THE MEAN VALUE OF THE MODULUS OF AN ANALYTIC FUNCTION

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1. Suppose that $f(x)$ is an analytic function of the complex variable x, regular for $|x| < \rho$, and that $M(r)$ denotes, as usual, the maximum of $|f(x)|$ on the circle $|x| = r < \rho$. Then it is known that $M(r)$ possesses the following properties:—

(i) $M(r)$ is a steadily increasing function of r;

(ii) $\log M(r)$ is a convex function of $\log r$, so that

$$\log M(r) \leqslant \frac{\log (r_2/r)}{\log (r_2/r_1)} \log M(r_1) + \frac{\log (r/r_1)}{\log (r_2/r_1)} \log M(r_2),$$

if
$$0 < r_1 \leqslant r \leqslant r_2 < \rho.$$

Further, when $f(x)$ is an integral function, so that $\rho = \infty$, it is known that

(iii) $M(r)$ tends to infinity with (r), and, unless $f(x)$ is a polynomial, more rapidly than any power of r.*

It was suggested to me by Dr. H. Bohr and Prof. E. Landau, rather more than a year ago, that the property (i) is possessed also by the *mean* value of $|f(x)|$ on the circle $|x| = r$, *i.e.*, by the function

$$\mu(r) = \frac{1}{2\pi} \int_0^{\pi} |f(re^{i\theta})| \, d\theta.$$

* The theorems (i) and (iii) are classical. Theorem (ii) was discovered independently by Blumenthal (*Jahresbericht der Deutschen Math.-Vereinigung*, Vol. 16, p. 97), Faber (*Math. Annalen*, Vol. 63, p. 549), and Hadamard (*Bulletin de la Soc. Math. de France*, Vol. 24, p. 186). The first statement of the theorem was due to Hadamard and the first proof to Blumenthal. The theorem is a corollary of one concerning the associated radii of convergence of a power series in two variables, due to Fabry (*Comptes Rendus*, Vol. 134, p. 1190), and Hartogs (*Math. Annalen*, Vol. 62, p. 1).

In the attempt to prove this I have been led to prove a good deal more, in particular that the function $\mu(r)$, and the more general function

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{\pi} |f(re^{i\theta})|^{\delta}\, d\theta,$$

where δ is any positive number, possesses *all* the properties (i)–(iii) characteristic of $M(r)$. It should be observed that this is obvious when $\delta = 1$ and $\sqrt{\{f(x)\}}$ is one-valued for $r < \rho$; for then we have

$$\sqrt{\{f(x)\}} = b_0 + b_1 x + b_2 x^2 + \ldots,$$

say, and $$\mu(r) = |b_0|^2 + |b_1|^2 r^2 + |b_2|^2 r^4 + \ldots .$$

2. The argument of the following paragraphs depends on two lemmas concerning conjugate functions*.

Suppose that $$x = \xi + i\eta,$$

and that $$X = \Xi + i\mathrm{H}$$

is a function of x regular for all values of x under consideration. Then Ξ and H are real conjugate functions of ξ and η.

Let ψ be a real function of Ξ and H, and so of ξ and η, with continuous second derivatives. Then the lemmas in question are expressed by the formulæ

$$\text{(A)} \qquad \frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = \left(\frac{\partial^2\psi}{\partial\Xi^2} + \frac{\partial^2\psi}{\partial\mathrm{H}^2}\right) M^2,$$

$$\text{(B)} \qquad \left(\frac{\partial\psi}{\partial\xi}\right)^2 + \left(\frac{\partial\psi}{\partial\eta}\right)^2 = \left\{\left(\frac{\partial\psi}{\partial\Xi}\right)^2 + \left(\frac{\partial\psi}{\partial\mathrm{H}}\right)^2\right\} M^2,$$

where $M = \left|\dfrac{dX}{dx}\right| = \sqrt{\left\{\left(\dfrac{\partial\Xi}{\partial\xi}\right)^2 + \left(\dfrac{\partial\mathrm{H}}{\partial\xi}\right)^2\right\}} = \sqrt{\left\{\left(\dfrac{\partial\Xi}{\partial\eta}\right)^2 + \left(\dfrac{\partial\mathrm{H}}{\partial\eta}\right)^2\right\}}.$

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* The use of these lemmas was suggested to me by Dr. Bromwich, at a time when the paper contained only a part of its present contents. The whole argument has been reconstructed in consequence of this suggestion, and is much more concise and elegant than it was before. I am also indebted to Dr. Bromwich and to a referee for a number of minor suggestions. The lemmas themselves are given in Clerk-Maxwell's *Electricity and Magnetism*, Vol. 1, p. 289, and Dr. Bromwich informs me that they are due to Lamé ("Mémoire sur les Lois de l'Équilibre du Fluide Éthéré", *Journal de l'École Polytechnique*, Vol. 3, cahier 23).

The formula (A) and (B) may be proved as follows. From the equations

$$\frac{\partial\psi}{\partial\xi} = \frac{\partial\psi}{\partial\Xi}\frac{\partial\Xi}{\partial\xi} + \frac{\partial\psi}{\partial H}\frac{\partial H}{\partial\xi}, \quad \dots, \quad \dots, \quad \dots,$$

$$\frac{dX}{dx} = \frac{\partial X}{\partial\xi} = -i\frac{\partial X}{\partial\eta} = \frac{\partial\Xi}{\partial\xi} + i\frac{\partial H}{\partial\xi} = -i\frac{\partial\Xi}{\partial\eta} + \frac{\partial H}{\partial\eta},$$

it is easy to deduce that

(1) $$\frac{\partial\psi}{\partial\xi} - i\frac{\partial\psi}{\partial\eta} = \left(\frac{\partial\psi}{\partial\Xi} - i\frac{\partial\psi}{\partial H}\right)\mu,$$

(2) $$\frac{\partial\psi}{\partial\xi} + i\frac{\partial\psi}{\partial\eta} = \left(\frac{\partial\psi}{\partial\Xi} + i\frac{\partial\psi}{\partial H}\right)\bar{\mu},$$

where $\mu = \dfrac{dX}{dx}$ and $\bar{\mu}$ is the conjugate of μ. The formula (B) follows at once by multiplication. To prove (A) we operate on (1) with the operator

$$\frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta},$$

and apply (2), observing that $$\left(\frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta}\right)\mu = 0.$$

3. Suppose now that $X = f(x)$ is regular for $|x| < \rho$, and that D is an annular region, defined by inequalities of the form

$$0 < r_1 \leqslant r = |x| \leqslant r_2 < \rho,$$

and including no zeros of $f(x)$.

Let $$\log x = \log r + i\theta = \zeta = \rho + i\theta,$$
$$\log X = \log R + i\Theta = Z = P + i\Theta,$$

where $$r > 0, \quad R > 0, \quad -\pi < \theta \leqslant \pi, \quad -\pi < \Theta \leqslant \pi.$$

Then P and Θ are conjugate functions of ρ and θ, with second derivatives continuous for all values of ρ and θ which correspond to values of x in D.

Let us take $$\psi = F(R) = \phi(P),$$

where $F(R)$ is a function with a continuous second differential coefficient. Applying Lemma A, we obtain

(1) $$\frac{\partial^2\phi}{\partial\rho^2} + \frac{\partial^2\phi}{\partial\theta^2} = \frac{\partial^2\phi}{\partial P^2}M^2,$$

where

(2) $$M^2 = \left|\frac{dZ}{d\zeta}\right|^2 = \left(\frac{\partial P}{\partial\rho}\right)^2 + \left(\frac{\partial\Theta}{\partial\rho}\right)^2.$$

Let us now suppose that $\log\phi(P)$ is a positive and convex function

of P, so that
$$\frac{\partial^2}{\partial P^2}\log\phi(P) \geqslant 0,$$
or
$$\phi\frac{\partial^2\phi}{\partial P^2} \geqslant \left(\frac{\partial\phi}{\partial P}\right)^2;$$
and let

(3)
$$\nu(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\phi(P)\,d\theta.$$
Then
$$\nu'(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial\phi}{\partial P}\,\frac{\partial P}{\partial\rho}\,d\theta,$$
$$|\nu'(\rho)| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\partial\phi}{\partial P}\right|\,\left|\frac{\partial P}{\partial\rho}\right|d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi}\sqrt{\left\{\phi\frac{\partial^2\phi}{\partial P^2}\right\}}\,\left|\frac{\partial P}{\partial\rho}\right|d\theta,$$
and so, by Schwarz's inequality,

(4)
$$\{\nu'(\rho)\}^2 \leqslant \frac{1}{4\pi^2}\int_0^{2\pi}\phi\,d\theta\int_0^{2\pi}\frac{\partial^2\phi}{\partial P^2}\left|\frac{\partial P}{\partial\rho}\right|^2 d\theta \leqslant \frac{\nu(\rho)}{2\pi}\int_0^{2\pi}\frac{\partial^2\phi}{\partial P^2}M^2 d\theta.$$
But
$$\nu''(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2\phi}{\partial\rho^2}\,d\theta$$
and
$$\frac{\partial^2\phi}{\partial\rho^2}+\frac{\partial^2\phi}{\partial\theta^2} = \frac{\partial^2\phi}{\partial P^2}M^2,$$
by (1). Hence

(5)
$$\nu''(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2\phi}{\partial P^2}M^2\,d\theta - \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2\phi}{\partial\theta^2}\,d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2\phi}{\partial P^2}M^2\,d\theta,$$
since ϕ is a function of P or of R only, and R is periodic in θ. From (4) and (5) it follows that

(6)
$$\nu(\rho)\,\nu''(\rho) \geqslant \{\nu'(\rho)\}^2,$$
or that $\log\nu$ is a convex function of ρ.

We have thus proved

THEOREM I.—*If* $$\log\{\phi(\log R)\}$$
is a convex function of $\log R$, *then*
$$\log\nu(\log r) = \log\left\{\frac{1}{2\pi}\int_0^{2\pi}\phi(\log R)\,d\theta\right\}$$
is, throughout any interval of values of r which includes no zeros of $f(x)$, *a convex function of* $\log r$.

In particular, we may take

$$F(R) = \phi(\mathrm{P}) = e^{\delta \mathrm{P}} = R^{\delta},$$

in which case $\phi\phi'' = \phi'^2$. It follows that $\log \mu_\delta(r)$, and in particular $\log \mu(r)$, is a convex function of $\log r$, throughout any interval of values of r which includes no zeros of $f(x)$. This case is indeed the critical case of Theorem I, the condition that $\phi(\mathrm{P})$ should be a convex function of P being only just satisfied.

4. With Theorem I we may associate another theorem, in which less is postulated and less proved.

THEOREM II.—*If* $\phi(\log R)$ *is a convex function of* $\log R$, *then* $\nu(\log r)$ *is a convex function of* $\log r$.

For $\nu''(\rho)$ is positive, by (5) of § 3. The critical case of Theorem II is that in which $\phi(\log R) = \log R$. In this case we have, by a well known theorem of Jensen*,

$$\nu(\log r) = \frac{1}{2\pi}\int_0^{2\pi} \log R\, d\theta = \log \left| \frac{cr^n}{a_{m+1} a_{m+2} \ldots a_n} \right|,$$

where
$$f(x) = cx^m + \ldots,$$

and $a_{m+1}, a_{m+2}, \ldots, a_n$ are the zeros of $f(x)$, other than the origin, whose moduli are not greater than r. In this case $\nu(\log r)$ is a linear function of $\log r$ throughout any interval of values of r which includes no zeros of $f(x)$.

5. In order to proceed further with our investigations concerning $\mu_\delta(r)$, we must examine the behaviour of $\mu_\delta(r)$ for the exceptional values of r which correspond to zeros of $f(x)$, and for $r = 0$. I shall prove that

$$r\frac{d\mu_\delta(r)}{dr}$$

is continuous without exception.

Let
$$x_0 = \rho e^{i\phi} \quad (\rho > 0)$$

be a zero of $f(x)$. We have to prove that

$$\frac{d\mu_\delta(r)}{dr}$$

* *Acta Mathematica*, Vol. 22, p. 359.

is continuous throughout an interval of values of r of the type

$$\rho-\eta \leqslant r \leqslant \rho+\eta.$$

I shall suppose, for simplicity, that x_0 is the only zero of modulus ρ. The proof is substantially the same when there are several such zeros. I shall prove that the integral

$$\frac{d\mu_\delta(r)}{dr} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial R^\delta}{\partial r}\, d\theta$$

is uniformly convergent throughout the interval $\rho-\eta \leqslant r \leqslant \rho+\eta$, if η is small enough.

We have $$f(x) = (x-x_0)^m f_1(x),$$

where m is a positive integer, and $f_1(x)$ has no zeros whose modulus lies between $\rho-\eta$ and $\rho+\eta$, so that $|f_1(x)|$ lies between positive constants H_1 and H_2.

Now, taking $\psi = F(R)$ in Lemma B, we have

$$\left(\frac{\partial F}{\partial \xi}\right)^2 + \left(\frac{\partial F}{\partial \eta}\right)^2 = \left\{\left(\frac{\partial F}{\partial \Xi}\right)^2 + \left(\frac{\partial F}{\partial \mathrm{H}}\right)^2\right\} \left|\frac{df}{dx}\right|^2.$$

In particular, if $$F(R) = R = \sqrt{(\Xi^2+\mathrm{H}^2)},$$

we have $$\left(\frac{\partial R}{\partial \xi}\right)^2 + \left(\frac{\partial R}{\partial \eta}\right)^2 = \left|\frac{df}{dx}\right|^2,$$

and $$\left|\frac{\partial R}{\partial r}\right| = \left|\cos\theta \frac{\partial R}{\partial \xi} + \sin\theta \frac{\partial R}{\partial \eta}\right| \leqslant \sqrt{\left\{\left(\frac{\partial R}{\partial \xi}\right)^2 + \left(\frac{\partial R}{\partial \eta}\right)^2\right\}} = \left|\frac{df}{dr}\right|.$$

But $$\frac{df}{dx} = m(x-x_0)^{m-1} f_1(x) + (x-x_0)^m \frac{df_1}{dx},$$

and so $$\left|\frac{df}{dx}\right| < K|x-x_0|^{m-1},$$

where K is a constant. Hence

(5) $$\left|\frac{\partial R}{\partial r}\right| < K|x-x_0|^{m-1}.$$

Also

(6) $$R^{\delta-1} < H_2^{\delta-1}|x-x_0|^{m(\delta-1)}$$

if $\delta > 1$, and

(6′) $$R^{\delta-1} < H_1^{\delta-1}|x-x_0|^{m(\delta-1)},$$

if $\delta < 1$. From (5) and (6) or (6′) it follows that

$$\left| R^{\delta-1} \frac{\partial R}{\partial r} \right| < K_1 \, | x - x_0 |^{m\delta - 1}, \tag{7}$$

where K_1 is a constant. If $m\delta - 1 \geqslant 0$, we have

$$\left| R^{\delta-1} \frac{\partial R}{\partial r} \right| < K_2,$$

where K_2 is a constant, and then the integral

$$\int_0^{2\pi} R^{\delta-1} \frac{\partial R}{\partial r} \, d\theta$$

is obviously uniformly convergent. If, on the other hand, $m\delta - 1 < 0$, we have

$$| x - x_0 | = \sqrt{(r^2 + \rho^2 - 2r\rho \cos \omega)},$$

where $\omega = \theta - \phi$, and so

$$| x - x_0 | > K_3 \, | \sin \tfrac{1}{2}\omega |,$$

where K_3 is a constant. The uniform convergence of the integral then follows at once when we compare it with

$$\int_0^{2\pi} | \sin \tfrac{1}{2}\omega |^{m\delta - 1} \, d\omega.$$

6. We have thus proved that $\log \mu_\delta(r)$ is a convex function of $\log r$ for all positive values of r save certain exceptional values, and that

$$\frac{d \log \mu_\delta(r)}{d \log r}$$

is continuous even for these values of r. It follows that $\log \mu_\delta(r)$ is a convex function of $\log r$ for all positive values of r without exception*. *A fortiori* is $\mu_\delta(r)$ a convex function of $\log r$, and

$$r \frac{d\mu_\delta(r)}{dr},$$

an increasing function of r.

It remains to consider the behaviour of $r \dfrac{d\mu_\delta(r)}{dr}$ as $r \to 0$. Suppose that the origin is a zero of $f(x)$ of order m. Then

$$R^\delta = r^{m\delta} R_1^\delta,$$

* A series of continuous convex arcs, fitted together so as to have the same tangents at the points of junction, forms a single convex curve.

where R_1 is positive and has continuous derivatives. Hence

$$r\frac{d\mu_\delta(r)}{dr} = m\delta\, r^{m\delta}\int_0^{2\pi} R_1^\delta\, d\theta + r^{m\delta+1}\int_0^{2\pi}\frac{\partial R_1^\delta}{\partial r}\, d\theta,$$

which plainly tends to zero as $r \to 0$.

Thus $r\dfrac{d\mu_\delta(r)}{dr}$ is continuous and steadily increasing for all positive values of r, and tends to zero as $r \to 0$. It follows that

$$r\frac{d\mu_\delta(r)}{dr} \geqslant 0$$

for all positive values of r.

We have thus proved

THEOREM III.—*The integral*

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{\pi} R^\delta\, d\theta \quad (\delta > 0)$$

is a positive, continuous, and steadily increasing function of r. The same is true of

$$r\frac{d\mu_\delta(r)}{dr}.$$

And $\log \mu_\delta(r)$, *and a fortiori* $\mu_\delta(r)$ *itself, is a convex function of* $\log r$.

7. The last theorem contains *inter alia* the answer to the question raised by Bohr and Landau. It should, however, be observed that the most appropriate measure of the "average increase" of $f(x)$ is not the mean value of R, or of any power of R, but of $\log R$; for the former means are not adequately affected by the occurrence of zeros of $f(x)$, or of arcs on which R is small.

8. It remains to discuss the analogues for $\mu_\delta(r)$ of the property (iii) of § 1.

We may suppose without loss of generality that $f(x)$ has infinitely many zeros. If it has not, it is of the form

$$P(x)\, e^{g(x)},$$

where $P(x)$ is a polynomial and $g(x)$ an integral function. Now

$$e^{\frac{1}{2}\delta g(x)} = b_0 + b_1 x + b_2 x^2 + \ldots,$$

say; and

$$\mu_\delta^{(1)}(r) = \frac{1}{2\pi}\int_0^{2\pi} |e^{g(x)}|^\delta \, d\theta = |b_0|^2 + |b_1|^2 r^2 + |b_2|^2 r^4 + \ldots,$$

certainly tends to infinity more rapidly than any power of r. It follows immediately that the same is true of $\mu_\delta(r)$.

Suppose, then, that $f(x)$ has an infinity of zeros, and that $r_{m+1}, r_{m+2}, \ldots, r_n$ are the moduli of those, other than the origin, whose moduli do not exceed r. Then, if $g(\theta)$ is any continuous function of θ, we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{g(\theta)} \, d\theta \geqslant e^{\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, d\theta};$$

and so $$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{2\pi} R^\delta \, d\theta \geqslant e^{\frac{\delta}{2\pi}\int_0^{2\pi}\log R\, d\theta} = \left|\frac{cr^n}{r_{m+1}\, r_{m+2} \ldots r_n}\right|^\delta,$$

by Jensen's theorem. It follows at once that $\mu_\delta(r)$ tends to infinity with r more rapidly than any power of r. We can indeed go further, and establish relations between the rate of increase of r_n, considered as a function of n, and $\mu_\delta(r)$, considered as a function of r, in every way analogous to those given by Jensen's theorem for $M(r)$.* For example, if the "real order" of $f(x)$ is ρ, we have

$$\mu_\delta(r) > e^{r^{\rho-\epsilon}}$$

for every positive ϵ and values of r surpassing all limit.

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* Lindelöf, *Acta Societatis Fennicae*, Vol. 31, No. 1; see also Borel, *Leçons sur les fonctions méromorphes*, p. 105.