*) pro N_r^p valentem etiam pro N_r^{p+1} et N_{r+1}^p valere, cui autem demonstrationi, cum nimis longa esset, supersedemus. Singulos nunc coëfficientium N valores, quia $0! = 1$ et ${}^x\mathfrak{B}_0 = 1$ invenitur, tabula sequenti exhibeamus tales:

p	$r=1$	$r=2$	$r=3$
1	$n+1$		
2	$n+1$	$1(n+1)^{n+4}\mathfrak{B}_1$	
3	$n+1$	$2(n+1)^{n+5}\mathfrak{B}_1$	$2.1(n+1)^{n+5}\mathfrak{B}_2-1(n+1)(n+2)$
4	$n+1$	$3(n+1)^{n+6}\mathfrak{B}_1$	$3.2(n+1)^{n+6}\mathfrak{B}_2-2(n+1)(n+2)$
5	$n+1$	$4(n+1)^{n+7}\mathfrak{B}_1$	$4.3(n+1)^{n+7}\mathfrak{B}_2-3(n+1)(n+2)$
6	$n+1$	$5(n+1)^{n+8}\mathfrak{B}_1$	$5.4(n+1)^{n+8}\mathfrak{B}_2-4(n+1)(n+2)$
7	$n+1$	$6(n+1)^{n+9}\mathfrak{B}_1$	$6.5(n+1)^{n+9}\mathfrak{B}_2-5(n+1)(n+2)$

p	$r=4$
4	$3.2.1(n+1)^{n+6}\mathfrak{B}_3-2.1(n+1)(n+2)^{n+6}\mathfrak{B}_1$
5	$4.3.2(n+1)^{n+7}\mathfrak{B}_3-3.2(n+1)(n+2)^{n+7}\mathfrak{B}_1$
6	$5.4.3(n+1)^{n+8}\mathfrak{B}_3-4.3(n+1)(n+2)^{n+8}\mathfrak{B}_1$
7	$6.5.4(n+1)^{n+9}\mathfrak{B}_3-5.4(n+1)(n+2)^{n+9}\mathfrak{B}_1$

p	$r=5$
5	$4.3.2.1(n+1)^{n+7}\mathfrak{B}_4-3.2.1(n+1)(n+2)^{n+7}\mathfrak{B}_2+2.1(n+1)(n+2)(n+3)$
6	$5.4.3.2(n+1)^{n+8}\mathfrak{B}_4-4.3.2(n+1)(n+2)^{n+8}\mathfrak{B}_2+3.2(n+1)(n+2)(n+3)$
7	$6.5.4.3(n+1)^{n+9}\mathfrak{B}_4-5.4.3(n+1)(n+2)^{n+9}\mathfrak{B}_2+4.3(n+1)(n+2)(n+3)$

etc. etc.

Denique, si denominatorem fractionis approximatae $p$tae in aequatione (g.) signo X_p denotes, ex formula recurrenti

$$X_p = X_{p-1}+(n+2p)xX_{p-1}-(p-1)(n+p)x^2X_{p-2}$$

facile demonstrabis, schema ibi relatum etiam pro X_{p+1} valere ideoque vim habere universalem; unde fit ut singulae quaeque fractiones approximatae functionis fx possint exhiberi.

§. 5.

Fractionem continuam (e.) ita etiam licet repraesentare, ut unitatem antecedentem in ipsas recipias fractiones approximatas. Quo facto eruimus

$$i.\ \begin{cases} fx_1 = \dfrac{1+1!x}{1+(n+2)x}, \quad fx_2 = \dfrac{1+(1!+n+4)x+2!x^2}{1+2(n+3)+(n+3)(n+2)x^2}, \\ fx_3 = \dfrac{1+(1!+2(n+5))x+(2!+1(n+5)(n+4)+1.4)x^2+3!x^3}{1+3(n+4)x+3(n+4)(n+3)x^2+(n+4)(n+3)(n+2)} \end{cases} \text{etc.,}$$

seu schema universale:

$$k.\quad fx_p =$$

$$\frac{'N_0^p+'N_1^px+'N_2^px^2+\ldots+'N_r^px^r+\ldots+'N_{p-1}^px^{p-1}+'N_p^px^p}{1+p^{n+p+1}\mathfrak{B}_1x+p(p-1)^{n+p+1}\mathfrak{B}_2x^2+\ldots+p(p-1)\ldots3.2^{n+p+1}\mathfrak{B}_{p-1}x^{p-1}+p!^{n+p+1}\mathfrak{B}_px^p},$$

cujus denominatorem eundem, quem supra in ($g.$) invenimus. Numeratoris vero coëfficientes $'N$ ex formulis recurrentibus evolvimus sequentes

$$\begin{aligned}
'N_0^p &= 1,\\
'N_1^p &= {'N_1^{p-1}} + (n+2p)\,{'N_0^{p-1}},\\
'N_2^p &= {'N_2^{p-1}} + (n+2p)\,{'N_1^{p-1}} - (p-1)(n+p)\,{'N_0^{p-2}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
'N_r^p &= {'N_r^{p-1}} + (n+2p)\,{'N_{r-1}^{p-1}} - (p-1)(n+p)\,{'N_{r-2}^{p-2}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
'N_{p-1}^p &= {'N_{p-1}^{p-1}} + (n+2p)\,{'N_{p-2}^{p-1}} - (p-1)(n+p)\,{'N_{p-3}^{p-2}},\\
'N_p^p &= \qquad (n+2p)\,{'N_{p-1}^{p-1}} - (p-1)(n+p)\,{'N_{p-2}^{p-2}},
\end{aligned}$$

qui valores sibi rite substituti schema nobis praebent hoc universale:

$$\begin{aligned}
l.\quad 'N_r^p = r! &+ (r-1)!(p-r)\,[{}^{n+p+r+1}\mathfrak{B}_1 - {}^{n+r}\mathfrak{B}_1]\\
&+ (r-2)!(p-r)(p-r+1)\,[{}^{n+p+r}\mathfrak{B}_2 - {}^{n+p+r-1}\mathfrak{B}_1.{}^{n+r-1}\mathfrak{B}_1 + {}^{n+p-1}\mathfrak{B}_2]\\
&+ (r-3)!(p-r)(p-r+1)(p-r+2)\,[{}^{n+p+r-1}\mathfrak{B}_3 - {}^{n+p+r-2}\mathfrak{B}_2.{}^{n+r-2}\mathfrak{B}_1\\
&\qquad + {}^{n+p+r-3}\mathfrak{B}_1.{}^{n+r-2}\mathfrak{B}_2 - {}^{n+r-2}\mathfrak{B}_3] + \ldots\\
&\ldots + (r-(r-1))!(p-r)(p-r+1)\ldots(p-r+(r-1))\,[{}^{n+p+3}\mathfrak{B}_{r-1} - {}^{n+p+2}\mathfrak{B}_{r-2}.{}^{n+2}\mathfrak{B}_1]\\
&+ (p-r)(p-r+1)\ldots(p-1)p\;{}^{n+p+2}\mathfrak{B}_r.
\end{aligned}$$

Quantitates uncis [ ] inclusae, quae forma communi

$$m.\quad {}^{a}\mathfrak{B}_u - {}^{a-1}\mathfrak{B}_{u-1}.{}^{b}\mathfrak{B}_1 + {}^{a-2}\mathfrak{B}_{u-2}.{}^{b}\mathfrak{B}_2 - \ldots \pm {}^{a-v}\mathfrak{B}_{u-v}.{}^{b}\mathfrak{B}_v \mp \ldots \pm {}^{a-u+1}\mathfrak{B}_1.{}^{b}\mathfrak{B}_{u-1} \mp {}^{b}\mathfrak{B}_u$$

utuntur, ita sunt definiendae, ut omnes termini, in quibus $u+v>r$, tollantur nec non ii solummodo retineantur, in quibus indicum u et v summa indicem r non superat. Demonstrationem formulae ($l.$) pro casu, quo $p+1$ et $r+1$ loco p et r positum est, facile quidem perpetrares; sed cum sit longior hoc loco eam omisimus. Expressionem ($m.$) cum constet esse aequalem ${}^{a-b}\mathfrak{B}_u$, aequationem ($l.$) paullo licet contrahere. Substituto enim valore ${}^{a-b}\mathfrak{B}_u$ pro omnibus terminis, qui schema ($m.$) integrum exhibent, proficiscuntur: pro valore indicis r pari aequatio

$$\begin{aligned}
n.\quad 'N_r^p = r! &+ (r-1)!(p-r)\,{}^{p+1}\mathfrak{B}_1 + (r-2)!(p-r)(p-r+1)\,{}^{p+1}\mathfrak{B}_2\\
&+ (r-3)!(p-r)(p-r+1)(p-r+2)\,{}^{p+1}\mathfrak{B}_3 + \ldots\\
&\ldots + (r-\tfrac{1}{2}r)!(p-r)(p-r+1)\ldots(p-r+\tfrac{1}{2}r-1)\,{}^{p+1}\mathfrak{B}_{\frac{1}{2}r}\\
&+ (\tfrac{1}{2}r-1)!(p-r)(p-r+1)\ldots(p-\tfrac{1}{2}r-1)(p-\tfrac{1}{2}r)\left[{}^{n+p+\frac{r}{2}+1}\mathfrak{B}_{\frac{r}{2}+1} - {}^{n+p+\frac{1}{2}r}\mathfrak{B}_{\frac{r}{2}}.{}^{n+\frac{1}{2}r}\mathfrak{B}_1 + \ldots\right]\\
&+ \text{etc.}
\end{aligned}$$

pro valore autem indicis r impari aequatio

$$o.\quad {}'N_r^p = r! + (r-1)!(p-r)\,{}^{p+1}\mathfrak{B}_1 + (r-2)!(p-r)(p-r+1)\,{}^{p+1}\mathfrak{B}_2$$
$$+ (r-3)!(p-r)(p-r+1)(p-r+2)\,{}^{p+1}\mathfrak{B}_3 + \dots$$
$$\dots + (r-\tfrac{1}{2}(r-1))!(p-r)(p-r+1)\dots(p-r+\tfrac{1}{2}(r-3))\,{}^{p+1}\mathfrak{B}_{\frac{1}{2}(r-1)}$$
$$+ (r-\tfrac{1}{2}(r+1))!(p-r)(p-r+1)\dots$$
$$\dots(p-r+\tfrac{1}{2}(r-1))[{}^{n+p+\frac{1}{2}(r+3)}\mathfrak{B}_{\frac{1}{2}(r+1)} - {}^{n+p+\frac{1}{2}(r+1)}\mathfrak{B}_{\frac{1}{2}(r-1)}.{}^{n+\frac{1}{2}(r+1)}\mathfrak{B}_1 + - + \dots]$$
$$+ \text{etc.}$$

Coëfficientes $'N$ cum eosdem inveniamus, si in aequatione ($g.$) fractionem ad dextram scriptam de unitate detrahimus, habemus etiam

$$p(p-1)\dots(p-r+1)\,{}^{n+p+1}\mathfrak{B}_r x^r - N_r^p x^r = {}'N_r^p x^r,$$

ideoque

$$p.\quad p(p-1)\dots(p-r+1)\,{}^{n+p+1}\mathfrak{B}_r - (r-1)!(n+1)\,{}^{p-1}\mathfrak{B}_{r-1}.{}^{n+p+2}\mathfrak{B}_{r-1}$$
$$+ (r-2)!(n+1)(n+2)\,{}^{p-2}\mathfrak{B}_{r-2}.{}^{n+p+2}\mathfrak{B}_{r-3} + \dots = {}'N_r^p,$$

quod adhibita aequatione ($l.$) relationes nobis praebet inter coëfficientes binomiales admodum memorabiles, quas hoc autem loco omitti necesse est. Primos nunc valores coëfficientis $'N_r^p$ tabula sequenti ante oculos ponamus:

p	$r=0$	$r=1$	$r=2$
1	1	$1!$	
2	1	$1!+1.{}^{n+4}\mathfrak{B}_1$	$2!$
3	1	$1!+2.{}^{n+5}\mathfrak{B}_1$	$2!+1.{}^{4}\mathfrak{B}_1+1.2.{}^{n+5}\mathfrak{B}_2$
4	1	$1!+3.{}^{n+6}\mathfrak{B}_1$	$2!+2.{}^{5}\mathfrak{B}_1+2.3.{}^{n+6}\mathfrak{B}_2$
5	1	$1!+4.{}^{n+7}\mathfrak{B}_1$	$2!+3.{}^{6}\mathfrak{B}_1+3.4.{}^{n+7}\mathfrak{B}_2$
6	1	$1!+5.{}^{n+8}\mathfrak{B}_1$	$2!+4.{}^{7}\mathfrak{B}_1+4.5.{}^{n+8}\mathfrak{B}_2$

p	$r=3$
3	$3!$
4	$3!+2!1.{}^{5}\mathfrak{B}_1+1!1.2[{}^{n+7}\mathfrak{B}_2-{}^{n+6}\mathfrak{B}_1.{}^{n+2}\mathfrak{B}_1]+1.2.3.{}^{n+6}\mathfrak{B}_3$
5	$3!+2!2.{}^{6}\mathfrak{B}_1+1!2.3[{}^{n+8}\mathfrak{B}_2-{}^{n+7}\mathfrak{B}_1.{}^{n+2}\mathfrak{B}_1]+2.3.4.{}^{n+7}\mathfrak{B}_3$
6	$3!+2!3.{}^{7}\mathfrak{B}_1+1!3.4[{}^{n+9}\mathfrak{B}_2-{}^{n+8}\mathfrak{B}_1.{}^{n+2}\mathfrak{B}_1]+3.4.5.{}^{n+8}\mathfrak{B}_3$

p	$r=4$
4	$4!$
5	$4!+3!1\,{}^{6}\mathfrak{B}_1+2!1.2.{}^{6}\mathfrak{B}_1+1!1.2.3[{}^{n+8}\mathfrak{B}_3-{}^{n+7}\mathfrak{B}_2.{}^{n+2}\mathfrak{B}_1]+1.2.3.4.{}^{n+7}\mathfrak{B}_4$
6	$4!+3!2\,{}^{7}\mathfrak{B}_1+3!2.3.{}^{7}\mathfrak{B}_1+1!2.3.4[{}^{n+9}\mathfrak{B}_3-{}^{n+8}\mathfrak{B}_3.{}^{n+2}\mathfrak{B}_1]+2.3.4.5.{}^{n+8}\mathfrak{B}_4$

etc. etc.

§. 6.

Jam restat ut fractioni continuae ($c.$) in productum infinitae multitudinis factorum studeamus transformandae. Habemus autem secundum virum cl. *Stern,* si fractionem aequationis ($g.$) solam per $'fx_p$ denotamus,

$$1-fx = 'fx_1 . \frac{'fx_2}{'fx_1} . \frac{'fx_3}{'fx_2} . \frac{'fx_4}{'fx_3} \ldots . \frac{'fx_p}{'fx_{p-1}} \ldots .$$

quare id agamus necesse est, ut expressionem universalem factoris $p$ti $\frac{'fx_p}{'fx_{p-1}}$ enucleemus. Quem ad finem ponamus $'fx_p = \frac{V_p}{X_p}$, et singulos fractionis continuae ($e.$) numeratores et denominatores particulares $= a_1\, a_2\, a_3 \ldots .$ et resp. $b_1\, b_2\, b_3 \ldots .,$ quo facto habemus

$$q. \quad \frac{V_p}{X_p} = \frac{V_{p-1}\, b_p - V_{p-2}\, a_p}{X_{p-1}\, b_p - X_{p-2}\, a_p} \text{ nec non}$$

$$\frac{'fx_p}{'fx_{p-1}} = \frac{X_{p-1}\, V_p}{V_{p-1}\, X_p} = \frac{X_{p-1}\, V_{p-1}\, b_p - X_{p-1}\, V_{p-2}\, a_p}{X_{p-1}\, V_{p-1}\, b_p - X_{p-2}\, V_{p-1}\, a_p}.$$

Valorem autem numeratoris $X_{p-1}\, V_{p-1}\, b_p - X_{p-1}\, V_{p-2}\, a_p$ ita quoque repraesentare licet, ut sit:

$$X_{p-1}\, V_{p-1}\, b_p - X_{p-2}\, V_{p-1}\, a_p - (X_{p-1}\, V_{p-2} - X_{p-2}\, V_{p-1})\, a_p,$$

unde, cum constet esse $-(X_{p-1}\, V_{p-2} - X_{p-2}\, V_{p-1}) = a_1\, a_2\, a_3 \ldots . a_{p-1},$ expressionem numeratoris illius deducimus sequentem

$$V_{p-1}\, X_p + a_1\, a_2\, a_3\, a_4 \ldots . a_{p-1} . a_p,$$

ex qua valor fractionis admodum elegans

$$r. \quad \frac{'fx_p}{'fx_{p-1}} = \frac{V_{p-1}\, X_p + a_1\, a_2\, a_3\, a_4 \ldots . a_p}{V_{p-1}\, X_p} = 1 + \frac{a_1\, a_2\, a_3\, a_4 \ldots . a_p}{V_{p-1}\, X_p}$$

descendit. Substitutione igitur perpetrata valoris $'fx_p$ in ($g.$) propositi, producti infiniti schema resultat sequens:

$$s. \quad fx = 1 - \frac{(n+1)x}{1+(n+2)x}\left[1 + \frac{1.(n+1)(n+2)x^2}{N_1^1[1+2(n+3)x+(n+3)(n+2)x^2]}\right]$$
$$\times\left[1 + \frac{1.2(n+1)(n+2)(n+3)x^4}{[N_1^2+N_2^2 x][1+3(n+4)x+3(n+4)(n+3)x^2+(n+4)(n+3)(n+2)x^3]}\right]$$
$$\times\left[1 + \frac{1.2.3(n+1)(n+2)(n+3)(n+4)x^6}{[N_1^3+N_2^3 x+N_3^3 x^2] \times [1+4(n+5)x+6(n+5)(n+4)x^2+4(n+5)(n+4)(n+3)x^3+(n+5)(n+4)(n+3)(n+2)x^4]}\right]$$
$$\times \text{ etc.},$$

cujus factores celerrime ad limitem 1 convergunt. Pari modo si valorem functionis fx_p in aequatione ($k.$) propositum adhibemus, aliud hoc nanciscimur producti infiniti schema:

$$t.\quad fx = \left[1 - \frac{(n+1)x}{1+(n+2)x}\right]\left[1 - \frac{1!(n+1)(n+2)x^3}{['N_0^1+'N_1^1x][1+2(n+3)x+(n+3)(n+2)x^2]}\right]$$
$$\times\left[1 - \frac{2!(n+1)(n+2)(n+3)x^5}{['N_0^2+'N_1^2x+'N_2^2x^2][1+3(n+4)x+3(n+4)(n+3)x^2+(n+4)(n+3)(n+2)x^3]}\right]$$
$\times$ etc.

Posito numeratore in aequatione $(k.) = 'V_p$ atque aequatione $(s.)$ per $(t.)$ divisa, proficiscitur:

$$\frac{1}{fx} - 1 = \frac{V_1}{'V_1} \cdot \frac{V_2 \cdot 'V_1}{V_1 \cdot 'V_2} \cdot \frac{V_3 \cdot 'V_2}{V_2 \cdot 'V_3} \cdots \frac{V_p \cdot 'V_{p-1}}{V_{p-1} \cdot 'V_p} \cdots$$

Porro, cum sit $'V_p = X_p - V_p$, habemus,

$$\frac{V_p \cdot 'V_{p-1}}{V_{p-1} \cdot 'V_p} = \frac{V_p X_{p-1} - V_p V_{p-1}}{V_{p-1} X_p - V_p V_{p-1}}$$
$$= \frac{V_{p-1} X_p - V_p V_{p-1} + (p-1)!(n+1)(n+2)\ldots(n+p)x^{2p-1}}{V_{p-1} X_p - V_p V_{p-1}}$$
$$= 1 + \frac{(p-1)!(n+1)(n+2)\ldots(n+p)x^{2p-1}}{'V_p \cdot V_{p-1}},$$

unde accipimus:

$$u.\quad \frac{1}{fx} = 1 + \frac{(n+1)x}{'V_1} \cdot \left(1 + \frac{1!(n+1)(n+2)x^3}{V_1 \cdot 'V_2}\right)\left(1 + \frac{2!(n+1)(n+2)(n+3)x^5}{V_2 \cdot 'V_3}\right)\cdots$$

Simili modo divisione inverse, nimirum $(t.)$ per $(s.)$ perpetrata, ad productum pervenimus hoc:

$$v.\quad \frac{1}{1-fx} = 1 - \frac{1+x}{V_1}\left(1 - \frac{1!(n+1)(n+2)x^3}{V_2 \cdot 'V_1}\right)\left(1 - \frac{2!(n+1)(n+2)(n+3)x^3}{V_3 \cdot 'V_2}\right)\cdots,$$

quas tamen evolutiones, cum rei nostrae alienae videantur, hoc loco omisimus.

§. 7.

Jam relabamur ad logarithmos integrales, quos posito $n=0$ atque $x=(lx)^{-1}$ invenimus $\mathrm{li}\frac{1}{x} = -\frac{1}{x \cdot lx} \cdot fx$, ideoque

$$34.\quad \mathrm{li}\frac{1}{x} = -\frac{1}{x}\left[\cfrac{1}{2x + \cfrac{lx}{lx + \cfrac{1}{1 + \cfrac{2}{lx + \cfrac{2}{1 + \cdots}}}}}\right] = -\frac{1}{x \cdot lx}\left[1 - \cfrac{1}{2 + lx - \cfrac{1 \cdot 2}{4 + lx - \cfrac{2 \cdot 3}{6 + lx - \cfrac{3 \cdot 4}{8 + lx - \cdots}}}}\right],$$

$$35.\quad \mathrm{li}\frac{1}{x} = -\frac{1}{x}\left[\frac{1}{lx+1} + \frac{1!\,1!}{[lx+1][(lx)^2+4lx+2]} + \frac{2!\,2!}{[(lx)^2+4lx+2][(lx)^3+9(lx)^2+18lx+6]} + \cdots\right].$$

Fractiones approximatas nunc habemus sequentes:

$$36.\quad \begin{cases} -\frac{1}{x.lx}\left(1-\frac{1}{lx+2}\right)=-\frac{1}{x.lx}\cdot\frac{lx+1}{lx+2}, \\ -\frac{1}{x.lx}\left(1-\frac{lx+4}{(lx)^2+6lx+6}\right)=-\frac{1}{x.lx}\cdot\frac{(lx)^2+5lx+2}{(lx)^2+6lx+6}, \\ -\frac{1}{x.lx}\left(1-\frac{(lx)^2+10lx+18}{(lx)^3+12(lx)^2+36lx+24}\right) \\ \qquad =-\frac{1}{x.lx}\cdot\frac{(lx)^3+11(lx)^2+26lx+6}{(lx)^3+12(lx)^2+36lx+24}, \\ -\frac{1}{x.lx}\left(1-\frac{(lx)^3+18(lx)^2+86lx+96}{(lx)^4+20(lx)^3+120(lx)^2+240lx+120}\right) \\ \qquad =-\frac{1}{x.lx}\cdot\frac{(lx)^4+19(lx)^3+102(lx)^2+154lx+24}{(lx)^4+20(lx)^3+120(lx)^2+240lx+120}, \\ -\frac{1}{x.lx}\left(1-\frac{(lx)^4+28(lx)^3+246(lx)^2+756lx+600}{(lx)^5+30(lx)^4+300(lx)^3+1200(lx)^2+1800lx+720}\right) \\ \qquad =-\frac{1}{x.lx}\cdot\frac{(lx)^5+29(lx)^4+272(lx)^3+954(lx)^2+1044lx+120}{(lx)^5+30(lx)^4+300(lx)^3+1200(lx)^2+1800lx+720}, \\ \text{etc.} \end{cases}$$

Denique nanciscimur producta infinita

$$37.\quad \text{li}\frac{1}{x}=-\frac{1}{x.lx}+\frac{1}{x.lx}\left(\frac{1}{lx+2}\right)\left(1+\frac{1!2!}{(lx)^2+6lx+6}\right)$$
$$\times\left(1+\frac{2!3!}{[lx+4][(lx)^3+12(lx)^2+36lx+24]}\right)\ldots.$$
$$=-\frac{1}{x.lx}\left(1-\frac{1}{lx+2}\right)\left(\frac{1!2!}{[lx+1][(lx)^2+6lx+6]}\right)$$
$$\times\left(1-\frac{2!3!}{[(lx)^2+5lx+2][(lx)^3+12(lx)^2+36lx+24]}\right)\ldots.$$

Quibus omnibus aequationibus ad li $\frac{1}{x}$ computationem satis commode utaris, si argumenti x valor admodum accreverit. Sed hanc ipsam formularum concinnitatem valde augere licet, ubi retentis omnibus seriei (12.) terminis, qui unitatem non superant, transformationem in antecedenti §pho doctam illis tantum terminis adhibueris, qui seriem reddunt divergentem. Quo facto evenit exempli gratia:

$$38.\quad \text{li}\frac{1}{x}=-\frac{1}{x.lx}\left[1-\frac{1!}{lx}+\frac{2!}{(lx)^2}-\frac{3!}{(lx)^3}+-\ldots\pm\frac{(n-1)!}{(lx)^{n-1}}\right]$$
$$\pm\frac{n!}{x(lx)^n}\left[\frac{1}{lx+(n+1)}+\frac{1!(n+1)}{[lx+(n+1)][(lx^2)+2(n+2)lx+(n+2)(n+1)]}+\ldots\right]$$
$$=-\frac{1}{x.lx}\left[1-\frac{1!}{lx}+\frac{2!}{(lx)^2}-\frac{3!}{(lx)^3}+-\ldots\pm\frac{(n-1)!}{(lx)^{n-1}}\right]\pm\frac{n!}{x(lx)^{n+1}}\cdot\frac{'V_p}{X_p}$$

etc. etc.

Facili negotio in his aequationibus pro quove variabilis x valore quantitatem n ita determinabis, ut minimum solummodo numerum dignitatum lx computandum habeas, qui logarithmi integralis valorem satis exactum praebeat.

§. 8.

Aequationes adhuc enucleatae quamvis aperte sint utilissimae, eo tamen incommodo laborant, quod singulorum numerorum logarithmi integrales singuli sunt computandi. Pro valoribus vero argumenti non magno discrimine diversis multo minor computationis erit labor, si logarithmum integralem alium ex alio componere vel si valorem $\text{li}(a+x)$ ex valore $\text{li}\, a$ derivare licet; cui rei constituendae haece inserviunt disquisitiones.

Si quae functio argumenti $a+x$ secundum dignitates functionis alius cujusdam argumenti x, quae tamen pro $x=0$ necesse est et ipsa evanescat, debet evolvi, optime theorema *Soldneri* elegantissimum adhibendum erit. Habes enim:

$$39. \quad F(a+x) = Fa + \frac{u}{1}A' + \frac{u^2}{2!}A'' + \frac{u^3}{3!}A''' + \dots,$$

qua in aequatione u functionem quamcunque argumenti x et formae $f(a+x)-fa$ indicat, coëfficientes vero A valores induunt hos

$$A' = \frac{\partial . Fa}{\partial . fa}, \quad A'' = \frac{\partial . A'}{\partial . fa}, \quad A''' = \frac{\partial . A''}{\partial . fa} \text{ etc.}$$

Quo in theoremate adhibendo id praecipue agendum est, ut functio u eligatur talis, ut coëfficientes A quam simplicissimi prodeant. Quod facillime ita effici posse videtur, ut $u = a+x-a =$ ponatur, quo facto theorema nostrum ad notum illud Taylorianum reducitur, nimirum posito $Fa = \text{li}\, a$,

$$40. \quad \text{li}(a+x) = \text{li}\, a + \frac{x}{1}.\frac{1}{la} + \frac{x^2}{2!}\frac{\partial \frac{1}{la}}{\partial a} + \frac{x^3}{3!}.\frac{\partial^2 \frac{1}{la}}{\partial a^2} + \frac{x^4}{4!}.\frac{\partial^3 \frac{1}{la}}{\partial a^3} + \dots.$$

Jam vero habes

$$41. \quad \frac{\partial^n \frac{1}{la}}{\partial a^n} = (-1)^n . \frac{1}{a^n}\left[\frac{n!\,{}^n\mathfrak{F}_0}{(la)^{n+1}} + \frac{(n-1)!\,{}^n\mathfrak{F}_1}{(la)^n} + \frac{(n-2)!\,{}^n\mathfrak{F}_2}{(la)^{n-1}} + \dots \right.$$
$$\left. \dots + \frac{2!\,{}^n\mathfrak{F}_{n-2}}{(la)^3} + \frac{1!\,{}^n\mathfrak{F}_{n-1}}{(la)^2}\right],$$

quem valorem si in (40.) substituas, solutis parenthesibus terminisque illis, qui easdem dignitates quantitatis la continent, in unum collectis, accipis

$$\mathrm{li}(a+x) = \mathrm{li}\,a + \frac{x}{la} - \frac{1!\,a}{(la)^2}\left(\frac{{}^1\mathfrak{F}_0}{2!}\cdot\frac{x^2}{a^2} - \frac{{}^2\mathfrak{F}_1}{3!}\cdot\frac{x^3}{a^3} + \frac{{}^3\mathfrak{F}_2}{4!}\cdot\frac{x^4}{a^4} - + \ldots\right)$$
$$+ \frac{2!\,a}{(la)^3}\left(\frac{{}^2\mathfrak{F}_0}{3!}\cdot\frac{x^3}{a^3} - \frac{{}^3\mathfrak{F}_1}{4!}\cdot\frac{x^4}{a^4} + \frac{{}^4\mathfrak{F}_2}{5!}\cdot\frac{x^5}{a^5} - + \ldots\right)$$
$$- \text{etc.}$$

et si pro seriebus in parenthesi inclusis earum valores ex aequatione (18.) substituas et brevitatis causa $1+\frac{x}{a} = u$ ponas,

42. $$\mathrm{li}(a+x) = \mathrm{li}\,au = \mathrm{li}\,a + au\left[\frac{lu}{la}C_1^{-lu} - \left(\frac{lu}{la}\right)^2 C_2^{-lu} + \left(\frac{lu}{la}\right)^3 C_3^{-lu} - + \ldots\right]$$

quae quidem series pro magnis valoribus quantitatis a cito convergit, et dummodo coëfficientes $(lu)^n C_n^{-lu}$ computaveris, facili negotio omni a adhiberi poterit. Tamen hanc seriem, cum sit maxima ad computandos logarithmos int. argumenti au utilitate, non sine magno labore ad computandos functionis valores pro numeris illis, qui a proxime insequuntur, adhibueris quoniam, mutato x seu u, omnes quoque coëfficientes $(lu)^n C_n^{-lu}$ denuo essent computandi. Aequatio (40.) huic rei efficiendae optime inserviret, cum ejus coëfficientes unius quantitatis a sint functiones; sed coëfficientes ipsi tardius comminuuntur ita ut via ulla gauderes sublevatione. Cujus rei importunitati ut mederetur vir cl. *Buzengeiger* hoc theorema integrale

43. $$\int f(a+x)\,\partial x =$$
$$\int fa.\partial a + x[(1+q'+q''+\ldots+q^{\mathrm{N}})fa - q'f(a+p'x) - q''f(a+p''x) - \ldots - q^{\mathrm{N}}f(a+p^{\mathrm{N}}x)]$$
$$+ \frac{x^{n-2}}{(n+1)!}\cdot\frac{\partial^{n+1}fa}{\partial a^{n+1}}\left(\frac{1}{n+2} + q'.p'^{n+1} + q''.p''^{n+1} + q'''.p'''^{n+1} + \ldots + q^{\mathrm{N}}.p^{\mathrm{N}\,n+1}\right)$$
$$+ \frac{x^{n+3}}{(n+2)!}\cdot\frac{\partial^{n+2}fa}{\partial a^{n+2}}\left(\frac{1}{n+3} + q'.p'^{n+2} + q''.p''^{n+2} + q'''.p'''^{n+2} + \ldots + q^{\mathrm{N}}.p^{\mathrm{N}\,n+2}\right)$$
$$+ \text{etc.}$$

proposuit, in quo quantitates p', p'', ... p^{N} prorsus ex arbitrio, quantitates autem q', q'', ... q^{N} ex illis ita definiuntur, ut n primi seriei terminorum evanescant. Prodeunt igitur, si combinationes $w$tae classis ex ν elementis, nulla permissa elementorum repetitione, significes per ${}^{\nu}\mathfrak{C}_w$, aequationes sequentes:

$$q' = -\frac{\frac{1}{2}{}^{N-1}\mathfrak{C}_{N-1} - \frac{1}{3}{}^{N-1}\mathfrak{C}_{N-2} + \frac{1}{4}{}^{N-1}\mathfrak{C}_{N-3} - \dots \pm \frac{1}{n+1}}{p'(p^{N}-p')(p^{N-1}-p)\dots(p'''-p')(p''-p')}, \text{ (elem. } p'', p''', p^{IV}, \dots p^{N});$$

$$q'' = -\frac{\frac{1}{2}{}^{N-1}\mathfrak{C}_{N-1} - \frac{1}{3}{}^{N-1}\mathfrak{C}_{N-2} + \frac{1}{4}{}^{N-1}\mathfrak{C}_{N-3} - \dots \pm \frac{1}{n+1}}{p''(p^{N}-p'')(p^{N}-p'')\dots(p'''-p'')(p'-p'')}, \text{ (elem. } p', p''', p^{IV}, \dots p^{N}),$$

. . . .

$$q^{N} = -\frac{\frac{1}{2}{}^{N-1}\mathfrak{C}_{N-1} - \frac{1}{3}{}^{N-1}\mathfrak{C}_{N-2} + \frac{1}{4}{}^{N-1}\mathfrak{C}_{N-3} - \dots \pm \frac{1}{n+1}}{p^{N}(p^{N-1}-p^{N})(p^{N-2}-p^{N})\dots(p''-p^{N})(p'-p^{N})}, \text{ (elem. } p', p'', p''', \dots p^{N-1}).$$

Jam sponte apparet, quatenus hoc theorema cum aliis quadraturae methodis congruat. Si adeo N quantitates p ita determinare velles, ut terminorum in (43.) adhuc reliquorum etiam N primi evanescerent, pro quantitatibus p et q valores illi resultarent a viro cl. *Gaufs* determinati. Qui cum alias magna cum utilitate adhibeantur, in nostro casu non satis magnum praestarent commodum, quia functiones $q^r f(a+p^r x) = q^r[l(a+p^r x)]^{-1}$ magnam tum in computando facerent importunitatem. Licet vero eundem assequi finem ita ut quantitates p', p'', ... plane ex arbitrio determines, dummodo in (43.) quantitates q', q'', ... ita constituas, ut ab $r$to demum seriei termino N termini evanescant. Quo facto proficiscitur aequatio

$$44. \quad \int f(a+x)\partial x =$$

$$\int fa.\partial a + x[(1+q'+q''+\dots+q^{N})fa - q'f(a+p'x) - q''f(a+p''x) - \dots - q^{N}f(a+p^{N}x)]$$
$$+ \frac{x^2}{1!}\cdot\frac{\partial fa}{\partial a}\left(\tfrac{1}{2} + q'.p' + q''.p'' + \dots + q^{N}.p^{N}\right)$$
$$+ \frac{x^3}{2!}\cdot\frac{\partial^2 fa}{\partial a^2}\left(\tfrac{1}{3} + q'.p'^2 + q''.p''^2 + \dots + q^{N}.p^{N}{}^2\right)$$
$$+ \dots\dots\dots\dots\dots$$
$$+ \frac{x^{r+1}}{r!}\cdot\frac{\partial^r fa}{\partial a^r}\left(\frac{1}{r+1} + q'.p'^r + q''.p''^r + \dots + q^{N}.q^{N\,r}\right)$$
$$+ \frac{x^{n+r+1}}{(n+r+1)!}\cdot\frac{\partial^{n+r+1} fa}{\partial a^{n+r+1}}\left(\frac{1}{n+r+2} + q'.p'^{n+r+1} + \dots + q^{N}.p^{N\,n+r+1}\right)$$
$$+ \text{ etc.},$$

nec non valores quantitatum q', q'', ... qui sequuntur:

$$q' = -\frac{\frac{1}{r+2}\,{}^{N-1}\mathfrak{C}_{N-1} - \frac{1}{r+3}\,{}^{N-1}\mathfrak{C}_{N-2} + \dots \pm \frac{1}{n+r+1}}{p'^{r+1}(p^N - p')(p^{N-1}-p')\dots(p'''-p')(p''-p')}, \quad (\text{elem. } p'', p''', \dots p^N),$$

$$q'' = -\frac{\frac{1}{r+2}\,{}^{N-1}\mathfrak{C}_{N-1} - \frac{1}{r+3}\,{}^{N-1}\mathfrak{C}_{N-2} + \dots \pm \frac{1}{n+r+1}}{p''^{r+1}(p^N - p'')(p^{N-1}-p'')\dots(p'''-p'')(p'-p'')}, \quad (\text{elem. } p', p''', \dots p^N),$$

. . . .

$$q^N = -\frac{\frac{1}{r+2}\,{}^{N-1}\mathfrak{C}_{N-1} - \frac{1}{r+3}\,{}^{N-1}\mathfrak{C}_{N-2} + \dots \pm \frac{1}{n+r+1}}{p^{N\,r+2}(p^{N-1} - p^N)(p^{N-2}-p^N)\dots(p''-p^N)(p'-p^N)}, \quad (\text{elem. } p', p'', \dots p^{N-1}).$$

Si a non inferius est 100 neque x superat 5, sufficit ut $r=3$ et $N=2$ ponas, ut logarithmos integrales argumenti $a+x$ usque ad 7 locos decimales accurate computes. Quamquam ita etiam satis operosa restat computatio, ut in alias inquiramus formulas necesse est.

§. 9.

Quodsi series omnes pro logarithmo integrali nobis propositas inter se conferimus, neminem fugere potest paene omnes progredi secundum dignitates logarithmorum, ipsasque illas, quae ab initio secundum dignitates ipsius argumenti evolutae erant, in alias posse mutari, quae secundum dignitates illius logarithmi procederent; quae quidem res aperte docet, logarithmum integralem per theorema viri cl. *Soldner* simplicissime evolvi, posito $fa = la$. Habes ita, cum sit $l(a+x) - la = l\left(1+\frac{x}{a}\right) = lu$,

$$45. \quad \mathrm{li}(a+x) = \mathrm{li}\,au = \mathrm{li}\,a + a\left[\frac{lu}{1}\cdot\frac{1}{la} + \frac{(lu)^2}{2!}\cdot\frac{la-1}{(la)^2} + \frac{(lu)^3}{3!}\cdot\frac{(la)^2-2la+1}{(la)^3} + \dots\right].$$

Coëfficientes terminorum $(lu)^n$ cum solius quantitatis a sint functiones, iis semel determinatis, ex hac formula logarithmi integrales totius seriei numerorum a subsequentium computari possunt neque requiritur, ut parvi tantum quantitatis x valores admittantur. Etenim cum adeo $x=a$ ponatur, $lu = l2$ satis exiguum habes, ut series ipsa cito convergat. Jam cum per formulam recurrentem

$$\frac{1-(n-1)A_{n-1}}{la} = A_n$$

coëfficientes posterior quique ex antecedenti derivari possint, seriem (45.) si adhibeas multum praebet commoditatis. Quare vir cl. *Soldner* seriem hanc non transformatam sed talem, qualem modo exhibuimus, ad computandam tabulam suam adhibuit. Licet vero multo augere ejus utilitatem, dispositis nimirum coëfficientibus A, et terminis inde ortis alium ad ordinem inter se junctis. Prodit enim ex (45.)

$$46. \quad \text{li}\,au = \text{li}\,a + a\left\{\begin{array}{l} \frac{lu}{la} - \frac{1}{2}\left(\frac{lu}{la}\right)^2 + \frac{1}{3}\left(\frac{lu}{la}\right)^3 - \frac{1}{4}\left(\frac{lu}{la}\right)^4 + \dots \\ + \frac{(lu)^2}{1!2la} - \frac{(lu)^3}{1!3(la)^2} + \frac{(lu)^4}{1!4(la)^3} - \frac{(lu)^5}{1!5(la)^4} + \dots \\ + \frac{(lu)^3}{2!3la} - \frac{(lu)^4}{2!4(la)^2} + \frac{(lu)^5}{2!5(la)^3} - \frac{(lu)^6}{2!6(la)^4} + \dots \\ + \text{etc.} \end{array}\right\}.$$

Collectis nunc terminis in serie horizontali collocatis, accipis aequationem:

$$\text{li}(au) = \text{li}\,a + a\left\{l\left(1+\frac{lu}{la}\right) - \frac{la}{1!}\left[l\left(1+\frac{lu}{la}\right) - \frac{lu}{la}\right]\right.$$
$$\left. - \frac{(la)^2}{2!}\left[l\left(1+\frac{lu}{la}\right) - \frac{lu}{la} + \frac{1}{2}\left(\frac{lu}{la}\right)^2\right] - \dots\right\},$$

quae solutis denuo parenthesibus, singulisque terminis rite conjunctis, abit in hanc

$$47. \quad \text{li}\,au = \text{li}\,a + l\left(\frac{lau}{la}\right) + a\left(\frac{lu}{1!}C_1^{-la} + \frac{(lu)^2}{2!}C_2^{-la} + \frac{(lu)^3}{3!}C_3^{-la} + \dots\right).$$

Rursus igitur habemus series C^z, nec non ad speciem quam maxime commodam conformatas, cum z hic sit negativum ideoque opposita sese excipiant singulorum terminorum signa. Eandem aequationem (47.) adhibitis formulis (23.) et (24.) immediate ex formula (45.) derivare licet, cum pateat esse

$$A_n = C_n^{la} \pm \frac{(n-1)!}{(la)^n}.$$

Contractis in (46.) terminis, linea perpendiculari sibi suppositis, relabimur ad seriem (42.), quam nunc ex *Taylori* et *Soldneri* theorematibus eandem oriri videmus. Solutis iterum in aequatione (42.) seriebus C^{lu} terminisque, qui aequales argumenti lu dignitates continent, inter se junctis, nova prodit formula:

$$48. \quad \text{li}\,au = \text{li}\,a + au\left[l\left(\frac{lau}{la}\right) - lu H_1 + (lu)^2 H_2 - (lu)^3 H_3 + \dots\right]$$

$$H_{n-1} = \frac{1}{1.2.3\dots n}\cdot\frac{lu}{la} - \frac{1}{2.3\dots(n+1)}\cdot\left(\frac{lu}{la}\right)^2 + \frac{1}{3.4\dots(n+2)}\left(\frac{lu}{la}\right)^3 - + \dots$$

Coëfficientes H ex quantitatibus X §phi tertiae hos sibi habent valores:

$$H_0 = l(1+x)$$
$$H_1 = \frac{1+x}{1.x}H_0 - \frac{1}{1.1!},$$
$$H_2 = \frac{1+x}{2.x}H_1 - \frac{1}{2.2!},$$
$$\dots\dots\dots\dots$$
$$H_n = \frac{1+x}{n.x}H_{n-1} - \frac{1}{n.n!} \quad \text{etc.}$$

unde accipimus et hos:

$$H_n = \frac{1}{(n+1)!}\cdot\frac{x}{1+x} + \frac{(n+1)x}{1+x} H_{n+1}, \quad H_{n+1} = \frac{1}{(n+2)!}\cdot\frac{x}{1+x} + \frac{(n+2)x}{1+x} H_{n+2} \text{ etc.}$$

Quos valores si in perpetuum sibi substituis, cum residuum

$(n+1)(n+2)\dots(n+r)\left(\frac{x}{1+x}\right)^r H_{n+r}$ pro $r=\infty$ plane evanescat, coëfficientis H_n valor prodit sequens:

$$49. \quad H_n = \frac{1}{n!}\left[\frac{1}{n+1}\cdot\frac{x}{1+x} + \frac{1}{n+2}\left(\frac{x}{1+x}\right)^2 + \frac{1}{n+3}\left(\frac{x}{1+x}\right)^3 + \dots\right] = \frac{1}{n!} K_n.$$

Habemus autem casu nostro $\frac{x}{1+x} = \frac{lu}{lau}$, qua de causa aequatio (48.) abit in hanc:

$$50. \quad \text{li}\, au = \text{li}\, a + au\left[l\left(\frac{lau}{la}\right) - \frac{lu}{1!}K_1 + \frac{(lu)^2}{2!}K_2 - \frac{(lu)^3}{3!}K_3 + \dots\right],$$

quae nonnunquam bene adhiberi poterit.

Denique hoc loco notare convenit, ut omnes formulae in §pho 8. et 9. propositae alia etiam simplicissima via investigari possint. Est enim

$$\text{li}\, au = \int\frac{a\,\partial u}{lau} = \int\frac{a\,\partial u}{la+lu} = a\int\partial u\left[\frac{1}{la} - \frac{lu}{(la)^2} + \frac{(lu)^2}{(la)^3} - \frac{(lu)^3}{(lu)^4} + \dots\right],$$

unde, cum sit

$$\int\partial u\,(lu)^n = u[(lu)^n - n(lu)^{n-1} + n(n-1)(lu)^{n-2} - \dots \pm n(n-1)\dots 3.2\,lu \mp n!]$$
$$= G_{n+1}^{lu}(lu)^{n+1} \pm n!,$$

aequationem habes:

$$\text{li}\, au = c + a\left[G_1^{lu}\frac{lu}{la} - G_2^{lu}\left(\frac{lu}{la}\right)^2 + G_3^{lu}\left(\frac{lu}{la}\right)^3 - \dots\right]$$
$$+ a\left[\frac{1}{la} + \frac{1!}{(la)^2} + \frac{2!}{(la)^3} + \frac{3!}{(la)^4} + \dots\right],$$

ex qua posito $u=1$, valorem constantis

$$c = \text{li}\, a - a\left[\frac{1}{la} + \frac{1!}{(la)^2} + \frac{2!}{(la)^3} + \dots\right]$$

accipis, qui suo loco substitutus aequationem nostram

$$51. \quad \text{li}\, au = \text{li}\, a + a\left[\frac{lu}{la}G_1^{lu} - \left(\frac{lu}{la}\right)^2 G_2^{lu} + \left(\frac{lu}{la}\right) G_3^{lu} - + \dots\right.$$

eandem praebet, quam supra in (42.) jam invenimus, ita ut evolutiones functionis $\text{li}(a+x)$ ex theoremate *Tayloriano* vel *Soldneriano* omnino non sint necessariae.

§. 10.

Formulam illam (42.), quam triplici modo investigatam semper eandem invenimus, in tota hac theoria esse fundamentalem neminem fugere potest, ita ut in alias novasque aequationes frustra inquirere videamur. Jam formula haec et illae in (47.) et (50.) propositae, quae ex ea descendunt, facultatem nobis praebent, logarithmum integralem pro quovis argumenti a valore satis commode computandi. Maximam autem utilitatem hae aequationes logarithmis integralibus dignitatum argumenti computandis suppeditant. Posito enim $a = a^n$ et $u = a$, prodeunt ex (42.) et (50.) aequationes

$$52.\quad \mathrm{li}\, a^{n+1} = \mathrm{li}\, a^n + a^{n+1}\left[\frac{1}{n} C_1^{-la} - \frac{1}{n^2} C_2^{-la} + \frac{1}{n^3} C_3^{-la} - \frac{1}{n^4} C_4^{-la} + \dots\right],$$

$$53.\quad \mathrm{li}\, a^{n+1} = \mathrm{li}\, a^n + a^{n+1}\left[l(n+1) - \frac{la}{1!} K'_1 + \frac{(la)^2}{2!} K'_2 - \frac{(la)^3}{3!} K'_3 + \dots\right],$$

$$K'_p = \frac{1}{p+1}\cdot\frac{1}{n+1} + \frac{1}{p+2}\left(\frac{1}{n+1}\right)^2 + \frac{1}{p+3}\left(\frac{1}{n+1}\right)^3 + \dots,$$

quarum ope aut logarithmos integrales dignitatum ejusdem radicis successive ascendentium, aut logarithmos integrales earundem dignitatum diversarum radicum commode computare poteris. Ex (52.) successive substituendo et haec prodit formula:

$$\begin{aligned}54.\quad \mathrm{li}\, a^{n+1} = \mathrm{li}\, a^{n-r} &+ C_1^{-la}\left[\frac{a^{n+1}}{n} + \frac{a^n}{n-1} + \frac{a^{n-1}}{n-2} + \dots + \frac{a^{n+1-r}}{n-r}\right] \\ &- C_2^{-la}\left[\frac{a^{n+1}}{n^2} + \frac{a^n}{(n-1)^2} + \frac{a^{n-1}}{(n-2)^2} + \dots + \frac{a^{n+1-r}}{(n-r)^2}\right] \\ &+ C_3^{-la}\left[\frac{a^{n+1}}{n^3} + \frac{a^n}{(n-1)^3} + \frac{a^{n-1}}{(n-2)^3} + \dots + \frac{a^{n+1-r}}{(n-r)^3}\right] \\ &- \text{etc.}\end{aligned}$$

ex qua licet ascendere a logarithmo integrali inferioris dignitatis argumenti a, omissis mediis terminis ad eum summae dignitatis argumenti; quae formula cum celerrime convergat omnibus iis satis commoda esse videatur, qui non nimia flagitant. Aequationem paucis admodum exemplis utilem series (47.) nobis praeberet, coëfficientibus C pro $\alpha = a^n$ tardissime decrescentibus. Ut vero logarithmos integrales argumenti $\frac{a}{u}$ compararet haec series aptissima est, quippe quae, lu posito negativo, aequationem

$$55.\quad \mathrm{li}\frac{a}{u} = \mathrm{li}\, a + l\left(\frac{la - lu}{la}\right) - a\left[\frac{lu}{1!} C_1^{-la} - \frac{(lu)^2}{2!} C_2^{-la} + \frac{(lu)^3}{3!} C_3^{-la} - + \dots\right]$$

suppeditat. Quodsi idem mutaveris in (42.) resultat:

$$56.\quad \mathrm{li}\frac{a}{u} = \mathrm{li}\, a - \frac{a}{u}\left[\frac{lu}{la} C_1^{lu} + \left(\frac{lu}{la}\right)^2 C_2^{lu} + \left(\frac{lu}{la}\right)^3 C_3^{lu} + \dots\right],$$

quae quidem eadem est series, quam vir cl. *Bessel*, quamvis aliam prorsus viam ingressus, nactus est et ex qua duas formulas enucleavit, aequationibus (52.) et (54.) simillimas. Quodsi vero ad usum spectes, nostras quas modo exposuimus formulas utilitate sua *Besselianas* superare putaverimus, cum opposita in nostris et terminorum et coëfficientium C sese excipiant signa, ideoque posito $a>1$ magis quam *Besselianae* convergant. Posito autem $a<1$ formulae nunc *Besselianae* antehabendae fuerint, cum iis adhibitis certe in coëfficientibus C opposita se signa excipiant. Pari denique modo ex serie (55.) transmutata novam habes aequationem sequentem:

$$57.\quad \operatorname{li}\frac{1}{au}=\operatorname{li}\frac{1}{a}+l\left(1+\frac{lu}{la}\right)-\frac{1}{a}\left[\frac{lu}{1!}C_1^{la}-\frac{(lu)^2}{2!}C_2^{la}+\frac{(lu)^3}{3!}C_3^{la}-+\ldots\right].$$

Quae aequationes, rite inter se junctae, ad tabulam logarithmorum integralium conscribendam optime sufficiunt, cum quantitates a et u valorem quemvis vel integrum vel fractum habere possint. Jam uti in logarithmorum naturalium computatione sufficit, ut logarithmos solummodo numerorum primorum investiges, ita ad tabulam logarithmorum integralium exstruendam id unum necesse est, ut valores coëfficientium $C_n^{\pm lu}$ pro $u=2, \frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{11}{10}, \frac{13}{12}, \frac{17}{16}, \frac{19}{18}, \frac{23}{22}$ etc. computes, quippe quorum ope totam functionis nostrae tabulam condere licet. His enim coëfficientibus indagatis aequatio (29.) logarithmos integrales suppeditat argumenti $2, \frac{3}{2}, \frac{5}{4}$ etc. nec non argumenti $\frac{1}{2}, \frac{2}{3}$ etc., qui cum totius tabulae fundamentum sint, ut ad computationem verificandam eos ex aequatione (2.) iterum evolvas operae, cum paene nulla sit, pretium magnopere facias. Jam vero ex (52.) seu (54.) logarithmos integrales dignitatis $n$tae omnium horum numerorum nancisceris, quos verificationis gratia iterum ex (53.) quaeras. Denique si logarithmos integrales computas numerorum 3, 4, 5, 6, 7, 8, 9, 10 eorumque dignitatum priorum, nunc tabulae schema habes, quod facili opera explere potes; nec non perpetuo providetur, ne errorem in computando committas.

§. 11.

Coronidis loco aequationes quaedam afferantur, quae ex substitutione $a=e$ seu $la=le=1$ proficiscuntur. Etenim est

$$58.\quad \operatorname{li}e^{\pm n}=\gamma+ln\pm\frac{n}{1.1!}+\frac{n^2}{2.2!}\pm\frac{n^3}{3.3!}+\frac{n^4}{4.4!}\pm \text{etc.}$$

$$=\gamma+ln\pm n[C_2^{\pm n}+\tfrac{1}{2}C_3^{\pm n}+\tfrac{1}{3}C_4^{\pm n}+\ldots],$$

59. $$\mathrm{li}\, e^{n+1} = \mathrm{li}\, e^n + e^{n+1}\left[\frac{1}{n}C_1^{-1} - \frac{1}{n^2}C_2^{-1} + \frac{1}{n^3}C_3^{-1} - + \ldots\right]$$
$$= \mathrm{li}\, e^n + e^{n+1}\left[l(1+n) - \frac{1}{1!}K_1' + \frac{1}{2!}K_2' - \frac{1}{3!}K_3' + \ldots\right],$$

60. $$\begin{cases} \mathrm{li}(e^n+x) = \mathrm{li}\, e^n + l\left[\frac{l(e^n+x)}{n}\right] + e^n\left[\frac{lv}{1!}C_1^{-n} + \frac{(lv)^2}{2!}C_2^{-n} + \frac{(lv)^3}{3!}C_3^{-n} + \ldots\right], \\ lv = l\left(1+\frac{x}{e^n}\right), \\ \mathrm{li}\, e^n u = \mathrm{li}\, e^n + e^n u\left[\frac{lu}{n}C_1^{-lu} - \left(\frac{lu}{n}\right)^2 C_2^{-lu} + \left(\frac{lu}{n}\right)^3 C_3^{-lu} - \ldots\right], \end{cases}$$

61. $$\mathrm{li}\frac{1}{e^n} = -\frac{1}{e^n}\left[\frac{1}{n+1} + \frac{1!1!}{(n+1)(n^2+4n+2)} + \frac{2!2!}{(n^2+4n+2)(n^3+9n^2+18n+6)} + \ldots\right]$$
$$= -\frac{1}{n.e^n} + \frac{1}{n.e^n}\left[\frac{1}{n+2}\right]\left[1+\frac{1!2!}{n^2+6n+6}\right]\left[1+\frac{2!3!}{(n+4)(n^3+12n^2+36n+24)}\right]\ldots$$
$$= -\frac{1}{n.e^n}\left[1-\frac{1}{n+2}\right]\left[1-\frac{1!2!}{(n+1)(n^2+6n+6)}\right]\left[1-\frac{2!3!}{(n^2+5n+2)(n^3+12n^2+36n+24)}\right]\ldots$$

Posito $n = 1$ valores inde resultant particulares:

62. $$\mathrm{li}\frac{1}{e} = -\frac{1}{e}\left[\frac{1}{2} + \frac{1!1!}{2.7} + \frac{2!2!}{7.34} + \frac{3!3!}{34.209} + \frac{4!4!}{209.1564} + \frac{5!5!}{1564.13327} + \frac{6!6!}{13327.130922} + \frac{7!7!}{130922.1441729} + \ldots\right]$$
$$= -\frac{1}{e} + \frac{1}{e}.\frac{1}{3}\left(1+\frac{1!2!}{1.13}\right)\left(1+\frac{2!3!}{5.73}\right)\left(1+\frac{3!4!}{29.501}\right)\left(1+\frac{4!5!}{201.1051}\right) \times \left(1+\frac{5!6!}{1631.37633}\right)\ldots$$
$$= -\frac{1}{e}\left(1-\frac{1}{3}\right)\left(1-\frac{1!2!}{2.13}\right)\left(1-\frac{2!3!}{8.73}\right)\left(1-\frac{3!4!}{44.501}\right)\left(1-\frac{4!5!}{300.4051}\right) \times \left(1-\frac{5!6!}{2420.37633}\right)\ldots$$

Fractionum subsidio approximatarum e fractione continua ($a.$) §phi quartae evolutarum haec quoque aequatio proficiscitur:

63. $$\mathrm{li}\frac{1}{e} = -\frac{1}{e}.\frac{1}{2}.\frac{4}{3}.\frac{6}{7}.\frac{14}{13}.\frac{65}{68}.\frac{374}{365}.\frac{2263}{2299}.\frac{15675}{15531}.\frac{115230}{117300}\ldots$$

Jam sine ullo labore lex, qua divisores factorum singulorum progrediuntur, ex formulis in §pho 6. et 7. propositis recurrentibus evolvi potest. Est enim ex. gr. $73 = 7.13 - 2.3.3$; $501 = 9.73 - 3.4.13$; $4051 = 11.501 - 4.5.73$. Pari modo habes $44 = 7.8 - 2.3.2$; $300 = 9.44 - 3.4.8$ etc. Valorem $\mathrm{li}\frac{1}{e}$ ipsum nunc sequentem enucleavimus:

$$\mathrm{li}\,\frac{1}{e} = -0{,}21938\ 39343\ 95520\ 27366\ 5\ldots,$$

cui addimus hunc correspondentem:

$$\mathrm{li}\,e = \quad 1{,}89511\ 78163\ 55936\ 75547\ 8\ldots,$$

Quibus omnibus praeparatis id solummodo nobis agendum videtur, ut tabulam functionis nostrae satis expeditam exstruamus, quod opus autem operosissimum, quamvis ejus fundamenta jam sint jacta, in aliud tempus nobis differendum est otiosius. Tunc vero non solum hanc tabulam sed theoriam etiam et tabulas aliarum quarundam functionum, cum logarithmo integrali arcte junctarum, exhibeamus.

Scripsi Gothae, die 13mo Oct. 1835.
