# ON THE DISCONTINUITIES OF A FUNCTION OF ONE OR MORE REAL VARIABLES

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1. The modern definition of a function of one or more real variables is of such a general nature that one is inclined at first to imagine that there can be, in the nature of things, little or no connection between the character of the discontinuity of the function at a point and the mode of distribution of the discontinuities. It was with some surprise therefore that I remarked that *the points at which the upper and lower limits of a function of single real variable can differ on the right and on the left are in no case more than countable*, so that, in particular, discontinuities of the first kind, sometimes called "ordinary discontinuities," are relatively speaking exceptional, while a symmetrical discontinuity of the second kind constitutes the normal type.*

Subsequently I perceived this result to be only a particular case of the more general one :—*the limits of a function of a single variable at a point, obtained by all modes of approach to the point on the right, are identical with those obtained by all modes of approach on the left, except at most at a countable set of points.*†

One is always glad to obtain a second proof of a new theorem, especially in the Theory of Sets of Points, where the reasoning is so subtle that almost all who have written on it have at one time or another stumbled. Such a second proof is furnished by the reasoning adopted in the present Note. My main object is, however, the proof of the following theorem, which, it will be seen, constitutes the analogous result for functions of two or more variables.

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* W. H. Young, "Some Results in the Theory of Functions of a Real Variable" (1907) *Report of the British Association*, p. 445 (abstract): "On the Distinction of Right and Left at Points of Discontinuity," *Quart. Jour. of Math.*, Vol. XXXIX, pp. 67–83.

† W. H. Young, "Sulle due funzioni a più valori costituite dai limiti d'una funzione d'una variabile a destra e a sinistra di ciascun punto" (1908), *Rend. della R. Acc. dei Lincei*, T. XVII, pp. 582-587.

*The whole set of limits of a function $f(x)$ of any number of variables, denoted in their ensemble by $x$, at a point $P$, may be obtained by different modes of approach to the point $P$ confined to a completely open neighbourhood of pre-assigned form with $P$ as boundary point, except possibly at a certain set of the first category,* depending on the particular function and on the pre-assigned form of neighbourhood.*

That this theorem is not a mere generalisation of the corresponding theorem for functions of a single variable will not be surprising if we reflect on one of the fundamental distinctions between the division of a straight line by means of segments and of the plane or higher space by means of compartments. In the former case the end-points of the intervals have the potency of the natural numbers; in the second case, however, the analogous points fill up lines, and have therefore, as also in higher space, the potency of the continuum. It is precisely this fact which is responsible for the modification in the theorem as we pass from the straight line to higher space. That the exceptional points considered in the theorem are in the one-dimensional case countable is, so to speak, an accidental fact; the general fact is that they form a set of the first category.

The following curious result, which seems to me not without importance in the general theory of limits, is an immediate consequence :—*Excepting only at the points of a certain set of the first category, all the limits of $f(x)$, where there are any number of variables $x$, may be obtained by means of sequences of points tangential at the point to a fixed direction, previously chosen.*

In particular, *if there is a fixed direction tangential to which $f(x)$ has an unique limit at every point, the function is necessarily only pointwise discontinuous.*

It should be noticed that in all these theorems the modes of approach are by means of regions. The main result is capable of the following mode of expression :—*There is symmetry for approach by means of regions at every point except at a set of the first category.* Here, however, it must be remembered that the set of the first category depends not only on the function, but also on the choice of the regions grouped round a point.

Another immediate result is that, *except possibly at a set of the first category, $f(P)$ lies between its upper and lower limits for all modes of approach to the point $P$ in a region of prescribed form with $P$ as boundary*

* For the definition of such a set see, for example, Young's *Theory of Sets of Points*, Camb. Univ. Press (1906), pp. 70, 71.

*point.* To obtain this from the main result, we require to assume that *$f(x)$ lies between the upper and lower limits of $f(x)$ at each point except at a countable set of points.* A proof of this theorem valid for any number of variables $x$,* is accordingly given in § 3.

2. The reasoning of the present paper depends on the systematic use of the two facts:

(A) *If a number $a$ is greater than a number $b$, there is a rational number greater than $b$ and less than $a$.*

(B) *The rational numbers are countable.*

The countable order of the rational numbers may be any conveniently chosen one; for instance, they may be read in columns from the following wedge:—

$$\left.\begin{array}{llllll} \frac{1}{2}, & \frac{1}{3}, & \frac{2}{3}, & \frac{1}{4}, & \frac{3}{4}, & \frac{1}{5}, \ldots \\ & 1+\frac{1}{2}, & 1+\frac{1}{3}, & 1+\frac{2}{3}, & 1+\frac{1}{4}, & 1+\frac{3}{4}, \ldots \\ & & -1+\frac{1}{2}, & -1+\frac{1}{3}, & -1+\frac{2}{3}, & -1+\frac{1}{4}, \ldots \\ & & \ldots & \ldots & \ldots & \ldots \quad \ldots \end{array}\right\} . \tag{1}$$

Here the first row is the usual order of the rational numbers between 0 and 1, in which the denominators form a monotone series, and so do the numerators corresponding to any the same denominator: the succeeding rows are got by adding in succession the various integers, alternately positive and negative. When arranged in this order we shall denote the rational numbers by $N_1, N_2, \ldots, N_r, \ldots$.

3. The principles employed are exhibited in their simplest form in the proof of the following theorem:—

THEOREM 1.—*If $f(x)$ is a function of any number of variables, denoted in their ensemble by $x$, and $\phi(x)$ is its associated upper limiting function,*† *the points at which*

$$\phi(x) < f(x), \tag{2}$$

*if any, form a countable set.*

Let $S_r$ denote the set of those points, if any, such that

$$\phi(x) < N_r < f(x),$$

* The original statement and proof of this theorem, given in the paper first quoted, was for one dimension.

† That is, the function whose value at each point is the upper limit of $f(x)$—that is, the lower bound of the upper bound of $f(x)$ at all points except $P$ in an interval or region with $P$ as internal point.

while no rational number of the series (1) with index less than $r$ is greater than $\phi(x)$ and less than $f(x)$. This set $S_r$ is then perfectly determined, and any point $P$ at which

$$\phi(P) < f(P)$$

belongs to one, and one only, of the sets $S_r$. Thus the whole set of points, if any, at which

$$\phi(x) < f(x) \tag{2}$$

is the sum $S_1+S_2+\ldots+S_r+\ldots$, of these sets, and is therefore countable if $S_r$ is countable for all values of the integer $r$; this we shall now prove to be the case.

Let $P$ be any point of the set $S_r$. Then, by the definition of $\phi(P)$, we can find a whole interval or region with $P$ as internal point, such that at every point $x$ of it, except $P$,

$$f(x) < N_r.$$

so that no point of it, except $P$, belongs to the set $S_r$. Thus $P$ is an isolated point of the set $S_r$. Hence $S_r$ consists entirely of isolated points, and is therefore countable,* which proves the theorem.

4. THEOREM 2.—*If $\theta(P)$ denote the upper limit at a point $P$ of a function $f(x)$ of two or more variables denoted in their ensemble by $x$, that upper limit being determined with respect to a certain neighbourhood having $P$ as boundary point, e.g., a $(+, +)$-axial neighbourhood,† and $\phi(x)$ is the associated upper limiting function of $f$, then*

$$\theta(x) = \phi(x), \tag{3}$$

*except at a set of points of the first category.*

Let $S_r$ denote the set of all those points $x$ at which (3) does not hold, and such that $N_r$ is the first of the succession (1) of rational numbers which is greater than $\theta(x)$ and less than $\phi(x)$. Then any point at which

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* By a theorem of Cantor's. Young's *Theory of Sets of Points*, p. 42.

† In the case of two independent variables we suppose the axes of coordinates to be horizontal and vertical, with positive values of $x_1$ on the right, and positive values of $x_2$ above. The parallels to the axes through any point $P$ are then said to form *the axial cross* through $P$, and divide the neighbourhood of $P$ into four quadrants, of which the $(+, +)$-quadrant has $P$ as left-hand bottom corner. By *a $(+, +)$-axial neighbourhood* of $P$ is meant any completely open rectangle in this $(+, +)$-quadrant, bounded on the left and at the bottom by the axial cross through $P$.

A similar notation enables us to distinguish axial neighbourhoods in the other quadrants. When there are more variables, we can in like manner distinguish the various axial neighbourhoods at a point.

(3) does not hold belongs to one, and only one, of the sets $S_r$, so that the set of points at which (3) does not hold is the sum $S_1+S_2+\ldots+S_r+\ldots$ of the sets $S_r$, and is therefore a set of the first category if $S_r$ is dense nowhere for every integer $r$. This we shall now prove to be the case.

Let $d$ denote any completely open region. Then either in $d$ there is no point of $S_r$, or there is such a point $P$. In the latter case

$$\theta(P) < N_r < \phi(P).$$

Hence, by the definition of $\theta$, there is a completely open region $d'$, with $P$ as one of its boundary points, sufficiently small to lie inside the region $d$, and such that at every internal point of $d'$,

$$f(x) < N_r;$$

and therefore $$\phi(x) \leqslant N_r,$$

so that no point internal to $d'$ is a point of $S_r$.

Thus, whatever region $d$ be taken, there is a region internal to it containing no points of the set $S_r$; that is, the set $S_r$ is dense nowhere, which proves the theorem.

COR.—*If there are two independent variables, and* $\phi_{++}$, $\phi_{+-}$, $\phi_{-+}$, $\phi_{--}$, *denote the associated upper limiting functions taken with respect to the axial neighbourhoods at P, then*

$$\phi_{++} = \phi_{+-} = \phi_{-+} = \phi_{--} = \phi,$$

*except at a set of the first category.*

5. In the case when there is only one independent variable, we can go a step further and prove the theorem already referred to, that

$$\phi_L = \phi_R,$$

*except at a countable set of points.*

For the small region $d'$ of the preceding proof is then an interval with $P$ as end-point, so that $P$ is an isolated point or only a one-sided limiting point of the set $S_r$. The set $S_r$ contains therefore no points which are limiting points of the set on both sides, and is therefore a countable set.* Hence the set $S_1+S_2+\ldots+S_r+\ldots$, at which (3) does not hold, is also a countable set.

6. THEOREM 3.—*If $f(x)$ is a function of two or more real variables $x$, its limits at any point $P$ obtained from any one of the axial neighbour-*

* Young's *Theory of Sets of Points*, p. 42, § 20, Cor.; the intervals of that corollary being taken to be the black intervals of the set got by closing $S_r$.

*hoods are the same as those obtained from any other axial neighbourhood, unless $P$ belongs to a certain set of the first category, depending on the particular function $f(x)$.*

Let $P$ be any point at which there is a limit obtained from one of the axial neighbourhoods, say a $(+, +)$-limit, which is not a limit obtained from some other, say not a $(+, -)$-limit. Then there is a gap in the closed set of the $(+, -)$-limits at $P$, that is, there are two $(+, -)$-limits $G(P)$ and $H(P)$, between which there do not lie any $(+, -)$-limits, while there does lie at least one $(+, +)$-limit.

This gap determines therefore a pair of integers $(i, j)$, such that $N_i$ is the first rational number of the succession (1), which is greater than $G(P)$ and less than $H(P)$, and such that between it and $H(P)$ there is at least one $(+, +)$-limit; while $N_j$ is the first rational number greater than $N_i$ and less than $H(P)$, and such that between it and $N_i$ there is at least one $(+, +)$-limit.

Hence, if $S_{i,j}$ denote the set of all those points $P$ at which the particular pair of integers $(i, j)$ is so determined, every point of every one of the sets $S_{i,j}$ for all integers $i$ and $j$, is an unsymmetrical discontinuity of $f(x)$ at which there is a $(+, +)$-limit which is not at the same time a $(+, -)$-limit. Conversely, every such unsymmetrical discontinuity belongs to *at least one* of the sets $S_{i,j}$. Since there is precisely a countably infinite number of the sets $S_{i,j}$, it follows that these unsymmetrical discontinuities form a set of the first category, provided each set $S_{i,j}$ is dense nowhere. This we shall now prove to be the case.

For this purpose it is sufficient to shew that if a completely open region contains any point $P$ of $S_{i,j}$, it contains a $(+, -)$-neighbourhood of $P$ free from points of $S_{i,j}$. This we shall prove by a *reductio ad absurdum.*

Suppose there were a sequence of points $P_1, P_2, \ldots$ lying in the $(+, -)$-neighbourhood of $P$, with $P$ as limiting point, and each, as well as $P$, belonging to the set $S_{i,j}$. Then there is a $(+, +)$-limit at $P_n$ lying in the closed interval $(N_i, N_j)$, and therefore a determinate one $g(P_n)$ which is nearest to $N_i$ since such limits form a closed set.

Since $g(P_n)$ is a limit of $f(x)$ at $P_n$, there is a sequence of points along which $f(x)$ has at $P_n$ the unique limit $g(P_n)$, and since

$$N_i \leqslant g(P_n) \leqslant N_j,$$

we may commence this sequence at such a point that at every point of it

$$N_i - 2^{-n} < f(x) < N_j + 2^{-n}.$$

Moreover, since $P_n$ is a point of a $(+, -)$-neighbourhood of $P$, we may take it that every point of this sequence belongs also to a $(+, -)$-neighbourhood of $P$. Let this sequence be

$$Q_{n,1}, \quad Q_{n,2}, \quad \ldots, \quad Q_{n,r}, \quad \ldots .$$

Since $P$ is the limiting point of the sequence $P_1, P_2, \ldots, P_n, \ldots$, and each point $P_n$ is the limiting point of the sequence

$$Q_{n,1}, \quad Q_{n,2}, \quad \ldots, \quad Q_{n,r}, \quad \ldots,$$

we can find a point from each of these sequences, so as to form a sequence,

$$Q_{1,r_1}, \quad Q_{2,r_2}, \quad \ldots, \quad Q_{n,r_n}, \quad \ldots,$$

having $P$ as limiting point. As we pass along this sequence the limits of $f(x)$ which we obtain, being $(+, -)$-limits, cannot lie in the closed interval $(N_i, N_j)$. But, by the preceding inequality,

$$N_i - 2^{-n} < f(Q_{n,r_n}) < N_j + 2^{-n},$$

so that the limits in question cannot lie outside the interval $(N_i, N_j)$. Thus the assumption that there is such a sequence $P_1, P_2, \ldots$ leads to an *impasse* and must be abandoned.

Hence there must be a $(+, -)$-neighbourhood of $P$ entirely free of points of the set $S_{i,j}$, which proves that $S_{i,j}$ is dense nowhere, and therefore that the set of points at which there is a $(+, +)$-limit which is not a $(+, -)$-limit is a set of the first category.

Taking the axial quadrants in pairs, this shews that the points at which the limits obtained from any one of the axial neighbourhoods are not precisely the same as those obtained from any other, form a finite number of sets of the first category, and therefore form a set of the first category. Q.E.D.

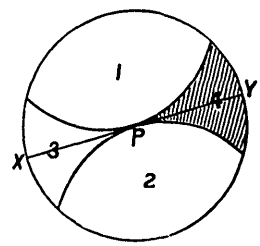
7. The preceding theorem has been stated for functions of two or more variables, and the proof has been given as if there were two variables. It is clear, however, that the proof is valid for any number of dimensions, only requiring a proper interpretation of the symbols $(+, +)$ and $(+, -)$, or the substitution for them of other symbols.

In particular, if there is only a single independent variable we get, as in § 5, an alternative proof of the theorem of the Lincei paper quoted in the introduction.

Again, it was only for the sake of clearness that the neighbourhoods used in defining the limits were taken to be axial neighbourhoods. The argument would hold as it stands if we assumed any other law of neighbourhoods by which the forms of neighbourhood at any point are non-overlapping.

In this case, in fact, there will be a finite or countable infinite set of forms of neighbourhood at a point, just as there were four axial neighbourhoods at a point in a plane. Hence, by the preceding proof, taking the neighbourhoods two by two, we get a countably infinite set of sets of the first category, that is, a set of the first category, of exceptional points.

In particular, let us choose the following law of neighbourhoods :—




Let a circle be drawn with $P$ as centre (see figure), whose diameter in a fixed direction is $XY$. Draw two other circles with the same radius, touching $XY$ at $P$. We thus determine four neighbourhoods at $P$, numbered 1, 2, 3, and 4 in the figure. Since the limits obtained from the neighbourhood 4 are the same as in the neighbourhoods 1, 2, and 3, except at a certain set of the first category, it follows that *excepting only at the points of a certain set of the first category, all the limits of a function of two variables at any point may be obtained by means of sequences of points tangential at the point to a fixed direction, previously chosen.*

This theorem is evidently true for functions of any number of variables, in the case of three variables we need only rotate the accompanying figure about the line $XY$ to get suitably forms of neighbourhood, and similarly we can define suitable forms of neighbourhood when there are more variables.