13.

Aequationes modulares pro transformatione Functionum Ellipticarum.

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Integrale $\int_0^y \frac{\partial y}{\sqrt{(1-y^2)}.\sqrt{(1-\lambda^2 y^2)}}$ substitutione adhibita:

$$y = \frac{x(a+a'x^2+a''x^4+\ldots+a^{(m)}x^{2m})}{1+b'x^2+b''x^4+\ldots+b^{(m)}x^{2m}}$$

in eius simile $\int_0^x \frac{\partial x}{\sqrt{(1-x^2)}.\sqrt{(1-\varkappa^2 x^2)}}$ transformari potest, quae transformatio $n^{ti}=(2m+1)^{ti}$ ordinis nominatur. Relationes analyticas inter y et x, λ et $\varkappa$ intercedentes Cl. *Jacobi* in libro suo: *Fundamenta nova* etc. docuit. Ibi pag. 39 invenitur:

$$\sqrt[4]{\lambda} = \sqrt[4]{\varkappa^n}.[\sin\operatorname{coam}4\omega.\sin\operatorname{coam}8\omega\ldots\sin\operatorname{coam}2(n-1)\omega],$$

ubi significant:

$$\omega=\frac{mK+m'iK'}{n},\quad K=\int_0^{\frac{\pi}{2}}\frac{\partial\varphi}{\sqrt{(1-\varkappa\varkappa\sin\varphi^2)}},\quad K'=\int_0^{\frac{\pi}{2}}\frac{\partial\varphi}{\sqrt{(1-\varkappa'\varkappa'\sin\varphi^2)}},$$
$$\varkappa\varkappa+\varkappa'\varkappa'=1.$$

Numerum n, qui ordinem transformationis indicat in sequentibus numerum primum ponamus, quem ad casum notum est reliquos revocari posse.

In fractione ω quantitatibus m et m' omnes valores et positivos et negativos, ipso n minores, tribui licet, verum l. c. pag. 49 demonstratur, omnes transformationes diversas, quae exstare possint, evasuras esse, si pro ω elegerimus valores hos:

$$\frac{K}{n},\quad \frac{iK'}{n},\quad \frac{K+iK'}{n},\quad \frac{K+2iK'}{n},\quad \frac{K+3iK'}{n},\quad \ldots\quad \frac{K+(n-1)iK'}{n},$$

qui in formula ipsius λ pro ω positi $(n+1)$ inter se diversos valores moduli λ per $\varkappa$ expressos praebent, unde elucet aequationis algebraicae inter λ et $\varkappa$, quae aequatio modularis dicitur, gradum esse $(n+1)$. Pro transformatione et tertii et quinti ordinis aequationem ejusdem gradus adeo inter radices quadraticas modulorum locum habere Cl. *Jacobi* docuit, nec non pro septimo ordine Dr. *Guetzlaff* (in hujus Diarii Tom. XII. p. 173): idem valere pro transformatione cujusvis ordinis ex sequentibus patet.

§. 1.

Formula laudata:

$$\sqrt[4]{\lambda} = \sqrt[4]{\varkappa^n}.[\sin\operatorname{coam}4\omega.\sin\operatorname{coam}8\omega\ldots\ldots\sin\operatorname{coam}2(n-1)\omega],$$

cum sit: $\sin\operatorname{coam}8p\omega = \sin\operatorname{coam}(4n\omega - 8p\omega) = \sin\operatorname{coam}4(n-2p)\omega$, ubi $(n-2p)$ est numerus impar, quoniam ponitur n numerus primus, etiam scribi potest:

$$\sqrt[4]{\lambda} = \sqrt[4]{\varkappa^n}.[\sin\operatorname{coam}4\omega.\sin\operatorname{coam}12\omega\ldots\ldots\sin\operatorname{coam}4(n-2)\omega],$$

qua in expressione alia multipla ipsius 4ω non inveniuntur, nisi imparia. Quorum sinus coamplitudinis tanquam functiones ipsius $\sin\operatorname{coam}4\omega$ repraesentari posse satis notum est, ita ut, denotante $f[\sin\operatorname{coam}4\omega]$ functionem ipsius $\sin\operatorname{coam}4\omega$ rationalem, efficiatur:

$$\sqrt[4]{\lambda} = \sqrt[4]{\varkappa^n}.f[\sin\operatorname{coam}4\omega], \quad \text{aut posito:} \quad \sqrt[4]{\lambda} = v, \quad \sqrt[4]{\varkappa} = u,$$

$$v = u^n.f[\sin\operatorname{coam}4\omega].$$

Haec vero in formula loco ω, valore ipsius v eodem manente, scribi posse ω, 2ω, 3ω, ... $\frac{n-1}{2}\omega$ facile intelligitur, unde prodit:

$$\frac{v}{u^n} = f[\sin\operatorname{coam}4\omega] = f[\sin\operatorname{coam}8\omega]\ \ldots\ldots = f[\sin\operatorname{coam}2(n-1)\omega],$$

vel etiam:

$$\frac{v^p}{u^{np}} = \{f[\sin\operatorname{coam}4\omega]\}^p = \{f[\sin\operatorname{coam}8\omega]\}^p\ldots\ldots = \{f[\sin\operatorname{coam}2(n-1)\omega]\}^p$$

$$= \frac{2}{n-1}.\Sigma\{f[\sin\operatorname{coam}4m\omega]\}^p$$

si numero m omnes valores 1, 2, 3, ... $\frac{n-1}{2}$ tribuuntur.

Habemus ergo singulos valores ipsius v sequentes:

$$v_1 = u^n.\left[\sin\operatorname{coam}\frac{4K}{n}.\sin\operatorname{coam}\frac{8K}{n}\ldots\ldots\sin\operatorname{coam}\frac{2(n-1)K}{n}\right] = u^n f\left[\sin\operatorname{coam}\frac{4K}{n}\right],$$

$$v_2 = u^n.\left[\sin\operatorname{coam}\frac{4iK'}{n}.\sin\operatorname{coam}\frac{8iK'}{n}\ldots\ldots\sin\operatorname{coam}\frac{2(n-1)iK'}{n}\right] = u^n.f\left[\sin\operatorname{coam}\frac{4iK'}{n}\right],$$

$$v_3 = u^n.\left[\sin\operatorname{coam}\frac{4K+4iK'}{n}.\sin\operatorname{coam}\frac{8K+8iK'}{n}\ldots\ldots\sin\operatorname{coam}\frac{2(n-1)K+2(n-1)iK'}{n}\right]$$

$$= u^n.f\left[\sin\operatorname{coam}\frac{4K+4iK'}{n}\right],$$

$$v_4 = u^n.\left[\sin\operatorname{coam}\frac{4K+8iK'}{n}.\sin\operatorname{coam}\frac{8K+16iK'}{n}\ldots\ldots\sin\operatorname{coam}\frac{2(n-1)K+4(n-1)iK'}{n}\right]$$

$$= u^n.f\left[\sin\operatorname{coam}\frac{4K+8iK'}{n}\right],$$

. . . .

$$v_{n+1} = u^n.\left[\sin\operatorname{coam}\frac{4K+4(n-1)iK'}{n}.\sin\operatorname{coam}\frac{8K+8(n-1)iK'}{n}\ldots\ldots\sin\operatorname{coam}\frac{2(n-1)K+2(n-1)^2iK'}{n}\right]$$

$$= u^n.f\left[\sin\operatorname{coam}\frac{4K+4(n-1)iK'}{n}\right]$$

aut:

$$\frac{n-1}{2}\cdot\frac{\nu_1^p}{u^{np}} = \Sigma\left\{f\left[\sin\operatorname{coam}\frac{4mK}{n}\right]\right\}^p,$$

$$\frac{n-1}{2}\cdot\frac{\nu_2^p}{u^{np}} = \Sigma\left\{f\left[\sin\operatorname{coam}\frac{4miK'}{n}\right]\right\}^p,$$

$$\frac{n-1}{2}\cdot\frac{\nu_3^p}{u^{np}} = \Sigma\left\{f\left[\sin\operatorname{coam}\frac{4m(K+iK')}{n}\right]\right\}^p,$$

$$\frac{n-1}{2}\cdot\frac{\nu_4^p}{u^{np}} = \Sigma\left\{f\left[\sin\operatorname{coam}\frac{4m(K+2iK')}{n}\right]\right\}^p,$$

. . . .

$$\frac{n-1}{2}\cdot\frac{\nu_{n+1}^p}{u^{np}} = \Sigma\left\{f\left[\sin\operatorname{coam}\frac{4m[K+(n-1)iK']}{n}\right]\right\}^p,$$

unde additione facta eruitur:

$$\frac{n-1}{2}\cdot\frac{\nu_1^p+\nu_2^p+\nu_3^p+\ldots+\nu_{n+1}^p}{u^{np}}$$

$$= \Sigma\left\{f\left[\sin\operatorname{coam}\frac{4m(K+m'iK')}{n}\right]\right\}^p + \Sigma\left\{f\left[\sin\operatorname{coam}\frac{4miK'}{n}\right]\right\}^p,$$

ubi numero m valores 1, 2, 3, ... $\frac{n-1}{2}$, numero m' valores 0, 1, 2, 3, ... $n-1$ conveniunt. Verumtamen, cum $\sin\operatorname{coam}(u+2iK') = \sin\operatorname{coam} u$ sit, formula ita transformari potest ut factor ipsius iK' ipso $2n$ minor fiat, quo obtinetur:

$$\frac{u-1}{2}\cdot\frac{\nu_1^p+\nu_2^p+\nu_3^p+\ldots+\nu_{n+1}^p}{u^{np}} = \Sigma\left\{f\left[\sin\operatorname{coam}\frac{4mK+4m'iK'}{n}\right]\right\}^p,$$

si pro m ponuntur 0, 1, 2, 3, ... $\frac{n-1}{2}$, pro m' 0, ±1, ±2, ±3, ... $\pm\frac{n-1}{2}$, ita tamen ut quoties $m=0$, et ipsi m' valores tantum positivi 1, 2, 3, ... $\frac{n-1}{2}$ tribuantur. Ut hanc formulam eandem esse ac praecedentem eluceat, animadverto et hanc et illam ex $\frac{nn-1}{2}$ terminis inter se diversis constare neque numerum m diversos quam in priori formula valores assumere.

Nunc demonstremus functiones symmetricas ipsius $\sin\operatorname{coam}\frac{4mK+4m'iK'}{n}$ esse functiones rationales ipsius $\varkappa$.

In auxilium vocamus ex *Fund.* pag. 42 formulam:

$$1-\lambda\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) =$$

$$\frac{(1-\varkappa.\sin\operatorname{am}u).[(1-\varkappa.\sin\operatorname{am}u.\sin\operatorname{coam}4\omega)(1-\varkappa.\sin\operatorname{am}u.\sin\operatorname{coam}8\omega)\ldots(1-\varkappa.\sin\operatorname{am}u.\sin\operatorname{coam}2(n-1)\omega)]^2}{(1-\varkappa^2.\sin^2\operatorname{am}u.\sin^2\operatorname{am}4\omega)(1-\varkappa^2.\sin^2\operatorname{am}u.\sin^2\operatorname{am}8\omega)\ldots(1-\varkappa^2.\sin^2\operatorname{am}u.\sin^2\operatorname{am}2(n-1)\omega)},$$

vel cum sit: $\frac{(1-\varkappa.\sin\mathrm{am}\,u.\sin\mathrm{coam}\,\alpha)^2}{1-\varkappa^2.\sin^2\mathrm{am}\,u.\sin^2\mathrm{am}\,\alpha} = \frac{(1-\varkappa.\sin\mathrm{am}(u+\alpha))(1-\varkappa.\sin\mathrm{am}(u-\alpha))}{\triangle^2\mathrm{am}\,\alpha}$:

$$(A)\qquad 1-\lambda\sin\mathrm{am}\left(\frac{u}{M},\lambda\right) =$$
$$\frac{(1-\varkappa.\sin\mathrm{am}\,u)(1-\varkappa.\sin\mathrm{am}(u+4\omega))(1-\varkappa.\sin\mathrm{am}(u+8\omega))\dots(1-\varkappa.\sin\mathrm{am}(u+4(n-1)\omega))}{\triangle^2\mathrm{am}\,4\omega.\triangle^2\mathrm{am}\,8\omega\dots\triangle^2\mathrm{am}\,2(n-1)\omega},$$

In hac aequatione poito $\omega = \frac{iK'}{n}$ ex notatione $\S^i$ 24. *Fund.* producitur:

$$1-\lambda_1\sin\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) =$$
$$\frac{[1-\varkappa.\sin\mathrm{am}\,u]\left[1-\varkappa.\sin\mathrm{am}\left(u+\frac{4iK'}{n}\right)\right]\left[1-\varkappa.\sin\mathrm{am}\left(u+\frac{8iK'}{n}\right)\right]\dots\left[1-\varkappa.\sin\mathrm{am}\left(u+\frac{4(n-1)iK'}{n}\right)\right]}{\triangle^2\mathrm{am}\frac{4iK'}{n}.\triangle^2\mathrm{am}\frac{8iK'}{n}\dots\triangle^2\mathrm{am}\frac{2(n-1)iK'}{n}}.$$

Mutando $\varkappa$ in λ, quo facto K' in Λ', λ_1 in $\varkappa$, M_1 in M' transit, et posito $\frac{u}{M}$ loco u, unde $\frac{u}{MM'} = nu$ loco $\frac{u}{M_1}$, obtinemus:

$$(B.)\qquad 1-\varkappa.\sin\mathrm{am}(nu,\varkappa) =$$
$$\frac{\left[1-\lambda.\sin\mathrm{am}\frac{u}{M}\right]\left[1-\lambda.\sin\mathrm{am}\left(\frac{u}{M}+\frac{4i\Lambda'}{n}\right)\right]\left[1-\lambda.\sin\mathrm{am}\left(\frac{u}{M}+\frac{8i\Lambda'}{n}\right)\right]\dots\left[1-\lambda.\sin\mathrm{am}\left(\frac{u}{M}+\frac{4(n-1)i\Lambda'}{n}\right)\right]}{\triangle^2\mathrm{am}\frac{4i\Lambda'}{n}.\triangle^2\mathrm{am}\frac{8i\Lambda'}{n}\dots\triangle^2\mathrm{am}\frac{2(n-1)i\Lambda'}{n}},$$

in cujus aequationis parte dextra Modulus λ valet.

Si porro in aequatione $(A.)$ loco u ponimus $u+\frac{4m'iK'}{n}$, unde $\frac{u}{M}$ transit in $\frac{u}{M}+\frac{4m'iK'}{nM} = \frac{u}{M}+\frac{4m'i\Lambda'}{n}$, posito $\omega = \frac{K}{n}$, prodit:

$$1-\lambda.\sin\mathrm{am}\left(\frac{u}{M}+\frac{4m'i\Lambda}{n}\right) = \frac{\Pi\left[1-\varkappa\ \sin\mathrm{am}\left(u+\frac{4mK+4m'iK'}{n}\right)\right]}{\left[\triangle\mathrm{am}\frac{4K}{n}.\triangle\mathrm{am}\frac{8K}{n}\dots\triangle\mathrm{am}\frac{2(n-1)K}{n}\right]^2}.$$

si ipsi m valores $0, 1, 2, 3, \dots n-1$ tribuuntur; jam ubi hac in aequatione ipsi m' tribuantur valores $0, 1, 2, 3, \dots n-1$, facto producto secundum aequationem $(B.)$ obtinemus:

$$1-\varkappa.\sin\mathrm{am}(nu,\varkappa) =$$
$$\frac{\Pi\left[1-\varkappa.\sin\mathrm{am}\left(u+\frac{4mK+4m'iK'}{n}\right)\right]}{\left[\triangle\mathrm{am}\frac{4K}{n}.\triangle\mathrm{am}\frac{8K}{n}\dots\triangle\mathrm{am}\frac{2(n-1)K}{n}\right]^2\left[\triangle\mathrm{am}\frac{4i\Lambda'}{n}.\triangle\mathrm{am}\frac{8i\Lambda'}{n}\dots\triangle\mathrm{am}\frac{2(n-1)i\Lambda'}{n}\right]}$$

ubi utrisque m, m' valores $0, 1, 2, 3, \dots n-1$ conveniunt.

Quae formula facile etiam in hanc formam redigitur:

$$1-\varkappa.\sin\mathrm{am}(nu,\varkappa)=(1-\varkappa.\sin\mathrm{am}\,u)\Pi\frac{\left[1-\varkappa.\sin\mathrm{am}\,u.\sin\mathrm{coam}\frac{4mK+4m'iK'}{n}\right]^2}{1-\varkappa^2.\sin^2\mathrm{am}\,u.\sin^2\mathrm{am}\frac{4mK+4m'iK'}{n}}$$

si numero m positivi tantum valores $0, 1, 2, 3, \ldots \frac{n-1}{2}$, numero m' valores $0, \pm 1, \pm 2, \pm 3, \ldots \pm\frac{n-1}{2}$ tribuuntur, ita vero ut quoties $m=0$, et ipsi m' valores tantum positivi $1, 2, 3, \ldots \frac{n-1}{2}$ conveniant.

Haec formula vero pro $n=4p+1$ modo valet. Nam pro $n=4p-1$, cum valor ipsius M sit negativus, ita ut $\frac{u}{MM'}=-nu$ fiat, signum ipsius u in contrarium mutemus necesse est, unde sub hac conditione habemus:

$$1-\varkappa.\sin\mathrm{am}(nu,\varkappa)=(1+\varkappa.\sin\mathrm{am}\,u)\Pi\frac{\left[1+\varkappa.\sin\mathrm{am}\,u.\sin\mathrm{coam}\frac{4mK+m'iK'}{n}\right]^2}{1-\varkappa^2.\sin^2\mathrm{am}\,u.\sin^2\mathrm{am}\frac{4mK+4m'iK'}{n}}.$$

Ut eandem quantitatem algebraice determinemus, ponamus in aequatione 18. *Fund.* pag. 33, quae est:

$$[1-\varkappa.\sin\mathrm{am}(u+v)].[1-\varkappa.\sin\mathrm{am}(u-v)]=\frac{[\Delta\,\mathrm{am}\,v-\varkappa.\sin\mathrm{am}\,u.\cos\mathrm{am}\,v]^2}{1-\varkappa^2.\sin^2\mathrm{am}\,u.\sin^2\mathrm{am}\,v},$$

$u=2pu$ et $v=(2p-1)u$, quo invenitur:

$$[1-\varkappa.\sin\mathrm{am}(4p-1)u].[1-\varkappa.\sin\mathrm{am}\,u]$$
$$=\frac{[1-\varkappa.\sin\mathrm{am}\,2pu.\sin\mathrm{coam}(2p-1)u]^2.\Delta^2\,\mathrm{am}(2p-1)u}{1-\varkappa^2.\sin^2\mathrm{am}\,2pu.\sin^2\mathrm{am}(2p-1)u},$$

aut $\sin\mathrm{am}\,u$ per x significato et per X_m functione rationali et integra ipsius x dimensionis m^{tae}, cum notum sit fieri:

$$\sin\mathrm{am}\,2pu=\frac{\sqrt{(1-x^2)}.\sqrt{(1-\varkappa^2x^2)}\,X_{4p^2-3}}{X_{4p^2}},\qquad \sin\mathrm{am}(2p-1)u=\frac{X_{(2p-1)^2}}{X_{(2p-1)^2-1}},$$

$$\cos\mathrm{am}\,2pu=\frac{X_{4p^2}}{X_{4p^2}},\qquad \cos\mathrm{am}(2p-1)n=\frac{\sqrt{(1-x^2)}.X_{(2p-1)^2-1}}{X_{(2p-1)^2-1}},$$

$$\Delta\,\mathrm{am}\,2pu=\frac{X_{4p^2}}{X_{4p^2}},\qquad \Delta\,\mathrm{am}(2p-1)u=\frac{\sqrt{(1-\varkappa^2x^2)}.X_{(2p-1)^2-1}}{X_{(2p-1)^2-1}},$$

$$\sin\mathrm{coam}\,2pu=\frac{X_{4p^2}}{X_{4p^2}},\qquad \sin\mathrm{coam}(2p-1)u=\frac{\sqrt{(1-x^2)}.X_{(2p-1)^2-1}}{\sqrt{(1-\varkappa^2x^2)}.X_{(2p-1)^2-1}},$$

excutitur:

$$1-\varkappa.\sin\mathrm{am}(4p-1)u=\frac{(1+\varkappa x).[X_{8p^2-4p}]^2}{X_{16p^2-8p}};$$

nec minus ex aequatione 19. *Fund.* pag. 33, posito $u=(2p+1)u$,

$y = 2pu$ deducitur:

$$1 - \varkappa . \sin\operatorname{am}(4p+1)u = \frac{(1-\varkappa x).[X_{8p^2+4p}]^2}{X_{16p^2+8p}}.$$

Invenitur exempli gratia:

$$1 - \varkappa . \sin\operatorname{am} 3u = \frac{[1+\varkappa x].[1 - 2\varkappa x + 2\varkappa x^3 - \varkappa^2 x^4]^2}{X_8},$$

$$1 - \varkappa . \sin\operatorname{am} 5u = \frac{\begin{array}{l}[1-\varkappa x].[1 - 2\varkappa x - 4\varkappa^2 x^2 + 10\varkappa x^3 + 5\varkappa^2 x^4 - 4\varkappa(2+3\varkappa^2)x^5 \\ \quad - 4\varkappa^2(1-\varkappa^2)x^6 + 4\varkappa^3(3+2\varkappa^2)x^7 - 5\varkappa^4 x^8 - 10\varkappa^5 x^9 \\ \quad + 4\varkappa^4 x^{10} + 2\varkappa^5 x^{11} - \varkappa^6 x^{12}]^2\end{array}}{X_{24}},$$

$$1 - \varkappa . \sin\operatorname{am} 7u = \frac{\begin{array}{l}[1+\varkappa x]\,[1 - 4\varkappa x - 4\varkappa^2 x^2 + 4\varkappa(7+2\varkappa^2)x^3 - 14\varkappa^2 x^4 \\ \quad - 28\varkappa(2+3\varkappa^2)x^5 + 28\varkappa^2(1+4\varkappa^2)x^6 \\ \quad + 4\varkappa(8+51\varkappa^2+16\varkappa^4)x^7 - \varkappa^2(16+305\varkappa^2+144\varkappa^4)x^8 \\ \quad - 8\varkappa^3(16+25\varkappa^2+4\varkappa^4)x^9 + 8\varkappa^4(46+57\varkappa^2+8\varkappa^4)x^{10} \\ \quad + 56\varkappa^5(1+2\varkappa^2)x^{11} - 4\varkappa^4(56+161\varkappa^2+56\varkappa^4)x^{12} \\ \quad + 56\varkappa^6(2+\varkappa^2)x^{13} + 8\varkappa^4(8+57\varkappa^2+46\varkappa^4)x^{14} \\ \quad - 8\varkappa^5(4+25\varkappa^2+16\varkappa^4)x^{15} \\ \quad - \varkappa^6(144+305\varkappa^2+16\varkappa^4)x^{16} \\ \quad + 4\varkappa^7(16+51\varkappa^2+8\varkappa^4)x^{17} + 28\varkappa^8(4+\varkappa^2)x^{18} \\ \quad - 28\varkappa^9(3+2\varkappa^2)x^{19} - 14\varkappa^{10}x^{20} + 4\varkappa^9(2+7\varkappa^2)x^{21} \\ \quad - 4\varkappa^{10}x^{22} - 4\varkappa^{11}x^{23} + \varkappa^{12}x^{24}]^2\end{array}}{X_{48}}.$$

Coefficientes ipsius x in functionibus X obvii omnes sunt expressiones rationales integrae ipsius $\varkappa$, sicuti e theoria algebraica multiplicationis functionum ellipticarum constat. Unde etiam patet expressionem, quae in numeratore ad quadratum elata invenitur quamque supra invenimus $= \Pi\left[1 \pm \varkappa . \sin\operatorname{am} u . \sin\operatorname{coam}\frac{4mK+4m'iK'}{n}\right]$, esse functionem rationalem integram ipsius x et $\varkappa$, quare concludi potest functiones symmetricas quantitatum $\sin\operatorname{coam}\frac{4mK+4m'iK'}{n}$, ubi ipsi m et m' valores supra dicti tribuuntur, esse functiones rationales integras ipsius $\varkappa$.

§. 2.

Valores omnes ipsius y sive radices aequationis modularis analytice expressae, ut jam dictum est, impetrantur, si in expressione

$$y = u^n . [\sin\operatorname{coam} 4\omega . \sin\operatorname{coam} 8\omega \ldots . \sin\operatorname{coam} 2(n-1)\omega]$$

pro ω ponuntur valores:

$$\frac{K}{n},\ \frac{iK'}{n},\ \frac{K+iK'}{n},\ \frac{K+2iK'}{n},\ \ldots.\ \frac{K+(n-1)iK'}{n}.$$

Cum quibus illos valores, qui secundum Cl. *Jacobi* eruuntur, si in formula 7. *Fund.* pag. 89 quae est

$$(C.)\qquad u = \sqrt{2}.\sqrt[8]{q}.\left[\frac{(1+q^2)(1+q^4)(1+q^6)\ldots.}{(1+q)(1+q^3)(1+q^5)\ldots.}\right]$$

pro q valores: q^n, $q^{\frac{1}{n}}$, $\alpha q^{\frac{1}{n}}$, $\alpha^2 q^{\frac{1}{n}}$, ... $\alpha^{n-1} q^{\frac{1}{n}}$, in quibus α radicem quamlibet aequationis $x^n = 1$ significat, ponuntur congruere jam demonstremus.

α) Posito primum $\omega = \frac{K}{n}$, erit:

$$\nu_1 = u^n.\left[\sin\operatorname{coam}\frac{4K}{n}.\sin\operatorname{coam}\frac{8K}{n}\ldots.\sin\operatorname{coam}\frac{2(n-1)K}{n}\right]$$

quod idem esse ac si in aequatione ($C.$) q^n loco q ponimus sequenti modo patet.

Si in *Fund.* pag. 88 formulam cos. am. per Δ am. dividimus, habemus:

$$\sin\operatorname{coam}\frac{2Kx}{\pi}$$

$$= \frac{2}{u^2}\sqrt[4]{q}.\cos x.\frac{(1+2q^2\cos 2x+q^4)(1+2q^4\cos 2x+q^8)(1+2q^6\cos 2x+q^{12})\ldots.}{(1+2q\cos 2x+q^2)(1+2q^3\cos 2x+q^6)(1+2q^5\cos 2x+q^{10})\ldots.}$$

$$= \frac{2}{u^2}\sqrt[4]{q}.\cos x\,\Pi\frac{1+2q^{2r}\cos 2x+q^{4r}}{1+2q^{2r-1}\cos 2x+q^{4r-2}},$$

ergo:

$$\nu_1 = u^n.\frac{2^{\frac{n-1}{2}}}{u^{n-1}}.\sqrt[4]{q^{\frac{n-1}{2}}}.\cos\frac{2\pi}{n}.\cos\frac{4\pi}{n}.\cos\frac{6\pi}{n}\ldots.\cos\frac{n-1}{\pi}\pi\times$$

$$\Pi\frac{\left(1+2q^{2r}\cos\frac{4\pi}{n}+q^{4r}\right)\left(1+2q^{2r}\cos\frac{8\pi}{n}+q^{4r}\right)\ldots..\left(1+2q^{2r}\cos\frac{2(n-1)\pi}{n}+q^{4r}\right)}{\left(1+2q^{2r-1}\cos\frac{4\pi}{n}+q^{4r-2}\right)\left(1+2q^{2r-1}\cos\frac{8\pi}{n}+q^{r-2}\right)\ldots.\left(1+2q^{2r-1}\cos\frac{2(n-1)\pi}{n}+q^{4r-2}\right)}.$$

Cum vero sit secundum theorema Cotesianum:

$$\left(1+2x\cos\frac{4\pi}{n}+x^2\right)\left(1+2x\cos\frac{8\pi}{n}+x^2\right)\ldots.\left(1+2x\cos\frac{2(n-1)\pi}{n}+x^2\right) = \frac{1+x^n}{1+x}$$

et $\cos\frac{2\pi}{n}.\cos\frac{4\pi}{n}.\cos\frac{6\pi}{n}\ldots.\cos\frac{n-1}{n}\pi = \pm\frac{1}{2^{\frac{n-1}{2}}}$, prout n sit $8p\pm 1$ aut $8p\pm 3$, eruitur

$$\nu_1 = \pm u\sqrt[4]{q^{n-1}}\,\Pi\frac{(1+q^{2nr})(1+q^{2r-1})}{(1+q^{n(2r-1)})(1+q^{2r})}$$

ubi signum multiplicatorium sic intelligendum est ut pro r deinceps 1, 2, 3, ... ad infinitum usque ponatur. Quod si revera fit nec non pro u valor ex aequatione scribitur, prodit:

$$\nu_1 = \pm\sqrt{2}.\sqrt[8]{q^n}.\left[\frac{(1+q^{2n})(1+q^{4n})(1+q^{6n})\ldots.}{(1+q^n)(1+q^{3n})(1+q^{5n})\ldots.}\right],$$

ubi positivum signum valet si n est formae $8p\pm1$, negativum autem pro $n=8p\pm3$.

β) Reliquae radices omnes sub hac forma comprehenduntur:

$$(D.)\qquad v=u^n.\left[\sin\operatorname{coam}\frac{4mK+4m'iK'}{n}.\sin\operatorname{coam}\frac{8mK+8m'iK'}{n}\dots\right.$$
$$\left.\dots\sin\operatorname{coam}\frac{2(n-1)mK+2(n-1)m'iK'}{n}\right]$$

si m' aliquem numerorum 1, 2, 3, ... $(n-1)$ significat et m unitatem, hac vero conditione addita, ut pro $m'=1$ sit m ponendum tum $=1$, tum $=0$.

Applicata formula pro sin. coam., qua jam supra usi sumus, fit:

$$(E.)\qquad v=2^{\frac{n-1}{2}}.u.\sqrt[8]{q^{n-1}}.\Pi\left\{\cos\left(\frac{2mp\pi}{n}+\frac{2m'pi\pi K'}{nK}\right)\frac{\Pi\left[1+2q^{2r}.\cos\left(\frac{4mp\pi}{n}+\frac{4m'pi\pi K'}{nK}\right)+q^{4r}\right]}{\Pi\left[1+2q^{2r-1}.\cos\left(\frac{4mp\pi}{n}+\frac{4m'pi\pi K'}{nK}\right)+q^{4r-2}\right]}\right\}$$

qua in aequatione prius signum multiplicatorium indicat productum esse formandum ex omnibus terminis, qui eruuntur si pro p deinceps 1, 2, 3, ... $\frac{n-1}{n}$ ponuntur, altera vero signa ad r referuntur ita ut pro r sit ponendum 1, 2, 3, ... ∞.

Ad productum accuratius dilucidandum adnotabo formulas:

$$\cos x=\frac{e^{ix}+e^{-ix}}{2}=\tfrac{1}{2}e^{-ix}[1+e^{2ix}]$$

et cum sit $q=e^{-\frac{\pi K'}{K}}$,

$$\frac{1+q^{2r}.\cos 2x+q^{4r}}{1+q^{2r-1}.\cos 2x+q^{4r-2}}=\frac{\left[1+e^{-\frac{2r\pi K'}{K}+2ix}\right].\left[1+e^{-\frac{2r\pi K'}{K}-2ix}\right]}{\left[1+e^{-\frac{(2r-1)\pi K'}{K}+2ix}\right].\left[1+e^{-\frac{(2r-1)\pi K'}{K}-2ix}\right]};$$

quibus adhibitis invenitur:

$$v=2^{\frac{n-1}{2}}.u.\sqrt[8]{q^{n-1}}.\Pi\left\{e^{-\frac{2mpi\pi}{n}+\frac{2m'p\pi K'}{nK}}.\left(1+e^{\frac{4mpi\pi}{n}-\frac{4m'p\pi K'}{nK}}\right)\times\right.$$
$$\left.\frac{\Pi\left[1+e^{-\frac{2r\pi K'}{K}+\frac{4mpi\pi}{n}-\frac{4m'p\pi K'}{nK}}\right].\left[1+e^{-\frac{2r\pi K'}{K}-\frac{4mpi\pi}{n}+\frac{4m'p\pi K'}{nK}}\right]}{\Pi\left[1+e^{-\frac{(2r-1)\pi K'}{K}+\frac{4mpi\pi}{n}-\frac{4m'p\pi K'}{nK}}\right].\left[1+e^{-\frac{(2r-1)\pi K'}{K}-\frac{4mpi\pi}{n}+\frac{4m'p\pi K'}{K}}\right]}\right\}.$$

Jam numerum integrum s determinemus ejusmodi qui satisfaciat congruentiae $2m's\equiv m$ (mod. n) aut si placet aequationi $2m's=m+\beta n$, denotante β numerum minimum, qui aequationem explere possit, et ponamus $e^{\frac{2si\pi}{n}}=\alpha$, quod α est radix aliqua aequationis $x^n=1$; tum $e^{-\frac{2r\pi K'}{K}\pm\frac{4mpi\pi}{n}\mp\frac{4m'p\pi K'}{nK}}$ aequabit

$$= \left(e^{-\frac{\pi K'}{nK}}\right)^{2nr\pm 4m'p} \left(e^{\frac{2si\pi}{n}}\right)^{2nr\pm 4m'p} = \left(\alpha q^{\frac{1}{n}}\right)^{2nr\pm 4m'p}$$

quoniam est:

$$\left(e^{\frac{2si\pi}{n}}\right)^{2nr} = 1.$$

Ita permutata aequatio ($E.$) formam induit hanc:

$$(F.)\quad \nu = u.\sqrt[8]{q^{n-1}}\,\Pi\left\{\left(\alpha q^{\frac{1}{n}}\right)^{-2m'p}.\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{4m'p}\right].\frac{\Pi\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{2nr+4m'p}\right]\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{2nr-4m'p}\right]}{\Pi\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{(2r-1)n+4m'p}\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{(2r-1)n-4m'p}\right]}\right\}.$$

Satis vero notum est per expressionem $2nr \pm 4m'p$, ratione signi non habita, omnes numeros pares repraesentari, si pro r omnes numeri $0, 1, 2, 3, \ldots \infty$, pro p vero omnes numeri $0, 1, 2, 3, \ldots \frac{n-1}{2}$ ponantur, quum congruentia $\pm 4m'p \equiv 2\gamma$ (mod. n), ubi γ numerum quemlibet integrum significat, semper solvi possit, idque unico tantum modo si n est numerus primus et p valorem ipso $\frac{n-1}{2}$ majorem assumere nequit. Itaque in numeratore:

$$\Pi\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{2nr+4m'p}\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{2nr-4m'p}\right]$$

pares potestates omnes inveniuntur, exceptis tum illis, pro quibus r valorem zero assumturum esset, tum illis, quorum exponentes ipsius $2n$ forent multipla; priores autem factore, qui huic producto praefixus est, $\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{4'mp}\right]$ exhibentur et reliquos accipies si numeratorem factoris u ($C.$) uti licet in hunc modum scribis:

$$\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{2n}\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{4n}\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{6n}\right]\ldots.$$

Illos factores, qui potestates negativas continent, ergo hujus formae sunt $\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{2\gamma}\right]$, mutamus in $\left(\alpha q^{\frac{1}{n}}\right)^{-2\gamma}\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{2\gamma}\right]$.

Quae in numeratore cum ita sint atque in denominatore res plane similiter se habeat, valorem ipsius ν sequenti modo scribi licet:

$$\nu = \sqrt{2}.\sqrt[8]{q^n}.\left(\alpha q^{\frac{1}{n}}\right)^{-2m'.\frac{n^2-1}{8}}.\left(\alpha q^{\frac{1}{n}}\right)^{x}.\frac{\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{2}\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{4}\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{6}\right]\ldots.}{\left[1+\left(\alpha q^{\frac{1}{n}}\right)\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{3}\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^{5}\right]\ldots.};$$

per $\left(\alpha q^{\frac{1}{n}}\right)^{x}$ significatur productum omnium factorum, qui in numeratore et in denominatore ex potestatibus negativis nascuntur. Ipsum x functionem esse ipsius m' facile perspicitur cum x valores diversos assumat pro diversis radicibus ν aut quod ad idem redit pro diverso valore ipsius m'.

Ponamus ergo $x = \Phi(m')$, tum erit:

$$(G.)\quad \nu = \sqrt{2}.\sqrt[8]{\left(\left(\alpha q^{\frac{1}{n}}\right)^{n^2-2m'(n^2-1)+8.\varphi(m')}\right)}.\frac{\left[1+\left(\alpha q^{\frac{1}{n}}\right)^2\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^4\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^6\right]\ldots.}{\left[1+\left(\alpha q^{\frac{1}{n}}\right)\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^3\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^5\right]\ldots.}.$$

Valor ipsius ν, qualis per aequationem ($D.$) repraesentatur, quoniam immutatus remanet, si $m'+zn$, cujus z numerum quemlibet integrum denotat, in locum ipsius m' cedit, inde consequitur, ut expressio ipsius ν quoque talis, qualem aequatio ($G.$) praebet, permutationi non sit obnoxia ulli, ubi $m'+zn$ pro m' scribitur. Atque cum praeterea vel α in eadem ipsa positione nullam permutationem subeat, aequatio ($G.$) transit in hanc:

$$\nu = \sqrt{2}.\sqrt[8]{\left(\left(\alpha q^{\frac{1}{n}}\right)^{n^2-2zn(n^2-1)-2m'(n^2-1)+8.\varphi(m'+zn)}\right)}.\frac{\left[1+\left(\alpha q^{\frac{1}{n}}\right)^2\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^4\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^6\right]\ldots.}{\left[1+\left(\alpha q^{\frac{1}{n}}\right)\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^3\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^5\right]\ldots.}.$$

Ita quidem comparatione facta cum ($G.$) patet

$n^2-2zn(n^2-1)-2m'(n^2-1)+8.\Phi(m'+zn)$ aequare $= n^2-2m'(n^2-1)+8.\Phi(m')$

vel:

$$\Phi(m'+zn)-\Phi(m') = \frac{zn(n^2-1)}{4}$$

vel ex theoremate Tayloriano:

$$\frac{\partial\varphi(m')}{\partial m'}.zn+\frac{\partial^2\varphi(m')}{\partial m'^2}.\frac{z^2n^2}{1.2}+\ldots. = \frac{zn(n^2-1)}{4},$$

quod per zn divisum posito deinde $z=0$ praebet:

$$\frac{\partial\varphi(m')}{\partial m'} = \frac{n^2-1}{4}.$$

Integratione facta eruitur:

$$\Phi(m') = \frac{m'(n^2-1)}{4}+C.$$

Ut Constantem C determinemus, ponimus in ($G.$) $m'=1$, $m=0$, quo fit $\alpha=1$, tunc efficitur:

$$\nu_2 = \sqrt{2}.\sqrt[8]{\left(\left(q^{\frac{1}{n}}\right)^{-n^2+2+8.\varphi(1)}\right)}.\frac{\left[1+q^{\frac{2}{n}}\right].\left[1+q^{\frac{4}{n}}\right].\left[1+q^{\frac{6}{n}}\right]\ldots.}{\left[1+q^{\frac{1}{n}}\right].\left[1+q^{\frac{3}{n}}\right].\left[1+q^{\frac{5}{n}}\right]\ldots.},$$

qua in aequatione $\left(q^{\frac{1}{n}}\right)^{\varphi(1)}$ productum esse scimus, quod ex negativis potestatibus, quae insunt in aequatione ($F.$) conflatum est. In qua aequatione ($F.$), statuens m' esse $=1$, $m=0$, ubi factores elegeris, qui negativas in se contineant potestates, non nisi tum, ubi $r=1$ ponitur praetereaque in solo denominatore a fine incipienti tibi sese praebent hi:

$$\left[1+q^{-\frac{n-2}{n}}\right].\left[1+q^{-\frac{n-6}{n}}\right].\left[1+q^{-\frac{n-10}{n}}\right]\ldots.\begin{matrix}\left[1+q^{-\frac{3}{n}}\right] \text{ si } n=4m+1\\ \left[1+q^{-\frac{1}{n}}\right] \text{ si } n=4m-1\end{matrix}$$

quorum productum facile hanc formam induit:

$$q^{-\frac{n^2-1}{8n}}.\left[1+q^{\frac{n-2}{n}}\right].\left[1+q^{\frac{n-6}{n}}\right].\left[1+q^{\frac{n-10}{n}}\right]\ldots.\begin{matrix}\left[1+q^{\frac{3}{n}}\right]\\ \left[1+q^{\frac{1}{n}}\right].\end{matrix}$$

Factor $q^{-\frac{n^2-1}{8n}}$ ex denominatore in numeratorem translatus gignit:

$$q^{\frac{n^2-1}{8n}} = \left(q^{\frac{1}{n}}\right)^{\varphi(1)},$$

unde producitur:

$$\varphi(1) = \frac{n^2-1}{8};$$

ex quibus colligitur:

$$\varphi(m') = \frac{m'(n^2-1)}{4} - \frac{n^2-1}{8}.$$

Qui valor in aequatione $(G.)$ positus, efficit ut ea transeat in:

$$v = \sqrt{2}.\sqrt[8]{\left(\alpha q^{\frac{1}{n}}\right)}.\frac{\left[1+\left(\alpha q^{\frac{1}{n}}\right)^2\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^4\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^6\right]\ldots.}{\left[1+\left(\alpha q^{\frac{1}{n}}\right)\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^3\right].\left[1+\left(\alpha q^{\frac{1}{n}}\right)^5\right]\ldots.}.$$

Hinc videmus transformationes singulas quamvis in diverso ordine nos lucrari sive ponamus in valore

$$v = u^n.[\sin\operatorname{coam}4\omega.\sin\operatorname{coam}8\omega\ldots.\sin\operatorname{coam}2(n-1)\omega]$$

pro ω:

$$\frac{K}{n},\ \frac{iK'}{n},\ \frac{K+iK'}{n},\ \frac{K+2iK'}{n},\ \ldots.\ \frac{K+(n-1)iK'}{n},$$

sive in aequatione $(C.)$:

$$(C.)\qquad u = \sqrt{2}.\sqrt[8]{q}.\left[\frac{(1+q^2)(1+q^4)(1+q^6)\ldots.}{(1+q)(1+q^3)(1+q^5)\ldots.}\right]$$

pro q scribamus: $q^n,\ q^{\frac{1}{n}},\ \alpha q^{\frac{1}{n}},\ \alpha^2 q^{\frac{1}{n}},\ \ldots.\ \alpha^{n-1}q^{\frac{1}{n}}$.

Ut denique eluceat, ipsum α suos valores omnes accipere, si pro m' singuli numeri 1, 2, 3, ... $n-1$ ponantur, in memoriam revocare sufficiet a congruentia $2m's\equiv 1 \pmod{n}$ pro certo valore ipsius m' unam tantum solutionem, ipso n minorem admitti, ita ut posito $m'=1, 2, 3, \ldots$ $\ldots n-1$ pro s expromantur $n-1$ valores ipso n minores, quos omnes inter se diversos esse patet.

3.

Aequatio modularis, quam inter u et v revera locum habere et cujus gradum esse $n+1$ in §° secunda vidimus, pluribus conditionibus accuratius est determinata, quas nunc afferam.

a) Forma aequationis est haec:

$$v^{n+1}+C_1 v^n + C_2 v^{n-1} + C_3 v^{n-2} + \ldots + C_n v + C_{n+1} = 0,$$

in qua coefficientes C_1, C_2, C_3, ... sunt functiones rationales integrae ipsius u. Terminus constans C_{n+1}, uti ex theoria aequationum algebraicarum notum est, productum radicum exprimit, ita ut secundum §. 1. fiat

$$C_{n+1} = u^{n(n+1)}.\Pi\left[\sin\operatorname{coam}\frac{4mK+4m'iK'}{n}\right],$$

ubi m et m' valores loco citato exactius definitos assumunt. Si vero aequationem 8. per 9. pag. 66 in *Fund.* dividimus nec non radicem quadraticam extrahimus, habemus

$$\Pi\left[\sin\operatorname{coam}\frac{4mK+4m'iK'}{n}\right] = \pm\left(\frac{1}{\varkappa}\right)^{\frac{nn-1}{4}} = \pm\left(\frac{1}{u}\right)^{nn-1},$$

ergo:

$$C_{n+1} = \pm\left(\frac{1}{u}\right)^{nn-1}.u^{n(n+1)} = \pm u^{n+1}.$$

Ut de signo hujus termini certiores fiamus, forma omnium radicum, secunda excepta, haec est:

$$v = u^n.\left[\sin\operatorname{coam}\frac{4K+4m'iK'}{n}.\sin\operatorname{coam}\frac{8K+8m'iK'}{n}\ldots\right.$$
$$\left.\ldots\sin\operatorname{coam}\frac{2(n-1)K+2(n-1)m'iK'}{n}\right]$$

quae expressio, valoribus diversis ipsi m' tributis, signum suum minime mutat, cum $\sin\operatorname{coam} u$ quovis multiplo ipsius iK' ad argumentum u addito signum suum servet. Radices igitur omnes idem signum habeant necesse est, quod est ipsius

$$v_1 = u^n.\left[\sin\operatorname{coam}\frac{4K}{n}.\sin\operatorname{coam}\frac{8K}{n}\ldots\sin\operatorname{coam}\frac{2(n-1)K}{n}\right].$$

Cum vero sit $\sin\operatorname{coam}(u+K) = -\sin\operatorname{am} u$, in hoc producto illos factores, qui argumentum ipso K majus continent, a ceteris separemus, unde evadit pro $n=8r+1$ et $=8r-3$:

$$v_1 = (-1)^{\frac{n-1}{4}}.u^n.\left[\sin\operatorname{coam}\frac{4K}{n}.\sin\operatorname{coam}\frac{8K}{n}\ldots\sin\operatorname{coam}\frac{(n-1)K}{n}\right]\times$$
$$\left[\sin\operatorname{am}\frac{3K}{n}.\sin\operatorname{am}\frac{7K}{n}\ldots\sin\operatorname{am}\frac{(n-2)K}{n}\right];$$

pro $n = 8r - 1$ et $= 8r + 3$:

$$v_1 = (-1)^{\frac{n+1}{4}}.u.\left[\sin\text{coam}\frac{4K}{n}.\sin\text{coam}\frac{8K}{n}\dots\sin\text{coam}\frac{(n-3)K}{n}\right]\times$$
$$\left[\sin\text{am}\frac{K}{n}.\sin\text{am}\frac{5K}{n}\dots\sin\text{am}\frac{(n-2)K}{n}\right],$$

qua ex re manifestum est, radicem primam, ideoque reliquas omnes, secunda excepta habere positiva signa pro $n = 8m \pm 1$, negativa vero pro $n = 8m \pm 3$. Radix secunda semper est positiva. Productum ergo radicum omnium sive C_{n+1} erit positivum sive $= + u^{n+1}$, si n habeat formam $8r \pm 1$, et negativum sive $= -u^{n+1}$, si $n = 8r \pm 3$.

b) Aequationes modulares, commutatis inter se $\varkappa$ et λ, immutatas manere ex *Fund.* §. 29. notum est. Exinde idem valere si u et v inter se commutentur sponte sequitur, dummodo, quod sub sequenti litera facturi sumus, utrum v pro u ipso sit ponendum an pro u negativo, accuratiori examini subjiciamus.

Jam vero ex his concludi potest aequationes respectu tam ad v quam ad u ejusdem gradus esse ita ut coefficientes C_1, C_2, C_3, ... altiorem potestatem ipsius u, quam $(n+1)^{\text{tam}}$ continere nequeant, cum summa potestas ipsius v sit $(n+1)^{\text{ta}}$. Hi coefficientes habent adeo formam certam hanc: $u^m.(\alpha + \beta u^8 + \gamma u^{16} + \delta u^{24} + \dots)$, ubi in parenthesi illae tantum potestates ipsius u inveniuntur, quarum exponentes per 8 divisibiles sunt. Quod ut probetur ex *Fund.* pag. 89 formulam 7. in usum nostrum liceat convertere

$$\sqrt[4]{\varkappa} = u = \sqrt[8]{2}.\sqrt{q}.\left[\frac{(1+q^2)(1+q^4)(1+q^6)\dots}{(1+q)(1+q^3)(1+q^5)\dots}\right]$$
$$= \sqrt{2}.\sqrt[8]{q}.[1 - q + 2q^2 - 3q^3 + 4q^4 - 6q^5 + 9q^6 - 12q^7 + \dots]$$
$$= \sqrt{2}.\sqrt[8]{q}.[f(q)]$$

ergo:

$$u^2 = 2.\sqrt[8]{q^2}.[f(q)]^2,$$
$$u^3 = 2.\sqrt{2}.\sqrt[8]{q^3}.[f(q)]^3,$$
$$u^4 = 4.\sqrt[8]{q^4}.[f(q)]^4,$$
$$\dots\dots\dots$$
$$u^8 = 16.q.[f(q)]^8,$$
$$\dots\dots\dots$$
$$u^{n+1} = \sqrt{2^{n+1}}.\sqrt[8]{q^{n+1}}.[f(q)]^{n+1} = \sqrt{2^{n+1}}.q^{\varepsilon}.\sqrt[8]{q^t}.[f(q)]^{n+1}$$

ubi $n+1=8s+t$ positum est, unde pro $n=8r+1$ fit $t=2$,
- $n=8r+3$ - $t=4$,
- $n=8r-1$ - $t=0$,
- $n=8r-3$ - $t=6$.

In paragrapho secundo unum valorem ipsius v vidimus erui si in valore ipsius u modo allato q^n loco q ponatur, quo facto impetramus:

$$v = \pm\sqrt{2}.\sqrt[8]{q^n}.[f(q^n)],$$
$$v^2 = 2 . \sqrt[8]{q^{2n}}.[f(q^n)]^2,$$
$$v^3 = \pm 2\sqrt{2}.\sqrt[8]{q^{3n}}.[f(q^n)]^3,$$
$$v^4 = 4 . \sqrt[8]{q^{4n}}.[f(q)^n]^4,$$
$$\cdots\cdots\cdots$$
$$v^8 = 16 . q^n.[f(q^n)]^8,$$
$$\cdots\cdots\cdots$$
$$v^{n+1} = \sqrt{2^{n+1}}.\sqrt[8]{q^{n(n+1)}}.[f(q^n)]^{n+1} = \sqrt{2^{n+1}}.q^{s'}.\sqrt[8]{q^t}.[f(q^n)]^{n+1},$$

qui exponens t ipsius q sub signo radicali ab illo t in valore ipsius u^{n+1} differre nequit, cum facile intelligatur numerum $n(n+1)$ per 8 divisum, si n ut impar numerus ponitur, idem residuum habere ac $(n+1)$. Ubi signa duplicia apposita inveniuntur, superius valet pro $n=8m\pm1$, inferius pro $n=8m\pm3$.

Si in aequatione modulari, quae jam tanquam perfecte composita statuenda est pro u et v valores modo dicti ponuntur, termini qui ex v^{n+1} et u^{n+1} nascuntur eandem quantitatem irrationalem $\sqrt[8]{q^t}$ continent neque aliam reliqui omnes termini debent involvere ne quis coëfficientium in aequatione per hanc irrationalitatem divisa irrationalis restet. Potestas igitur quaedam u^m in talem tantum potestatem v^p ducta inveniri potest ut $m+p\equiv t$ (mod. 8) fiat; ideo coefficiens C potestatis v^p in universum habet formam hanc: $\alpha u^m+\beta u^{m+8}+\gamma u^{m+16}+\delta u^{m+24}+\ldots$.

c) Haec productum quod ex prima potestate ipsius u et prima potestate ipsius v formatur in unaquaque aequatione modulari necessario inesse debere satis demonstrant. Tria igitur membra adhuc definita hanc aequationis formam produnt:

$$v^{n+1}+\ldots+\alpha uv\pm u^{n+1} = 0.$$

Cum vero $\varkappa$ et λ inter se commutatis, quo aequationis mutationem quampiam fieri non licet, terminus αuv in se ipsum redire debeat, facile perspicitur *pro* $n = 8m + 1$ *aut* $n = 8m - 1$, cum in ultimo termino aequationis superius signum valeat, u *et* v *ipsa esse inter se commutanda, pro* $n = 8m + 3$ *aut* $n = 8m - 3$, ubi negativum signum locum habet, u *esse ponendum loco* v, *loco* u *autem* $-v$; nullo enim alio pacto terminus $\alpha u v$ signum suum servare potest.

d) Ex illa observatione, quod aequatio modularis u et v inter se commutatis immutata manet, *coefficientes terminorum* $u^m v^p$ *et* $u^p v^m$ *aequales esse* sponte sequitur, *hac vero conditione adjecta ut eorum signa sint contraria, si* n *habeat formam* $8r \pm 3$ *et* p *sit in numeris paribus.*

e) Ex alia nota proprietate (*Fund.* pag. 31) aequationes modulares immutatae manent, si loco u, v ponatur $\frac{1}{u}$, $\frac{1}{v}$ unde *coefficientes terminorum* $u^m . v^p$ *et* $u^{n+1-m} . v^{n+1-p}$ *non diversos esse* patet. Inde sequitur hoc theorema: *Si aequationem secundum potestates descendentes ipsius* v *et factores singulorum terminorum secundum potestates ascendentes ipsius* u *ordinaveris, in terminis qui aeque longo intervallo ab initio atque a fine distant, aequales habebis coefficientes, qui vero pro* $n = 8r \pm 3$ *contrario signo affecti sunt.*

f) Posito $u = 1$, fit etiam $v = 1$ et tali quidem modo ut pro $n = 8r \pm 1$ omnes $(n+1)$ valores ipsius v fiant $= +1$, pro $n = 8r \pm 3$ vero n valores $= -1$, unicus $= +1$. Expressiones enim ipsius v comprehenduntur in hac forma:

$$v = u^n . \Big[\sin \operatorname{coam} \frac{4mK + 4m'iK'}{n} . \sin \operatorname{coam} \frac{8mK + 8m'iK'}{n} \dots$$
$$\dots \sin \operatorname{coam} \frac{2(n-1)mK + 2(n-1)m'iK'}{n} \Big];$$

pro $\sin \operatorname{coam} \frac{4mK + m'iK'}{n}$ vero scribi licet

$$\sin \operatorname{am} \frac{(n-4mp)K - 4m'piK'}{n} = i . \operatorname{tang} \operatorname{am} \left[\frac{n-4mp}{n} . \frac{K}{i} - \frac{4m'p}{n} . K' \right] \text{ (mod. } \varkappa').$$

Posito autem $u = 1$, unde $\varkappa = 1$, $\varkappa' = 0$ transit $\operatorname{tang} \operatorname{am} w$ (mod. $\varkappa'$) in in $\operatorname{tang} w$; ideoque fit:

$$i . \operatorname{tang} \operatorname{am} \left[\frac{n-4mp}{n} . \frac{K}{i} - \frac{4m'p}{n} . K' \right] \text{ (mod. } \varkappa') = i . \operatorname{tang} \left[\frac{n-4mp}{n} . \frac{K}{i} - \frac{4m'p}{n} . K' \right]$$
$$= \frac{e^{\frac{n-4mp}{n} . K - \frac{4m'p}{n} . iK'} - e^{-\frac{n-4mp}{n} . K + \frac{4m'p}{n} . iK'}}{e^{\frac{n-4mp}{n} . K - \frac{4m'p}{n} . iK'} + e^{-\frac{n-4mp}{n} . K + \frac{4m'p}{n} . iK'}} = \frac{1 - e^{-\frac{2(n-4mp)}{n} . K + \frac{8m'p}{n} . iK'}}{1 + e^{-\frac{2(n-4mp)}{n} . K + \frac{8m'p}{n} . iK'}},$$

cum vero pro $\varkappa' = 0$ sit $K = \infty$ et $K' = \frac{\pi}{2}$, efficitur

$$i.\text{tang am}\left[\frac{n-4mp}{n}\cdot\frac{K}{i} - \frac{4m'p}{n}\cdot K'\right] (\text{mod. } 0) = 1;$$

ergo etiam v ad unitatem reducitur, de cujus signo idem prorsus valet, quod sub lit. *a.* hujus §i diximus. Hinc patet *aequationem modularem, ubi ponas* $u = 1$, *pro* $n = 8r \pm 1$ *transire in* $(v-1)^{n+1} = 0$ *et pro* $n = 8r \pm 3$ *in* $(v+1)^n.(v-1) = 0$. Inde summa coefficientium, quibus diversae potestates ipsius u affectae sunt, in quoque termino aequationis secundum potestates ipsius v ordinatae cognoscitur, ita ut singuli quique coefficientes e reliquis determinari possint. Summa enim coefficientium termini v^p:

$$\text{pro } n = 8r \pm 1 \text{ erit } = (-1)^p.P_{n+1}^{n+1-p},$$
$$\text{- } n = 8r \pm 3 \text{ - } = [P_n^{n+1-p} - P_n^{n-p}],$$

ubi in universum per P_m^s coefficientem termini $(s+1)^{ti}$ in evolutione m^{tae} potestatis binomii cujusvis significamus.

§. 4.

His conditionibus rite perpensis et collectis aequationes modulares facili negotio derivantur. Exstant vero duae methodi ad aequationes obtinendas aptae, quarum alteram paucis tantum verbis adumbrare et uno tantummodo exemplo illustrare est animus, quia calculus prolixior est quam in altera.

Coefficientes aequationis modularis quae secundum §. 3. lit. *b.* formam sequentem habet

$$v^{n+1} + u^s v^n(\alpha + \beta u^8 + \gamma u^{16} + \ldots) + u^{s'} v^{n-1}(\alpha' + \beta' u^8 + \gamma' u^{16} + \ldots) \ldots \pm u^{n+1} = 0$$

hoc modo se accuratius definitos praebent.

Signum ultimi termini ex §. 3. lit. *a.* determinatur. Secundum §. 3. lit. *d.* termini $u^m.v^p$ et $u^p.v^m$, secundum §. 3. lit. *e.* termini, qui ab initio atque a fine aequo distant aequales habent factores, ex §. 3. lit. *f.* summa coefficientium cujuscunque termini definitur.

Ex his numerum coefficientium determinandorum valde minui manifestum est. Si igitur aequationem aliquam revera calculo indagare volumus primo loco ex §. 3. lit. *b.* scimus quaenam potestates ipsius u et v in singulis quibusque terminis contineantur nec non plures conditiones quas inter horum coefficientes intercedere e modo dictis elucet; deinde ponamus pro u et v valores (ex §. 3. lit. *b.*) in series infinitas evolutos. Summa coefficientium uniuscujusqué potestatis ipsius q ipsi zero aequiposita aequationes suppeditat conditionales inter factores terminorum aequationis modularis ex quibus conditionibus hi factores derivari possunt.

Potestates ipsius u aut serie

$$u = \sqrt{2}.\sqrt[8]{q}.[1-q+2q^2-3q^3+4q^4-6q^5+9q^6-12q^7+\ldots]$$

in potestates suas elata inveniendae sunt aut singuli termini potestatum e theoremate polynomico, quod dicitur derivandi.

Calculo satis prolixo potestates ipsius u ad vicesimam usque formavi, quas ad alium etiam usum aptas hic apponam. Hae potestates sunt:

$$\begin{aligned}
\sqrt[4]{\varkappa} = u = \sqrt{2}.\sqrt[8]{q}.\{1 & -q+2q^2-3q^3+4q^4-6q^5+9q^6-12q^7+16q^8-22q^9 \\
& +29q^{10}-38q^{11}+50q^{12}-64q^{13}+82q^{14}-105q^{15}+132q^{16} \\
& -166q^{17}+208q^{18}-258q^{19}+320q^{20}-395q^{21}+484q^{22} \\
& -592q^{23}+722q^{24}-876q^{25}+1060q^{26}-\ldots\},
\end{aligned}$$

$$\begin{aligned}
u^2 = 2\sqrt[8]{q^2}.\{1 & -2q+5q^2-10q^3+18q^4-32q^5+55q^6-90q^7+144q^8-226q^9 \\
& +346q^{10}-522q^{11}+777q^{12}-1138q^{13}+1648q^{14}-2362q^{15} \\
& +3348q^{16}-4704q^{17}+6554q^{18}-9056q^{19}+12425q^{20}-16932q^{21} \\
& +\ldots\},
\end{aligned}$$

$$\begin{aligned}
u^3 = 2\sqrt{2}.\sqrt[8]{q^3}.\{1 & -3q+9q^2-22q^3+48q^4-99q^5+194q^6-363q^7+657q^8 \\
& -1155q^9+1977q^{10}-3312q^{11}+5443q^{12}-8787q^{13}+13968q^{14} \\
& -21894q^{15}+33873q^{16}-51795q^{17}+78345q^{18}-117412q^{19} \\
& +174033q^{20}-255945q^{21}+\ldots\},
\end{aligned}$$

$$\begin{aligned}
u^4 = 4.\sqrt[8]{q^4}.\{1 & -4q+14q^2-40q^3+101q^4-236q^5+518q^6-1080q^7+2162q^8 \\
& -4180q^9+7840q^{10}-14328q^{11}+25591q^{12}-44776q^{13}+76918q^{14} \\
& -129952q^{15}+216240q^{16}-354864q^{17}+574958q^{18}-\ldots\},
\end{aligned}$$

$$\begin{aligned}
u^5 = 4\sqrt{2}.\sqrt[8]{q^5}.\{1 & -5q+20q^2-65q^3+185q^4-481q^5+1165q^6-2665q^7 \\
& +5820q^8-12220q^9+24802q^{10}-48880q^{11}+93865q^{12} \\
& -176125q^{13}+323685q^{14}-583798q^{15}+1035060q^{16} \\
& -1806600q^{17}+3108085q^{18}-\ldots\},
\end{aligned}$$

$$\begin{aligned}
u^6 = \sqrt[8]{q^6}\{1 & -6q+27q^2-98q^3+309q^4-882q^5+2330q^6-5784q^7+13644q^8 \\
& -30826q^9+67107q^{10}-141444q^{11}+289746q^{12}-578646q^{13} \\
& +1129527q^{14}-2159774q^{15}+4052721q^{16}-7474806q^{17} \\
& +3108085q^{18}-\ldots\},
\end{aligned}$$

$$\begin{aligned}
u^7 = 8\sqrt{2}.\sqrt[8]{q^7}.\{1 & -7q+35q^2-140q^3+483q^4-1498q^5+4277q^6-11425q^7 \\
& +28889q^8-69734q^9+161735q^{10}-362271q^{11}+786877q^{12} \\
& -1662927q^{13}+3428770q^{14}-6913760q^{15}+13660346q^{16} \\
& -26492361q^{17}+50504755q^{18}-\ldots\},
\end{aligned}$$

$$u^8 = 16.q.\{1-8q+44q^2-192q^3+718q^4-2400q^5+7352q^6-20992q^7 +56549q^8-145008q^9+356388q^{10}-844032q^{11}+1934534q^{12} -4306368q^{13}+9337704q^{14}-19771392q^{15}+40965362q^{16} -83207976q^{17}+165944732q^{18}-\ldots\},$$

$$u^9 = 16\sqrt{2}.q\sqrt[8]{q}.\{1-9q+54q^2-255q^3+1026q^4-3672q^5+11997q^6 -36414q^7+103977q^8-281911q^9+730953q^{10} -1822689q^{11}+4390824q^{12}-10256508q^{13}+23303025q^{14} -51631227q^{15}+111804966q^{16}-237074742q^{17} +493063403q^{18}-\ldots\},$$

$$u^{10} = 32.q\sqrt[8]{q^2}.\{1-10q+65q^2-330q^3+1420q^4-5412q^5+18765q^6-60270q^7 +181645q^8-518660q^9+1413465q^{10}-3697960q^{11} +9331565q^{12}-22800050q^{13}+54112825q^{14}-125090220q^{15} +282298020q^{16}-623185010q^{17}+1348033540q^{18}-\ldots\},$$

$$u^{11} = 32\sqrt{2}.q\sqrt[8]{q^3}.\{1-11q+77q^2-418q^3+1914q^4-7733q^5+28336q^6 -95931q^7+304062q^8-911240q^9+2601786q^{10} -7120136q^{11}+18766759q^{12}-47830486q^{13} +118270746q^{14}-284527793q^{15}+667553898q^{16} -1530587256q^{17}+3435726536q^{18}-\ldots\},$$

$$u^{12} = 64q\sqrt[8]{q^4}.\{1-12q+90q^2-520q^3+2523q^4-10764q^5+41534q^6 -147720q^7+490869q^8-1539472q^9+4592430q^{10} -13111632q^{11}+36006362q^{12}-95497116q^{13}+245457000q^{14} -613183064q^{15}+1492474572q^{16}-3546915228q^{17} +8245677110q^{18}-\ldots\},$$

$$u^{13} = 64\sqrt{2}.q\sqrt[8]{q^5}.\{1-13q+104q^2-637q^3+3263q^4-14651q^5+59345q^6 -221091q^7+768131q^8-2514551q^9+7818200q^{10} -23233535q^{11}+66328964q^{12}+182681916q^{13} -487098378q^{14}-1261118313q^{15}+3178449222q^{16} -7815313766q^{17}+18783535199q^{18}-\ldots\},$$

$$u^{14} = 128.q\sqrt[8]{q^6}.\{1-14q+119q^2-770q^3+4151q^4-19558q^5+82936q^6 -322828q^7+1169847q^8-3988292q^9+12896562q^{10} -39809574q^{11}+117921321q^{12}-336630840q^{13} +929461993q^{14}-2489690882q^{15}+6486711301q^{16} -16475721276q^{17}+40874694490q^{18}-\ldots\},$$

$$u^{15} = 128\sqrt{2}.q\sqrt[8]{q^7}.\{1-15q+135q^2-920q^3+5205q^4-25668q^5+113675q^9$$
$$-461265q^7+1739710q^8-6164345q^9+20690964q^{10}$$
$$-66222405q^{11}+203173760q^{12}-600165795q^{13}$$
$$+1713196575q^{14}-4740491107q^{15}+12748926285q^{16}$$
$$-33400680615q^{17}+85415669230q^{18}-\ldots.\},$$

$$u^{16} = 256.q^2.\{1-16q+152q^2-1088q^3+6444q^4-33184q^5+153152q^6$$
$$-646528q^7+2533070q^8-9311664q^9+32387616q^{10}$$
$$-107299904q^{11}+340436664q^{12}-1039026144q^{13}$$
$$+3061896704q^{14}-8739810688q^{15}+24229115109q^{16}$$
$$-65390485328q^{17}+172155210320q^{18}-\ldots.\},$$

$$u^{17} = 256\sqrt{2}.q^2\sqrt[8]{q}.\{1-17q+170q^2-1275q^3+7888q^4-42330q^5$$
$$+203201q^6-890800q^7+3619334q^8-13780540q^9$$
$$+49590581q^{10}-169812320q^{11}+556366922q^{12}$$
$$-1752038020q^{13}+5323089708q^{14}-15653783345q^{15}$$
$$+44679433473q^{16}-124069449335q^{17}$$
$$+335888162944q^{18}-\ldots.\},$$

$$u^{18} = 512.q^2\sqrt[8]{q^2}.\{1-18q+189q^2-1482q^3+9558q^4-53352q^5+265923q^6$$
$$-1208610q^7+5084478q^8-20021534q^9+74438388q^{10}$$
$$-263104686q^{11}+889020813q^{12}-2884990266q^{13}$$
$$+9026077050q^{14}-27314626158q^{15}+8017703378q^{16}$$
$$-228831885054q^{17}+636376573943q^{18}-\ldots.\},$$

$$u^{19} = 512\sqrt{2}.q^2\sqrt[8]{q^3}.\{1-19q+209q^2-1710q^3+11476q^4-66519q^5$$
$$+343710q^6-1617147q^7+7034047q^8-28607673q^9$$
$$+109745767q^{10}-\ldots.\},$$

$$u^{20} = 1024.q^2\sqrt[8]{q^4}.\{1-20q+230q^2-1960q^3+13665q^4-82124q^5+439270q^6$$
$$-2136600q^7+9596460q^8-40260300q^9+159174524q^{10}$$
$$+\ldots.\}.$$

§. 5.

Exempli loco, secundum quod aequatio modularis cujusvis ordinis deduci possit, calculum pro aequatione decimi tertii ordinis determinanda per partes consumatum addam.

Habemus nimirum ex §. 3. lit. *b*.

$$u = \sqrt{2}.\sqrt[8]{q}.[f(q)], \qquad v = -\sqrt{2}.q.\sqrt[8]{q^5}.[f(q^{13})],$$
$$u^2 = 2.\sqrt[8]{q^2}.[f(q)]^2, \qquad v^2 = +\ 2.q^3.\sqrt[8]{q^2}.[f(q^{13})]^2,$$

$$u^3 = 2\sqrt{2}\,.\,\sqrt[8]{q^3}\,.\,[f(q)]^3, \qquad v^3 = -\,2\sqrt{2}\,.\,q^4\,.\,\sqrt[8]{q^7}\,.\,[f(q^{13})]^3,$$
$$u^4 = 4\,.\,\sqrt[8]{q^4}\,.\,[f(q)]^4, \qquad v^4 = +\,4\,.\,q^6\,.\,\sqrt[8]{q^4}\,.\,[f(q^{13})]^4,$$
$$u^5 = 4\sqrt{2}\,.\,\sqrt[8]{q^5}\,.\,[f(q)]^5, \qquad v^5 = -\,4\sqrt{2}\,.\,q^8\,.\,\sqrt[8]{q}\,.\,[f(q^{13})]^5,$$
$$u^6 = 8\,.\,\sqrt[8]{q^6}\,.\,[f(q)]^6, \qquad v^6 = +\,8\,.\,q^9\,\sqrt[8]{q^6}\,.\,[f(q^{13})]^6,$$
$$u^7 = 8\sqrt{2}\,.\,\sqrt[8]{q^7}\,.\,[f(q)]^7, \qquad v^7 = -\,8\sqrt{2}\,.\,q^{11}\,.\,\sqrt[8]{q^3}\,.\,[f(q^{13})]^7,$$
$$u^8 = 16\,.\,q\,.\,[f(q)]^8, \qquad v^8 = +\,16\,.\,q^{13}\,.\,[f(q^{13})]^8,$$
$$u^9 = 16\sqrt{2}\,.\,q\sqrt[8]{q}\,.\,[f(q)]^9, \qquad v^9 = -16\sqrt{2}\,.\,q^{14}\,.\,\sqrt[8]{q^5}\,.\,[f(q^{13})]^9,$$
$$u^{10} = 32\,.\,q\sqrt[8]{q^2}\,.\,[f(q)]^{10}, \qquad v^{10} = +\,32\,.\,q^{16}\,.\,\sqrt[8]{q^2}\,.\,[f(q^{13})]^{10},$$
$$u^{11} = 32\sqrt{2}\,.\,q\sqrt[8]{q^3}\,.\,[f(q)]^{11}, \qquad v^{11} = -32\sqrt{2}\,.\,q^{17}\,.\,\sqrt[8]{q^7}\,.\,[f(q^{13})]^{11},$$
$$u^{12} = 64\,.\,q\sqrt[8]{q^4}\,.\,[f(q)]^{12}, \qquad v^{12} = +\,64\,.\,q^{19}\,.\,\sqrt[8]{q^4}\,.\,[f(q^{13})]^{12},$$
$$u^{13} = 64\sqrt{2}\,.\,q\sqrt[8]{q^5}\,.\,[f(q)]^{13}, \qquad v^{13} = -64\sqrt{2}\,.\,q^{21}\,.\,\sqrt[8]{q}\,.\,[f(q^{13})]^{13},$$
$$u^{14} = 128\,.\,q\sqrt[8]{q^6}\,.\,[f(q)]^{14}, \qquad v^{14} = +\,128\,.\,q^{22}\,.\,\sqrt[8]{q^6}\,.\,[f(q^{13})]^{14},$$

Quoniam potestas v^{14} quantitatem irrationalem $\sqrt[8]{q^6}$ continet ex lit. *b.* §^{i} 3. scimus tales tantum potestates ipsius u et v conjunctim positas in aequatione inveniri posse, quae eandem quantitatem irrationalem faciant, unde pro aequatione modulari decimi tertii ordinis sequens forma evadit:

$$v^{14} + v^{13}u^5(\alpha_1 + \alpha_2 u^8) + v^{12}u^2(\beta_1 + \beta_2 u^8) + \gamma v^{11}u^7 + v^{10}u^4(\delta_1 + \delta_2 u^8)$$
$$+ v^9 u(\varepsilon_1 + \varepsilon_2 u^8) + \zeta v^8 u^6 + v^7 u^3(\eta_1 + \eta_2 u^8) + \theta v^6 u^8 + v^5 u^5(\iota_1 + \iota_2 u^8)$$
$$+ v^4 u^2(\varkappa_1 + \varkappa_2 u^8) + \lambda v^3 u^7 + v^2 u^4(\mu_1 + \mu_2 u^8) + v u(\nu_1 + \nu_2 u^8) - u^{14} = 0.$$

Ultimus terminus hujus aequationis habet secundum lit. *a.* §^{i} 3. signum negativum, quoniam 13 est numerus formae $8r-3$.

Ex lit. *d.* §^{i} 3. sequuntur:

$$\iota_2 = \alpha_1; \quad \mu_2 = -\beta_1; \quad \delta_2 = -\beta_2; \quad \eta_2 = \gamma; \quad \varkappa_2 = -\delta_1; \quad \nu_2 = \varepsilon_1;$$
$$\theta = -\zeta; \quad \lambda = \eta_1; \quad \mu_1 = -\varkappa_1.$$

Ex lit. *e.* §^{i} 3. sequuntur:

$$\nu_2 = -\alpha_1; \quad \nu_1 = -\alpha_2; \quad \mu_2 = -\beta_1; \quad \mu_1 = -\beta_2; \quad \lambda = -\gamma; \quad \varkappa_2 = -\delta_1;$$
$$\varkappa_1 = -\delta_2; \quad \iota_2 = -\varepsilon_1; \quad \iota_1 = -\varepsilon_2; \quad \theta = -\zeta; \quad \eta_2 = -\eta_1.$$

Ex lit. *f.* §^{i} 3. sequitur aequationem, ubi in ea $u = 1$ ponitur, transire in $(v+1)^{13}(v-1) = 0$ sive in

$$v^{14} + 12v^{13} + 65v^{12} + 208v^{11} + 429v^{10} + 572v^9 + 429v^8 - 429v^6 - 572v^5$$
$$-429v^4 - 208v^3 - 65v^2 - 12v - 1 = 0.$$

Quae cum aequatione antecedente comparata has praebet aequationes conditionales:

$$\alpha_1+\alpha_2=12;\quad \beta_1+\beta_2=65;\quad \gamma=208;\quad \delta_1+\delta_2=429;\quad \varepsilon_1+\varepsilon_2=572;$$
$$\zeta=429;\quad \eta_1+\eta_2=0.$$

Ex his elucet aequationem posse scribi sequenti modo:

$$\begin{aligned}&v^{14}+v^{13}u^5.[(12-\alpha_2)+\alpha_2 u^8]+v^{12}u^2[(65-\beta_2)+\beta_2 u^8]\\&+208v^{11}u^7+v^{10}u^4[(429+\beta_2)-\beta_2 u^8]-v^9 u[(12-\alpha_2)-(584-\alpha_2)u^8]\\&+429v^8u^6-v^7u^3(208-208u^8)-429v^6u^8-v^5u^5[(584-\alpha_2)-(12-\alpha_2)u^8]\\&+v^4u^2[\beta_2-(429+\beta_2)u^8]-208v^3u^7-v^2u^4[\beta_2+(65-\beta_2)u^8]\\&-vu[\alpha_2+(12-\alpha_7)u^8]-u^{14}=0.\end{aligned}$$

Restant igitur duo tantum coefficientes determinandi: α_2 et β_2. Ut hi determinentur pro diversis potestatibus ipsius u ponimus valores, quos in §° antecedente dedimus; pro v vero quod est

$$=-\sqrt{2}.q\sqrt[8]{q^5}.[1-q^{13}+2q^{26}-3q^{39}+\ldots.],$$

scribere sufficit $-\sqrt{2}.q\sqrt[8]{q^5}$, quoniam reliqui ejus termini potestates continent decima tertia majores. Eruitur ergo:

$$\left.\begin{array}{rcl}-\quad u^{14} &=& 128[-q+14q^2-119q^3+\ldots.]\\-(12-\alpha_2)vu^9 &=& (12-\alpha_2).32[\quad q^2-\ 9q^3+\ldots.]\\-\quad \alpha_2.vu &=& \alpha_2\ .\ 2[\ q-\ q^2+\ 2q^3+\ldots.]\\-\quad \beta_2.v^2u^4 &=& \beta_2\ .\ 8[\quad -\ q^3+\ldots.]\end{array}\right\}=0.$$

Reliquos aequationis terminos omnes omitti licet, quoniam infimae potestates ipsius q, quae in iis continentur, tertiam superant, ex qua jam β_2 determinari potest. Additione facta et coefficientibus singularum potestatum ipsius q ipsi zero aequipositis evadunt aequationes:

$$-128+2\alpha_2=0,$$
$$-119.128-9.32(12-\alpha_2)+4\alpha_2-8\beta_2=0,$$

ex quibus deducitur: $\alpha_2=64;\quad \beta_2=0;$

unde denique aequatio modularis decimi tertii ordinis eruitur sequens:

$$\begin{aligned}&v^{14}-4v^{13}u^5(13-16u^8)+65v^{12}u^2+208v^{11}u^7+429v^{10}u^4+52v^9u(1+10u^8)\\&+429v^8u^6-208v^7u^3(1-u^8)-429v^6u^8-52v^5u^5(10+u^8)-429v^4u^{10}\\&-208v^3u^7-65v^2u^{12}-4vu(16-13u^8)-u^{14}=0.\end{aligned}$$

Nota. Facile intelligitur infimam potestatem ipsius q, quae in aequatione quavis modulari inveniri possit in duobus tantum terminis in series evolutis contineri posse, in his dico: u^{n+1} et $\alpha.uv$, qui posito $n+1=8s+t$ praebent:

$$2^{\frac{n+1}{2}}.[q^s+\ldots.], \qquad \alpha.2.[q^s+\ldots.];$$

qua ex re concluditur aequationem conditionalem, ex qua coefficiens ipsius $u\nu$ determinetur, esse hanc:

$$2\alpha = -(2)^{\frac{n+1}{2}}, \quad \text{ergo:} \quad \alpha = -(2)^{\frac{n-1}{2}}.$$

§. 6.

Altera methodus coefficientes inveniendi in eo nititur quod coefficientes cujusvis aequationis algebraicae ex summis potestatum radicum componi possunt. Habemus enim, ut omnibus notum est, in aequatione:

$$\nu^{n+1}+C_1\nu^n+C_2\nu^{n-1}+\ldots.+C_n\nu\pm u^{n+1}=0,$$

ad coefficientes determinandos has aequationes conditionales:

$$S_1+C_1=0,$$
$$S_2+C_1S_1+2C_2=0,$$
$$S_3+C_1S_2+C_2S_1+3C_3=0,$$
$$S_4+C_1S_3+C_2S_2+C_3S_1+4C_4=0$$

et cetera,

ubi per S_m significatur summa m^{tarum} potestatum radicum.

Si $q=r^8$ et

$$\frac{(1+r^{16})(1+r^{32})(1+r^{48})\ldots.}{(1+r^8)(1+r^{24})(1+r^{40})\ldots.} = 1+A_1r^8+A_2r^{16}+A_3r^{24}+\ldots. = f(r^8)$$

ponimus, fit secundum *Fund.* pag. 89, 7.

$$u=\sqrt{2}.r.f(r^8).$$

Si n est numerus primus, impetramus secundum §. 2. omnes $(n+1)$ valores ipsius ν hos:

$$\pm\sqrt{2}.r^n.f(r^{8n}) \quad \text{et} \quad \sqrt{2}.r^{\frac{1}{n}}.f\left(r^{\frac{8}{n}}\right),$$

si pro $r^{\frac{1}{n}}$ omnes ejus valores in numero n ponuntur. Quod ad primam radicem attinet, utrum signum positivum an negativum sit sumendum; ea quaestio jam supra §. 2. lit. α. absoluta est. Erit ergo:

$$S_1(\nu)=\sqrt{2}.\left[\pm r^n.f(r^{8n})+\Sigma\, r^{\frac{1}{n}}.f\left(r^{\frac{8}{n}}\right)\right],$$

in qua expressione illi tantum termini sumendi sunt, qui irrationalitatem non continent, quoniam ex §. 1. scimus aequationis coefficientes omnes ergo etiam S functiones esse rationales ipsius u ideoque ipsius r'. Si igitur α et β tali modo determinamus ut sit $8\alpha_1+1=n.\beta_1$, habemus

$$S_1(\nu)=\sqrt{2}.[\pm r^n.f(r^{8n})+n(A_{\alpha_1}.r^{\beta_1}+A_{\alpha_1+n}.r^{\beta_1+8}+A_{\alpha_1+2n}.r^{\beta_1+16}+\ldots.)].$$

Pro $n = 3, 5, 7, 11, 13, 17, 19, \ldots.$
fiunt: $\alpha_1 = 1, 3, 6, 4, 8, 2, 7, \ldots.$
et $\beta_1 = 3, 5, 7, 3, 5, 1, 3, \ldots.$

Exponentes β_1, $\beta_1 + 8$, $\beta_1 + 16$, ... tantum ad illum usque valorem continuare opus est, qui minime ab n differt et ipsius $f(r^{8n})$ primus tantum terminus qui unitatem aequat sumendus est quoniam sequentes termini potestates n^{ta} superiores continent.

Hinc nanciscimur:

pro: $n = 3$, $S_1^{(3)} v = \sqrt{2}.[-1 + 3A_1] r^3$,
- $n = 5$, $S_1^{(5)} v = \sqrt{2}.[-1 + 5A_3] r^5$,
- $n = 7$, $S_1^{(7)} v = \sqrt{2}.[+1 + 7A_6] r^7$,
- $n = 11$, $S_1^{(11)} v = \sqrt{2}.[11A_4 + (-1 + 11A_{15}) r^8] r^3$,
- $n = 13$, $S_1^{(13)} v = \sqrt{2}.[13A_8 + (-1 + 13A_{21}) r^8] r^5$,
- $n = 17$, $S_1^{(17)} v = \sqrt{2}.[17A_2 + 17A_{19} r^8 + (+1 + 17A_{36}) r^{16}] r$,
- $n = 19$, $S_1^{(19)} v = \sqrt{2}.[19A_7 + 19A_{26} r^8 + (-1 + 19A_{45}) r^{16}] r^3$,

. . . .

Similia valent de reliquis summis. Fiunt enim:

$$S_2^{(n)} v = 2\left\{r^{2n}.[f(r^{8n})]^2 + \Sigma r^{\frac{2}{n}}\left[f\left(r^{\frac{8}{n}}\right)\right]^2\right\}$$
$$= 2n[A_{\alpha_2}^{(2)}.r^{\beta_2} + A_{\alpha_2+n}^{(2)}.r^{\beta_2+8} + A_{\alpha_2+2n}^{(2)}.r^{\beta_2+16} + \ldots.],$$

si α_2 et β_2 tales valores assumunt, qui aequationi $8\alpha_2 + 2 = n\beta_2$ satisfaciunt;

$S_3^{(n)} v = 2\sqrt{2}.n.[A_{\alpha_3}^{(3)}.r^{\beta_3} + A_{\alpha_3+n}^{(3)}.r^{\beta_3+8} + A_{\alpha_3+2n}^{(3)}.r^{\beta_3+16} + \ldots.]$, si $8\alpha_3 + 3 = n\beta_3$;

$S_4^{(n)} v = \quad 4.n.[A_{\alpha_4}^{(4)}.r^{\beta_4} + A_{\alpha_4+n}^{(4)}.r^{\beta_4+8} + A_{\alpha_4+2n}^{(4)}.r^{\beta_4+16} + \ldots.]$, si $8\alpha_4 + 4 = n\beta_4$;

et cetera.

Per $A^{(2)}$, $A^{(3)}$, $A^{(4)}$, ... coefficientes evolutionum secundae, tertiae, quartae etc. potestatis ipsius u significantur, quales in §. 4. inveniuntur.

§. 7.

Deducitur aequatio modularis pro transformatione tertii ordinis.

Forma hujus aequationis ex §. 3. lit. *b.* est:

$$v^4 + a u^3 v^3 + b u v - u^4 = 0.$$

Secundum §. 3. lit. *f.* fiunt:

$$a = P_3^1 - P_3^0 = 3 - 1 = 2, \qquad b = P_3^3 - P_3^2 = 1 - 3 = -2;$$

ergo aequatio ipsa est haec:

$$v^4 + 2u^3v^3 - 2uv - u^4 = 0, \quad \text{sive:} \quad v^4 - u^4 - 2uv(1 - u^2v^2) = 0,$$
$$\text{aut:} \quad (v-u)^3(v+u) - 2vu(1+u^2)(1-v^2) = 0.$$

Deducitur aequatio modularis pro transformatione quinti ordinis.

Forma hujus aequationis est ex §. 3. lit. *a*. et *b*.

$$v^6+a u^5 v^5+b u^2 v^4+c u^4 v^2+d u v-u^6 = 0.$$

Secundum §. 3. lit. *e*. habemus:

$$d=-a; \quad c=-b.$$

Secundum §. 3. lit. *f*. est:

$$a=\mathrm{P}_5^1-\mathrm{P}_5^0=5-1=4, \qquad b=\mathrm{P}_5^2-\mathrm{P}_5^1=10-5=5;$$

ergo aequatio quaesita:

$$v^6+4 u^5 v^5+5 u^2 v^4-5 u^4 v^2-4 u v-u^6 = 0,$$

sive:

$$v^6-u^6-4 u v(1-u^4 v^4)+5 u^2 v^2 (v^2-u^2) = 0,$$

aut:

$$(v-u)^5 (v+u)-4 u v (1+u^4)(1-v^4) = 0.$$

Deducitur aequatio modularis pro transformatione septimi ordinis.

Forma hujus aequationis est:

$$v^8+a u^7 v^7+b u^6 v^6+c u^5 v^5+d u^4 v^4+e u^3 v^3+f u^2 v^2+g u v+u^8 = 0.$$

Ex §. 3. lit. *e*. sequuntur:

$$g=a; \quad f=b; \quad e=c.$$

Ex §. 3. lit. *f*. sequuntur:

$$a=-\mathrm{P}_8^1=-8, \quad b=\mathrm{P}_8^2=28, \quad c=-\mathrm{P}_8^3=-56, \quad d=\mathrm{P}_8^4=70,$$

ergo aequatio quaesita:

$$v^8-8 u^7 v^7+28 u^6 v^6-56 u^5 v^5+70 u^4 u^4-56 u^3 v^3+28 u^2 v^2-8 u v+u^8=0,$$

sive:

$$(1-u^8)(1-v^8) = (1-u v)^8.$$

Deducitur aequatio modularis pro transformatione undecimi ordinis.

Forma hujus aequationis est:

$$v^{12}+u^3 v^{11}(a_1+a_2 u^8)+b u^6 v^{10}+u v^9 (c_1+c_2 u^8)+d u^4 v^8+e u^7 v^7+u^2 v^6 (f_1+f_2 u^8)$$
$$+g u^5 v^5+h u^8 v^4+u^3 v^3 (i_1+i_2 u^8)+k u^6 v^2+u v (l_1+l_2 u^8)-u^{12} = 0.$$

Ex §. 3. lit. *e*. sequuntur:

$$l_2=-a_1; \quad l_1=-a_2; \quad k=-b; \quad i_2=c_1; \quad i_1=c_2; \quad h=-d;$$
$$g=-e; \quad f_2=-f_1.$$

Ex §. 3. lit. *d*. sequuntur:

$$i_2=a_1; \quad f_2=-b; \quad l_2=c_1; \quad h=-d; \quad k=-f_1.$$

Ex §. 3. lit. *f*. sequuntur:

$$a_1+a_2 = \mathrm{P}_{11}^1-\mathrm{P}_{11}^0 = 11-1 = 10,$$
$$b = \mathrm{P}_{11}^2-\mathrm{P}_{11}^1 = 55-11 = 44,$$

$$c_1 + c_2 = P^3_{11} - P^2_{11} = 165 - 55 = 110,$$
$$d = P^4_{11} - P^3_{11} = 330 - 165 = 165,$$
$$e = P^5_{11} - P^4_{11} = 462 - 330 = 132,$$
$$f_1 + f_2 = P^6_{11} - P^5_{11} = 462 - 462 = 0.$$

Ex §. 5. *Nota.* est $l_1 = -32$.

Ex his conditionibus jam coefficientes omnes sunt quantitates notae; aequatio igitur componi potest:

$$v^{12} - u^3 v^{11}(22 - 32u^8) + 44u^6 v^{10} + 22uv^9(1 + 4u^8) + 165u^4 v^8 + 132u^7 v^7$$
$$+ 44u^2 v^6(1 - u^8) - 132u^5 v^5 - 165u^8 v^4 - 22u^3 v^3(4 + u^8) - 44u^6 v^2$$
$$- uv(32 - 22u^8) - u^{12} = 0,$$

aut si placet:

$$(v-u)^{11}(v+u) + 44u^2 v^2(v^4 - u^4)(1 - u^4)(1 - v^4) - 32uv(1 + u^{10})(1 - v^{10})$$
$$-22uv(1+u^2)(1-v^2).\{(v^2+u^2).[4u^2v^2 - (u^2-v^2)^2] + 4u^2v^2(1-u^2v^2)(1-u^2)(1+v^2)\}$$
$$= 0 \text{ *)}.$$

Deducitur aequatio modularis pro transformatione decimi tertii ordinis.

Forma hujus aequationis est:

$$v^{14} + u^5 v^{13}(a_1 + a_2 u^8) + u^2 v^{12}(b_1 + b_2 u^8) + cu^7 v^{11} + u^4 v^{10}(d_1 + d_2 u^8) + uv^9(e_1 + e_2 u^8)$$
$$+ fu^6 v^8 + u^3 v^7(g_1 + g_2 u^8) + hu^8 v^6 + u^5 v^5(i_1 + i_2 u^8) + u^2 v^4(k_1 + k_2 u^8) + lu^7 v^3$$
$$+ u^4 v^2(m_1 + m_2 u^8) + uv(n_1 + n_2 u^8) - u^{14} = 0.$$

Ex §. 3. lit. *e*. sequuntur:

$n_2 = -a_1$; $n_1 = -a_2$; $m_2 = -b_1$; $m_1 = -b_2$; $l = -c$; $k_2 = -d_1$;
$k_1 = -d_2$; $i_2 = -e_1$; $i_1 = -e_2$; $h = -f$; $g_2 = -g_1$.

Ex §. 3. lit. *d*. sequuntur:

$i_2 = a_1$; $m_2 = -b_1$; $d_2 = -b_2$; $g_2 = c$; $k_2 = -d_1$; $n_2 = e_1$;
$h = -f$; $l = g_1$; $m_1 = -k_1$.

Ex §. 3. lit. *f*. sequuntur:

$$a_1 + a_2 = P^1_{13} - P^0_{11} = 13 - 1 = 12,$$
$$b_1 + b_2 = P^2_{13} - P^1_{13} = 78 - 13 = 65,$$
$$c = P^3_{13} - P^2_{13} = 286 - 78 = 208,$$
$$d_1 + d_2 = P^4_{13} - P^3_{13} = 715 - 286 = 429,$$
$$e_1 + e_2 = P^5_{13} - P^4_{13} = 1287 - 715 = 572,$$
$$f = P^6_{13} - P^5_{13} = 1716 - 1287 = 429,$$
$$g_1 + g_1 = P^7_{13} - P^6_{13} = 1716 - 1716 = 0.$$

*) In hujus Diarii Tom. XII. pag. 178, ubi hanc aequationem sine demonstratione dedi error typographicus invenitur; illo enim loco in secundo termino juxta $(v^2 + u^2)$ factor omissus est hic: $[4u^2v^2 - (u^2 - v^2)^2]$.

Ex §. 5. *Nota.* est $n_1 = -64$.

Aequatio ergo transit in sequentem:

$$v^{14} - u^5 v^{13}(52 - 64u^8) + u^2 v^{12}(b_1 + b_2 u^8) + 208 u^7 v^{11} + u^4 v^{10}[(429 + b_2) - b_2 u^8] + u v^9(52 + 520 u^8) + 429 u^6 v^8 - u^3 v^7 (208 - 208 u^8) - 429 u^8 v^6 - u^5 v^5 (520 + 52 u^8) + u^2 v^2 [b_2 - (429 + b_2) u^8] - 208 u^7 v^3 - u^4 v^2 (b_2 + b_1 u^8) - u v (64 - 52 u^8) - u^{14} = 0.$$

Restat igitur ut coefficientem secundum, qui est $= \frac{S_1^{(13)} S_1^{(13)}}{2} - \frac{S_2^{(13)}}{2}$, determinemus.

Secundum §. 6. est

$$S_2^n = 2n.[A_{\alpha_2}^{(2)}.r^{\beta_2} + A_{\alpha_2+n}^{(2)}.r^{\beta_2+8} + \ldots.],$$

si α_2 et β_2 tales valores accipiunt ut aequationi $8\alpha_2 + 2 = n.\beta_2$ satisfaciant, ergo:

$$S_2^{(13)} = 26.[A_3^{(2)}.r^2 + A_{16}^{(2)}.r^{10}],$$

sive posito secundum §. 4. $A_3^{(2)} = -10$, $A_{16}^{(2)} = 3348$,

$$S_2^{(13)} = 26[-10 r^2 + 3348 r^{10}],$$

qui termini ex aliis potestatibus ipsius u nasci nequeunt nisi ex secunda et decima, ita ut aequiponendum sit:

$$S_2^{(13)} = m u^2 + m' u^{10};$$

si hic valor, qui substitutione $u = \sqrt{2}.r.[1 - r^8 + \ldots.]$ adhibita, transit in: $2 m r^2 + (32 m' - 4 m) r^{10} \ldots.$ cum praecedente comparatur, eruitur:

$$2m = -260 \quad \text{et} \quad 32 m' - 4m = 26.3348,$$

ergo:

$$m = -130; \quad m' = 2704 \quad \text{unde} \quad S_2^{(13)} = -130 u^2 + 2704 u^{10}.$$

$S_1^{(13)}$ jam ex aequatione ipsa notum est, nimirum $S_1^{(13)} = -52 u^5 + 64 u^{13}$, ergo fit coefficiens secundi termini:

$$b_1 u^2 + b_2 u^{10} = \frac{S_1^{(13)}.S_1^{(13)}}{2} - \frac{S_2^{(13)}}{2} = \frac{2704 u^{10}}{2} + \frac{130 u^2}{2} - \frac{2704 u^{10}}{2} = 65 u^2,$$

ergo $b_1 = 65$, $b_2 = 0$.

Aequatio igitur modularis decimi tertii ordinis erit:

$$v^{14} - u^5 v^{13}(52 - 64 u^8) + 65 u^2 v^{12} + 208 u^7 v^{11} + 429 u^4 v^{10} + 52 u v^9 (1 + 10 u^8) + 429 u^6 v^8 - 208 u^3 v^7 (1 - u^8) - 429 u^8 v^6 - 52 u^5 v^5 (10 + u^8) - 429 u^{10} v^4 - 208 u^7 v^3 - 65 u^{12} v^2 - u v (64 - 52 u^8) - u^{14} = 0,$$

aut:

$$(v - u)^{13}(v + u) - 4 u v (1 + u^4)(1 - v^4).[13(3 u^4 v^4 + 4 u^2 v^2 (v^2 + u^2)^2 + (1 + v^4)(1 - u^4)(1 - u^4 v^4)) + 3(1 - u^4 + u^8)(1 + v^4 + v^8)] = 0.$$

Deducitur aequatio modularis pro transformatione decimi septimi ordinis.

Forma hujus aequationis est:

$$v^{18}+uv^{17}(a_1+a_2u^8+a_3u^{16})+u^2v^{16}(b_1+b_2u^8)+u^3v^{15}(c_1+c_2u^8)+u^4v^{14}(d_1+d_2u^8)$$
$$+u^5v^{13}(e_1+e_2u^8)+u^6v^{12}(f_1+f_2u^8)+u^7v^{11}(g_1+g_2u^8)+u^8v^{10}(h_1+h_2u^8)$$
$$+uv^9(i_1+i_2u^8+i_3u^{16})+u^2v^8(k_1+k_2u^8)+u^3v^7(l_1+l_2u^8)+u^4v^6(m_1+m_2u^8)$$
$$+u^5v^5(n_1+n_2u^8)+u^6v^4(o_1+o_2u^8)+u^7v^3(p_1+p_2u^8)+u^8v^2(q_1+q_2u^8)$$
$$+uv(r_1+r_2u^8+r_3u^{16})+u^{18}=0.$$

Ex §. 3. lit. *e.* sequuntur:

$r_3=a_1$; $r_2=a_2$; $r_1=a_3$; $q_2=b_1$; $q_1=b_2$; $p_2=c_1$; $p_1=c_2$; $o_2=d_1$; $o_1=d_2$; $n_2=e_1$; $n_1=e_2$; $m_2=f_1$; $m_1=f_2$; $l_2=g_1$; $l_1=g_2$; $k_2=h_1$; $k_1=h_2$; $i_3=i_1$.

Ex §. 3. lit. *d.* sequuntur:

$r_3=a_1$; $i_3=a_2$; $q_2=b_1$; $h_2=b_2$; $p_2=c_1$; $g_2=c_2$; $o_2=d_1$; $f_2=d_2$; $n_2=e_1$; $m_2=f_1$; $l_2=g_1$; $k_2=h_1$; $r_2=i_1$; $q_1=k_1$; $p_1=l_1$; $o_1=m_1$.

Ex §. 3. lit. *f.* sequuntur:

$$\begin{aligned}
a_1+a_2+a_3 &= -P^1_{18} = -\ 18,\\
b_1+b_2 &= +P^2_{18} = +\ 153,\\
c_1+c_2 &= -P^3_{18} = -\ 816,\\
d_1+d_2 &= +P^4_{18} = +\ 3060,\\
e_1+e_2 &= -P^5_{18} = -\ 8568,\\
f_1+f_2 &= +P^6_{18} = +18564,\\
g_1+g_2 &= -P^7_{18} = -31824,\\
h_1+h_2 &= +P^8_{18} = +43758,\\
i_1+i_2+i_3 &= -P^9_{18} = -48620.
\end{aligned}$$

Ex §. 5. *Nota.* est $r_1=-256$.

Aequatio ergo sequentem formam induit:

$$v^{18}+uv^{17}[a_1+(238-a_1)u^8-256u^{16}]+u^2v^{16}[b_1+(153-b_1)u^8]$$
$$+u^3v^{15}[c_1-(816+c_1)u^8]+u^4v^{14}[d_1+(3060-d_1)u^8]$$
$$+u^5v^{13}[e_1-(8568+e_1)u^8]+u^6v^{12}[(15504+d_1)+(3060-d_1)u^8]$$
$$+u^7v^{11}[(-31008+c_1)-(816+c_1)u^8]+u^8v^{10}[(43605+b_1)+(153-b_1)u^8]$$
$$+uv^9[(238-a_1)+(-49096+2a_1)u^8+(238-a_1)u^{16}]$$
$$+u^2v^8[(153-b_1)+(43605+b_1)u^8]+u^3v^7[-(816+c_1)+(-31008+c_1)u^8]$$
$$+u^4v^6[(3060-d_1)+(15504+d_1)u^8]+u^5v^5[-(8568+e_1)+e_1u^8]$$
$$+u^6v^4[(3060-d_1)+d_1u^8]+u^7v^3[-(816+c_1)+c_1u^8]$$
$$+u^8v^2[(153-b_1)+b_1u^8]+uv[-256+(238-a_1)+a_1]+u^{18}=0.$$

Ut coefficientem primum $a_1 u + (238 - a_1) u^9 + 256 u^{17} = C_1 = -S_1^{(17)}$ determinemus, ex §. 6. scimus esse:

$$S_1^{(17)} = \sqrt{2}.[17 A_2 + 17 A_{19} r^8 + (1 + 17 A_{36}) r^{16}].r$$
$$= \sqrt{2}.[17.2r - 17.258 r^9 + \ldots.]$$

atque

$$S_1^{(17)} = -a_1 u - (238 - a_1) u^9 - 256 u^{17}$$
$$= \sqrt{2}[-a_1 r + a_1 r^9 - 2a_1 r^{17} + \ldots. - 16(238 - a_1)(r^9 - 9r^{17} + \ldots.) - \ldots.];$$

si hi ambo valores ipsius $S_1^{(17)}$ inter se comparantur, evadit:

$$a_1 = -34,$$

ergo:

$$S_1^{(17)} = 34u + \ldots. \quad \text{et} \quad C_1 = 34u + \ldots.$$

Quod ad secundum coefficientem, est:

$$b_1 u^2 + (153 - b_1) u^{10} = C_2 = -\frac{S_2^{(17)}}{2} - \frac{C_1.S_1^{(17)}}{2}.$$

Ex §. 6. vero fit:

$$S_2^{(17)} = 34.[A_4^{(2)}.r^2 + A_{21}^{(2)}.r^{10} + \ldots.]$$
$$= 34.[18 r^2 + \ldots.],$$

atque:

$$S_2^{(17)} = m u^2 + m' u^{10}$$
$$= 2m r^2 + \ldots.$$

unde ex amborum comparatione oritur:

$$m = 306, \quad \text{ergo} \quad S_2^{(17)} = 306 u^2 + \ldots.$$

unde:

$$C_2 = b_1 u^2 + (153 - b_1) u^{10} = -153 u^2 + 578 u^2 + \ldots. = +425 u^2 + \ldots.,$$

ergo:

$$b_1 = +425.$$

Tertius coefficiens est

$$c_1 u^3 - (816 + c_1) u^{11} = C_3 = -\frac{S_3}{3} - \frac{C_1.S_2}{3} - \frac{C_2.S_1}{3}:$$

Ex §. 6. fit:

$$S_3^{(17)} = 34\sqrt{2}.[A_6^{(3)}.r^3 + A_{23}^{(3)}.r^{11}]$$
$$= 34\sqrt{2}.[194 r^3 + \ldots.]$$

atque

$$S_3^{(17)} = m u^3 + m' u^{11}$$
$$= 2\sqrt{2}.m r^3 - \ldots.;$$

alter valor ipsius S_3 cum altero comparatus praebet:

$$m = 3298, \quad \text{ergo} \quad S_3^{(17)} = 3298 u^3 + \ldots.,$$

unde:

$$C_3 = c_1 u^3 - (816 + c_1) u^{11} = -1099\tfrac{1}{3} \cdot u^3 + 3468 u^3 - 4816\tfrac{2}{3} u^3 \dots$$
$$= -2448 u^3 \dots,$$

ergo:

$$c_1 = -2448.$$

Quartus coefficiens est:

$$d_1 u^4 + (3060 - d_1) u^{12} = C_4 = -\frac{S_4}{4} - \frac{C_1 S_3}{4} - \frac{C_2 S_2}{4} - \frac{C_3 S_1}{4}.$$

Ex §. 6. fit:

$$S_4^{(17)} = 68.[A_8^{(4)}.r^4 + A_{25}^{(4)}.r^{12}]$$
$$= 68.[2162 r^4 + \dots]$$

atque

$$S_4^{(17)} = m u^4 + m' u^{12}$$
$$= 4 m r^4 + \dots$$

alter valor ipsius S_4 cum altero comparatus praebet:

$$m = 36754,$$

ergo:

$$S_4^{(17)} = 36754 u^4 - \dots,$$

unde:

$$C_4 = d_1 u^4 + (3060 - d_1) u^{12}$$
$$= -9188\tfrac{1}{2} u^4 + 28033 u^4 - 32512\tfrac{1}{2} u^4 + 20808 u^4 \dots$$
$$= 7140 u^4 \dots,$$

ergo:

$$d_1 = 7140.$$

Quintus coefficiens est:

$$e_1 u^5 - (8568 + e_1) u^{13} = C_5 = -\frac{S_5}{5} - \frac{C_1 S_4}{5} - \frac{C_2 S_3}{5} - \frac{C_3 S_2}{5} - \frac{C_4 S_1}{5}.$$

Ex §. 6. fit;

$$S_5^{(17)} = 68\sqrt{2}.[A_{10}^{(5)}.r^5 + A_{27}^{(5)}.r^{13}]$$
$$= 68\sqrt{2}.[24802 r^5 + \dots]$$

atque

$$S_5^{(17)} = m u^5 + m' u^{13}$$
$$= 4\sqrt{2}.m r^5 + \dots$$

alter valor ipsius S_5 cum altero comparatus praebet:

$$m = 421634,$$

ergo:

$$S_5^{(17)} = 421634 u^5 + \dots$$

unde:

$$C_5 = e_1 u^5 - (8568 + e_1) u^{13}$$
$$= -84326\tfrac{4}{5} u^5 + 249927\tfrac{1}{5} u^5 - 280330 u^5 + 149817\tfrac{3}{5} u^5 - 48552 u^5$$
$$= -13464 u^5 - \dots$$

ergo:

$$e_1 = -13464.$$

Si denique hi valores in aequatione prius dicta ponuntur, aequatio modularis decimi septimi ordinis eruitur:

$$v^{18}-(34-272u^8+256u^{16})uv^{17}+17(25-16u^8)u^2v^{16}-816(3-2u^8)u^3v^{15}$$
$$+1020(7-4u^8)u^4v^{14}-1224(11-4u^8)u^5v^{13}+204(111-20u^8)u^6v^{12}$$
$$-816(41-2u^8)u^7v^{11}+34(1295-8u^8)u^8v^{10}+68(4-723u^8+4u^{16})uv^9$$
$$-34(8-1295u^8)u^2v^8+816(2-41u^8)a^3v^7-204(20-111u^8)u^4v^6$$
$$+1224(4-11u^8)u^5v^5-1020(4-7u^8)u^6v^4+816(2-3u^8)u^7v^3$$
$$-17(16-25u^8)u^8v^2-(256-272u^8+34u^{16})uv+u^{18}=0,$$

aut:

$$(v-u)^{18}-16uv(1-u^8)(1-v^8).[17uv(v-u)^6-(v^4-u^4)^2+16(1-u^4v^4)^2]=0.$$

Deducitur aequatio modularis pro transformatione undevicesimi ordinis.

Forma hujus aequationis est:

$$v^{20}+(a_1+a_2u^8+a_3u^{16})u^3v^{19}+(b_1+b_2u^8)u^6v^{18}+(c_1+c_2u^8+c_3u^{16})uv^{17}$$
$$+(d_1+d_2u^8)u^4v^{16}+(e_1+e_2u^8)u^7v^{15}+(f_1+f_2u^8+f_3u^{16})u^2v^{14}$$
$$+(g_1+g_2u^8)u^5v^{13}+(h_1+h_2u^8)u^8v^{12}+(i_1+i_2u^8+i_3u^{16})u^3v^{11}$$
$$+(k_1+k_2u^8)u^6v^{10}+(l_1+l_2u^8+l_3u^{16})uv^9+(m_1+m_2u^8)u^4v^8$$
$$+(n_1+n_2u^8)u^7v^7+(o_1+o_2u^8+o_3u^{16})u^2v^6+(p_1+p_2u^8)u^5v^5$$
$$+(q+q_2u^8)u^8v^4+(r_1+r_2u^8+r_3u^{16})u^3v^3+(s_1+s_2u^8)u^6v^2$$
$$+(t_1+t_2u^8+t_3u^{16})u^{20}=0.$$

Ex §. 3. lit. *e*. sequuntur:

$t_3=-a_1$; $t_2=-a_2$; $t_1=-a_3$; $s_2=-b_1$; $s_1=-b_2$; $r_3=-c_1$;
$r_2=-c_2$; $r_1=-c_3$; $q_2=-d_1$; $q_1=-d_2$; $p_2=-e_1$; $p_1=-e_2$;
$o_3=-f_1$; $o_2=-f_2$; $o_1=-f_3$; $n_2=-g_1$; $n_1=-g_2$; $m_2=-h_1$;
$m_1=-h_2$; $l_3=-i_1$; $l_2=-i_2$; $l_1=-i_3$; $k_2=-k_1$.

Ex §. 3. lit. *d*. sequuntur:

$r_3=a_1$; $i_3=a_2$; $o_3=-b_1$; $f_3=-b_2$; $t_3=c_1$; $l_3=c_2$; $q_2=-d_1$;
$h_2=-d_2$; $n_2=e_1$; $s_2=-f_1$; $k_2=-f_2$; $p_2=g_1$; $m_2=-h_1$;
$r_2=i_1$; $o_2=-k_1$; $t_2=l_1$; $q_1=-m_1$; $s_1=-o_1$.

Ex §. 3. lit. *f*. sequuntur:

$$a_1+a_2+a_3 = P^1_{19}-P^0_{19} = 19-1 = 18,$$
$$b_1+b_2 = P^2_{19}-P^1_{19} = 171-19 = 152,$$
$$c_1+c_2+c_3 = P^3_{19}-P^2_{19} = 969-171 = 798,$$
$$d_1+d_2 = P^4_{19}-P^3_{19} = 3876-969 = 2907,$$
$$e_1+e_2 = P^5_{19}-P^4_{19} = 11628-3876 = 7752,$$

$$f_1+f_2+f_3 = P_{19}^6 - P_{19}^5 = 27132-11628 = 15504,$$
$$g_1+g_2 = P_{19}^7 - P_{19}^6 = 50388-27132 = 23256,$$
$$h_1+h_2 = P_{19}^8 - P_{19}^7 = 75582-50388 = 25194,$$
$$i_1+i_2+i_3 = P_{19}^9 - P_{19}^8 = 92378-75582 = 16796,$$
$$k_1+k_2 = P_{19}^{10} - P_{19}^9 = 92378-92378 = 0.$$

Ex §. 5. *Nota.* est:

$$t_1 = -512.$$

Aequatio ergo sequentem formam induit:

$$\begin{aligned}
&v^{20}+[a_1-(494+a_1)u^8+512u^{16}]u^3v^{19}+[b_1+(152-b_1)u^8]u^6v^{18}\\
&+[-a_1+c_2u^8+(798+a_1-c_2)u^{16}]uv^{17}+[d_1+(2907-d_1)u^8]u^4v^{16}\\
&+[e_1+(7752-e_1)u^8]u^7v^{15}+[b_1+(15656-2b_1)u^8-(152-b_1)u^{16}]u^2v^{14}\\
&+[-e_1+(23256+e_1)u^8]u^5v^{13}+[(28101-d_1)-(2907-d_1)u^8]u^8v^{12}\\
&+[-c_2+(17290+a_1+c_2)u^8-(494+a_1)u^{16}]u^3v^{11}\\
&+[(15656-2b_1)-(15656-2b_1)u^8]u^6v^{10}\\
&+[(494+a_1)-(17290+a_1+c_2)u^8+c_2u^{16}]uv^9\\
&+[(2907-d_1)-(28101-d_1)u^8]u^4v^8-[(23256+e_1)-e_1u^8]u^7v^7\\
&+[(152-b_1)-(15656-2b_1)u^8-b_1]u^2v^6-[(7752-e_1)+e_1u^8]u^5v^5\\
&-[(2907-d_1)+d_1u^8]u^8v^4-[(798+a_1-c_2)+c_2u^8-a_1u^{16}]u^3v^3\\
&-[(152-b_1)+b_1u^8]u^6v^2-[512-(494+a_1)u^8+a_1]uv-u^{20}=0.
\end{aligned}$$

Ut primus coefficiens determinetur est:

$$C_1 = a_1u^3-(494+a_1)u^{11}+512u^{19} = -S_1.$$

Ex §. 6. vero fit:

$$\begin{aligned}
S_1^{(19)} &= \sqrt{2}.[19A_7r^3+19A_{26}r^{11}+(-1+19A_{45})r^{19}]\\
&= \sqrt{2}.[-19.12.r^3+\ldots.]
\end{aligned}$$

atque

$$\begin{aligned}
S_1^{(19)} &= mu^3+m'u^{11}+m''u^{19}\\
&= 2\sqrt{2}.mr^3+\ldots.,
\end{aligned}$$

alter valor ipsius S_1 cum altero comparatus praebet:

$$m = -114,$$

ergo

$$S_1^{(19)} = -114u^3+\ldots.,$$

unde:

$$C_1 = \quad 114u^3+\ldots.$$

Secundus coefficiens est:

$$b_1u^6+(152-b_1)u^{14} = C_2 = -\frac{S_2}{2}-\frac{C_1S_1}{2}.$$

Ex §. 6. fit:

$$S_2^{(19)} = 38\,[A_{14}^{(2)}.r^6 + A_{33}^{(2)}.r^{14}]$$
$$= 38.[1648\,r^6 + \ldots.]$$

atque

$$S_2^{(19)} = m\,u^6 + m'\,u^{14}$$
$$= 8\,m\,r^6 + \ldots.,$$

alter valor ipsius S_2 cum altero comparatus praebet:

$$m = 7828,$$

ergo:

$$S_2^{(19)} = 7828\,u^6 + \ldots.,$$

unde:

$$C_2 = -3914\,u^6 + 6498\,u^6 + \ldots. = 2584\,u^6 + \ldots.,$$

ergo:

$$b_1 = 2584.$$

Tertius coefficiens est:

$$-a_1 u + c_2 u^9 + (798 + a_1 - c_2)u^{17} = C_3 = -\frac{S_3}{3} - \frac{C_1 S_2}{3} - \frac{C_2 S_1}{3}.$$

Ex §. 6. fit:

$$S_3^{(19)} = 38\sqrt{2}.[A_2^{(3)}r + A_{21}^{(3)}r^9 + A_{40}^{(3)}r^{17}]$$
$$= 38\sqrt{2}.[9\,r - 255945\,r^9 + \ldots.]$$

atque

$$S_3^{(19)} = m\,u + m'\,u^9 + m''\,u^{17}$$
$$= \sqrt{2}.\{m\,r - m\,r^9 + 2\,m\,r^{17} + 16\,m'\,r^9 - 144\,m'\,r^{17} + 256\,m''\,r^{17}\},$$

alter valor ipsius S_3 cum altero comparatus praebet:

$$m = 9.\ 38,$$
$$-m + 16\,m' = -38.255945,$$

ergo:

$$m = 342,$$
$$m' = -607848,$$

ergo:

$$S_3^{(19)} = 342\,u - 607848\,u^9,$$

unde:

$$C_3 = -114\,u + 202616\,u^9 - 297464\,u^9 + 98192\,u^9 + \ldots.$$
$$= -114\,u + 3344\,u^9 + \ldots.,$$

ergo:

$$c_2 = 3344.$$

Quartus coefficiens est:

$$d_1 u^4 + (2907 - d_1) u^{12} = C_4 = -\frac{S_4}{4} - \frac{C_1 S_3}{4} - \frac{C_2 S_2}{4} - \frac{C_3 S_1}{4}.$$

Ex §. 6. fit:

$$S_4^{(19)} = 76.[A_9^{(4)} r^4 + A_{28}^{(4)} r^{12}]$$
$$= 76.[-4180 r^4 + \ldots.],$$

atque

$$S_4^{(19)} = m u^4 + m' u^{12}$$
$$= 4 m r^4 + \ldots.,$$

alter valor ipsius S_4 cum altero comparatus praebet:

$$m = -79420,$$

ergo:

$$S_4^{(19)} = -79420 u^4 + \ldots.,$$

unde:

$$C_1 = 19855 u^4 - 9747 u^4 - 3249 u^4 \ldots. = 6859 u^4,$$

ergo:

$$d_1 = 6859.$$

Quintus coefficiens est:

$$e_1 u^7 + (7752 - e_1) u^{15} = C_5 = -\frac{S_5}{5} - \frac{C_1 S_4}{5} - \frac{C_2 S_3}{5} - \frac{C_3 S_2}{5} - \frac{C_4 S_1}{5}.$$

Ex §. 6. fit:

$$S_5^{(19)} = 76\sqrt{2}.[A_{16}^{(5)} r^7 + A_{35}^{(5)} r^{15}]$$
$$= 76\sqrt{2}.[1035060 r^7 + \ldots.]$$

atque

$$S_5^{(19)} = m u^7 + m' u^{15}$$
$$= \sqrt{2}.[8 m r^7 + \ldots.],$$

alter valor ipsius S_5 cum altero comparatus praebet:

$$m = 9833070,$$

ergo:

$$S_5^{(19)} = 9833070 u^7 + \ldots.$$

unde:

$$C_1 = -1966614 u^7 + 1810776 u^7 - \frac{883728}{5} u^7 + \frac{892392}{5} u^7 + \frac{781926}{5} u^7 + \ldots.$$
$$= 2280 u^7 + \ldots.,$$

ergo:

$$e_1 = 2280.$$

Ex his aequatio modularis undevicesimi ordinis oritur haec:

$$v^{20}+(114-608u^8+512u^{16})u^3v^{19}+152(17-16u^8)u^6v^{18}$$
$$-38(3-88u^8+64u^{16})uv^{17}+19(361-208u^8)u^4v^{16}+456(5+12u^8)u^7v^{15}$$
$$+152(17+69u^8+16u^{16})u^2v^{14}-456(5-56u^8)u^5v^{13}+494(43+8u^8)u^8v^{12}$$
$$-76(44-273u^8+8u^{16})u^3v^{11}+10488(1-u^8)u^6v^{10}$$
$$+76(8-273u^8+44u^{16})uv^9-494(8+43u^8)u^4v^8-456(56-5u^8)u^7v^7$$
$$-152(16+69u^8+17u^{16})u^2v^6-456(12-5u^8)u^5v^5+19(208-361u^8)u^8v^4$$
$$+38(64-88u^8+3u^{16})u^3v^3+152(16-17u^8)u^6v^2$$
$$-(512-608u^8+114u^{16})uv-u^{20}=0.$$

His exemplis, quomodo aequatio modularis pro transformatione cujusvis ordinis deduci possit, satis demonstratur.

Halae, mens. Mart. 1836.