viduals and pluralities of which he conceives that subject-matter to consist, we seem to be within measurable distance of a reply to our question—What is mathematics?

It is notoriously dangerous to attempt to formulate definitions; but, on the other hand, it must not be forgotten that a bad definition may provoke a good one. I would therefore venture provisionally to suggest that:—

Mathematics is the science by which we investigate those characteristics of any subject-matter of thought which are due to the conception that it consists of a number of differing and non-differing individuals and pluralities.

If the result of this attempt be only to elicit conclusive proof that mathematics is something else, and an indication of what it really is, my main object in this brief address will have been attained.

Third Memoir on the Expansion of certain Infinite Products. By L. J. Rogers. Received October 31st, 1894. Read November 8th, 1894.

Abbreviations used in the following Memoir.

(1) $(x) \equiv (1-x)(1-xq)(1-xq^2)\ldots.$

(2) $P(x) \equiv 1/(xe^{\theta i})(xe^{-\theta i}) \equiv 1+\dfrac{A_1(\theta)}{q_1}x+\dfrac{A_2(\theta)}{q_2!}x^2+\ldots$, where

$$q_r \equiv 1-q^r, \quad \text{and} \quad q_r! \equiv (1-q)(1-q^2)\ldots(1-q^r).$$

(3) $(-xqe^{\theta i})(-xqe^{-\theta i}) \equiv 1+\dfrac{B_1(\theta)}{q_1}x+\dfrac{B_2(\theta)}{q_2!}x^2+\ldots.$

(4) $P(x\lambda)/P(x) \equiv 1+\dfrac{L_1(\theta)}{q_1}x+\ldots.$

(5) $\lambda_r \equiv 1-\lambda q^{r-1}$, $\mu_r \equiv 1-\mu q^{r-1}$; $\lambda_r! \equiv \lambda_1\lambda_2\ldots\lambda_r$, $\mu_r! \equiv \mu_1\mu_2\ldots\mu_r$, and

$$\lambda_r^2! \equiv (1-\lambda^2)(1-\lambda^2 q)\ldots(1-\lambda^2 q^{r-1}), \text{ \&c.}$$

(6) $H_r(a, b, \ldots / \alpha, \beta, \ldots) \equiv$ coefficient of x^r in $(ax)(bx)\ldots/(\alpha x)(\beta x)\ldots.$

(7) $\phi\{a, b, c, q, x\} \equiv 1+\dfrac{a_1 b_1}{q_1 c_1}x+\dfrac{a_2!\, b_2!}{q_2!\, c_2!}x^2+\ldots.$

(8) $f_r \equiv 1-2q^{r-1}\cos 2\phi+q^{2r-1}.$

1. In investigating the properties of the function $A_r(\theta)$, we have hitherto been concerned in establishing relations connecting the coefficients in a supposed identity of the form

$$a_0+a_1A_1(\theta)+a_2A_2(\theta)+\ldots = b_0+2b_1\cos\theta+2b_2\cos 2\theta+\ldots \quad \ldots(1),$$

and applying such relations in a manner so as to obtain various identities involving a single quantity q.

It is easily seen that the method of application (§ 7 of the Second Memoir) implies an identity

$$(q)\left\{1+\frac{qA_2(\theta)}{q_2!}+\frac{q^4A_4(\theta)}{q_4!}+\ldots\right\} = 1+2q\cos 2\theta+2q^4\cos 4\theta+\ldots \quad \ldots\ldots\ldots\ldots\ldots(2),$$

and that all the deductions from the subsequent "examples" could have been derived from such an identity.

It is the object of the present memoir to prove identities of type (1) to which we may apply relations of the type

$$a_0+m_1a_1+m_2a_2+\ldots = b_0+n_1b_1+n_2b_2+\ldots,$$

always remembering that we may separate into independent identities those terms which contain even suffixes from those which contain odd.

For this purpose we shall investigate some of the properties of a generalized function similar to $A_r(\theta)$, obtained by expanding the quotient $P(\lambda x)/P(x)$ in powers of x.

Suppose this expansion denoted by

$$1+\frac{L_1(\theta)}{q_1}x+\frac{L_2(\theta)}{q_2!}x^2+\ldots,$$

or by

$$1+\Sigma\frac{L_r}{q_r!}x^r,$$

when the omission of θ involves no ambiguity.

Now, since

$$\frac{P(\lambda x)}{P(x)}(1-2x\cos\theta+x^2) = \frac{P(\lambda xq)}{P(xq)}(1-2\lambda x\cos\theta+\lambda^2x^2),$$

it is easy to see that

$$L_r-2\cos\theta\,.\,L_{r-1}(1-\lambda q^{r-1})+L_{r-2}(1-q^{r-1})(1-\lambda^2q^{r-2}) = 0\ldots(3).$$

We may, moreover, notice a few obvious properties of L_r.

Since, by definition,

$$\lambda_1 + \frac{\lambda_2 L_1}{q_1!} x + \frac{\lambda_3 L_2}{q_2!} x^2 + \ldots = \frac{P(x\lambda)}{P(x)} - \lambda \frac{P(xq\lambda)}{P(x)}$$

$$= (1-\lambda)(1-x^2\lambda) \frac{P(xq\lambda)}{P(x)} \quad \ldots\ldots\ldots (4),$$

we may express L_r in terms of similar coefficients in which λq replaces λ.

The following values of L_r for special values of λ are also worthy of notice.

If $\lambda = 0$, L_r becomes A_r,

$\lambda = q$, $\dfrac{P(\lambda x)}{P(x)}$ becomes $\dfrac{1}{1-2x\cos\theta + x^2}$,

so that $L_r = \dfrac{\sin(r+1)\theta}{\sin\theta} q_r!$;

if $\lambda = 1$, $\dfrac{\lambda_{r+1} L_r}{\lambda_1}$ becomes $2\cos r\theta \,.\, q_r!$, by (4),

$\lambda = \infty$, $\dfrac{L_r}{(-\lambda)^r}$ becomes $\dfrac{B_r}{q^r}$.

Moreover, if $\lambda = q^n$, and q is made equal to unity, then $\dfrac{L_r}{q_r!}$ is the coefficient of x^r in the expansion of $(1-2x\cos\theta + x^2)^{-n}$.

It is easy to obtain the values of L_r by multiplying together the series for $\dfrac{(\lambda x e^{\theta i})}{(x e^{\theta i})}$ and $\dfrac{(\lambda x e^{-\theta i})}{(x e^{-\theta i})}$, and collecting the coefficients of $\dfrac{x_r}{q_r!}$.

It will be seen that L_r can be expressed linearly in terms of $\cos r\theta$, $\cos(r-2)\theta$, &c., the last being $\cos\theta$, or $\cos 0$, according as θ is odd or even.

2. Suppose now that

$$P(\mu x) / P(x) \equiv 1 + \Sigma \frac{M_r}{q_r!} x^r,$$

so that M_r is the same function of μ as L_r is of λ. Then it is clear that M_r may be linearly expressed in terms of L_r, L_{r-2}, ...

Thus, since

$$L_1 = \lambda_1 \,.\, 2\cos\theta, \quad \text{and} \quad L_2 = \lambda_1\lambda_2 \,.\, 2\cos 2\theta + \frac{q_2}{q_1}\lambda_1^2,$$

by direct calculation,

$$\frac{L_1}{q_1} = \frac{M_1}{q_1}\frac{\lambda_1}{\mu_1},$$

$$\frac{L_2}{q_2!} = \frac{M_2}{q_2!}\frac{\lambda_2!}{\mu_2!} + \frac{(\mu-\lambda)\lambda_1}{\mu_1 q_1}.$$

We may derive the general formula for L_r by induction.

Assume

$$\frac{L_r}{q_r!} = \frac{M_r\mu_{r+1}}{q_r!}\frac{\lambda_r!}{\mu_{r+1}!} + \frac{M_{r-2}\mu_{r-1}}{q_{r-2}!}\frac{(\mu-\lambda)\lambda_{r-1}!}{q_1!\,\mu_r!}$$

$$+\frac{M_{r-4}\mu_{r-3}}{q_{r-4}!}\frac{(\mu-\lambda)(\mu-\lambda q)\lambda_{r-2}!}{q_2!\,\mu_{r-1}!}+\dots \qquad \dots(1),$$

the general term being

$$\frac{M_{r-2s}\mu_{r-2s+1}}{q_{r-2s}}\frac{(\mu-\lambda)(\mu-\lambda q)\dots(\mu-\lambda q^{s-1})}{q_s!\,\mu_{r-s+1}!}\lambda_{r-s}!\,.$$

Using a similar formula for L_{r-1}, and combining it with that above in the expression

$$2\cos\theta\,.\,L_r\lambda_{r+1} - L_{r-1}q_{r+1}(1-\lambda^2 q^{r-1}),$$

which, by § 2 (1), is equal to L_{r+1}, we shall obtain a series of M's and M's multiplied by $2\cos\theta$. However, $2\cos\theta\,.\,M_{r-1}$ may also be transformed by § 2 (1), and we shall finally reduce L_{r+1} to a series of M_{r+1}.

The coefficient of M_{r-2s+1} in $\frac{L_{r+1}}{q_r!}$ is then found to consist in general of three terms, these three reducing to two when

$$r-2s+1 = 0$$

These will be found, after some reductions and dividing by q_{r+1}, to reduce to the same function of $r+1$ as the above coefficient of M_{r-2s} is of r, thus establishing the identity.

The formula (1) is of great generality, and will be found to be of special importance when the particular values 0, q, 1 are given to λ or μ, since we then obtain expressions for $A_r(\theta)$, $\frac{\sin(r+1)\theta}{\sin\theta}$, $2\cos r\theta$, $L_r(\theta)$, each one in terms of functions of any one other kind.

As a general case, let us expand $P(\lambda x)/P(x)$ in terms of ascending orders of M's. Replacing the L's in $1+\Sigma \frac{L_r}{q_r!}x^r$ by their equivalent series, we get

$$\phi\left\{\frac{\lambda}{\mu},\ \lambda,\ \mu q,\ q,\ \mu x^2\right\}+\frac{M_1}{q_1}\frac{\lambda_1!}{\mu_1!}x\phi\left\{\frac{\lambda}{\mu},\ \lambda q,\ \mu q^2,\ q,\ \mu x^2\right\}$$
$$+\frac{M_2}{q_2!}\frac{\lambda_2!}{\mu_2!}x^2\phi\left\{\frac{\lambda}{\mu},\ \lambda q^2,\ \mu q^3,\ q,\ \mu x^2\right\}+\dots.$$

If
$$\lambda^2 x^2 = \mu q,$$

the coefficient series may all be expressed as products by Heine's formula

$$\phi\left\{a,\ b,\ abx,\ q,\ x\right\}=\frac{(ax)(bx)}{(abx)(x)},$$

so that, finally,

$$\frac{P(\sqrt{\mu q})}{P(x)}=\frac{(x\sqrt{\mu q})(\mu x\sqrt{\mu q})}{(\mu)(\mu x^2)}\left\{\mu_1+\frac{M_1\mu_2}{q_1!}\frac{x-\sqrt{\mu q}}{1-x\mu\sqrt{\mu q}}\right.$$
$$\left.+\frac{M_2\mu_3}{q_2!}\frac{(x-\sqrt{\mu q})(x-\sqrt{\mu q^3})}{(1-x\mu\sqrt{\mu q})(1-x\mu\sqrt{\mu q^3})}+\dots\right\}.$$

When $\mu=1$, we get a formula established by Heine

$$\frac{P(q^{\frac{1}{2}})}{P(x)}=\frac{(xq^{\frac{1}{2}})^2}{(q)(x^2)}\left\{1+\frac{x-q^{\frac{1}{2}}}{1-xq^{\frac{1}{2}}}2\cos\theta+\frac{(x-q^{\frac{1}{2}})(x-q^{\frac{3}{2}})}{(1-xq^{\frac{1}{2}})(1-xq^{\frac{3}{2}})}2\cos 2\theta+\dots\right\}$$
$$\dots\dots\dots\dots\dots(2).$$

Again, if
$$\lambda x^2 = -q,$$

all the Heinean series have product forms, and we get an M-expansion for $P(-q/x)/P(x)$, which, being independent of M, gives us equivalent series in A, $\frac{\sin(r+1)\theta}{\sin\theta}$, and $2\cos r\theta$. We have then an A-series which is equal to a cosine series, and consequently we have a means of employing the formulæ in the second memoir.

We shall, however, make use of a more general formula connecting an A-series with a cosine series, which we now proceed to establish.

3. By putting $\lambda=0$ in § 2, we get an expression for A_r in terms of M. Replacing these in the series

$$1+\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{q_1q_2}A_2+\frac{(u-q^{\frac{1}{2}})(u-q^{\frac{3}{2}})(v-q^{\frac{1}{2}})(v-q^{\frac{3}{2}})}{q_4!}A_4+\dots\quad\dots(1),$$

and collecting the coefficients of the several M's, we have

$$\phi\left\{\frac{q^{\frac{1}{2}}}{u},\ \frac{q^{\frac{1}{2}}}{v},\ \mu q,\ q,\ \mu uv\right\}+\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{q_2!\,\mu_3!}\,\phi\left\{\frac{q^{\frac{3}{2}}}{u},\ \frac{q^{\frac{3}{2}}}{v},\ \mu q^3,\ q,\ \mu uv\right\}M_2$$

$$+\frac{(u-q^{\frac{1}{2}})(u-q^{\frac{3}{2}})(v-q^{\frac{1}{2}})(v-q^{\frac{3}{2}})}{q_4!\,\mu_5!}\,\phi\left\{\frac{q^{\frac{5}{2}}}{u},\ \frac{q^{\frac{5}{2}}}{v},\ \mu q^5,\ q,\ \mu uv\right\}M_4+\dots .$$

All these Heinean series are of the form

$$\phi\,\{a,\ b,\ abx,\ q,\ x\},$$

which $$=(ax)(bx)/(abx)(x),$$

so that (1) finally becomes

$$\frac{(q^{\frac{1}{2}}\mu u)(q^{\frac{1}{2}}\mu v)}{(\mu)(\mu uv)}\left\{\mu_1+\frac{\mu_3 M_2}{q_2!}\,\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{(1-\mu uq^{\frac{1}{2}})(1-\mu vq^{\frac{1}{2}})}\right.$$

$$\left.+\frac{\mu_5 M_4}{q_4!}\,\frac{(u-q^{\frac{1}{2}})(u-q^{\frac{3}{2}})(v-q^{\frac{1}{2}})(v-q^{\frac{3}{2}})}{(1-\mu uq^{\frac{1}{2}})(1-\mu uq^{\frac{3}{2}})(1-\mu vq^{\frac{1}{2}})(1-\mu vq^{\frac{3}{2}})}+\dots\right\}\ \dots(2).$$

If we put $\mu=1$, this latter expression becomes

$$\frac{(q^{\frac{1}{2}}u)(q^{\frac{1}{2}}v)}{(q)(uv)}\left\{1+2\cos 2\theta\,\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{(1-uq^{\frac{1}{2}})(1-vq^{\frac{1}{2}})}+\dots\right\}\ \dots\dots\dots(3).$$

Equating the series (1) and (3) together, we have an identity of great generality, to which the formulæ of the second memoir may be applied.

If $$u=v=0,$$

we have the formula § 1 (1), from which the q-identities were obtained. It is also interesting to note that a large number of elliptic functions may be expanded in ascending orders of M, or more particularly Λ.

If we put $\mu=q$, we have

$$\frac{(q^{\frac{3}{2}}u)(q^{\frac{3}{2}}v)}{(q)(quv)}\,\frac{1}{\sin\theta}\left\{(1-q)\sin\theta+\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{(1-uq^{\frac{3}{2}})(1-vq^{\frac{3}{2}})}\,(1-q^3)\sin 3\theta+\dots\right\}$$

$$\dots\dots\dots\dots(4).$$

Now, if $a_0+a_2\Lambda_2(\theta)+\dots=b_0+2b_2\cos 2\theta+\dots$

$$=\frac{1}{\sin\theta}\,(\beta_0\sin\theta+\beta_2\sin 3\theta+\dots),$$

and any such series as

$$a_0+m_2a_2+m_4a_4+\dots$$

has an equivalent b-series which can be thrown into the form

$$(b_0-b_2)+(b_2-b_4)\,\mu_2+\dots,$$

we obviously get

$$a_0+m_2a_2+m_4a_4+\dots=\beta_0+\mu_2\beta_2+\mu_4\beta_4+\dots \qquad \dots\dots\dots(5).$$

From the equality of (1) and (4) we therefore have

$$1+m_2\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{q_1q_2}+m_4\frac{(u-q^{\frac{1}{2}})(u-q^{\frac{3}{2}})(v-q^{\frac{1}{2}})(v-q^{\frac{3}{2}})}{q_4!}+\dots$$

$$=\frac{(q^{\frac{1}{2}}u)(q^{\frac{1}{2}}v)}{(q)(quv)}\left\{q_1+\mu_2\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{(1-uq^{\frac{1}{2}})(1-vq^{\frac{1}{2}})}q_3+\dots\right\}\quad\dots(6).$$

Let $\qquad u=e^{2\phi i}$ and $v=e^{-2\phi i}$;

then, if we write f_r for $1-2q^{r-\frac{1}{2}}\cos 2\phi+q^{2r-1}$, we get

$$1+m_2\frac{f_1}{q_2!}+m_4\frac{f_1f_2}{q_4!}+\dots=\frac{P(q^{\frac{1}{2}},2\phi)}{(q)^2}\left\{\frac{q_1}{f_1}+\mu_2\frac{q_3}{f_3}+\mu_4\frac{q_5}{f_5}+\dots\right\}\quad\dots(7).$$

As specimens of this identity we may quote the following: from p. 322 (12), we have

$$1=\frac{P(q^{\frac{1}{2}},2\phi)}{(q)^2}\left\{\frac{q_1}{f_1}-q\frac{q_3}{f_3}+q^3\frac{q_5}{f_5}-\dots\right\};$$

as is shown by Jacobi, from p. 325 (9), we have

$$1+\Sigma\frac{f_r!\,q^r}{q_{2r}!}=\frac{P(q^{\frac{1}{2}},2\phi)}{(q)^2}\left\{\frac{q_1}{f_1}-q^4\frac{q_5}{f_5}+q^8\frac{q_7}{f_7}-q^{20}\frac{q_{11}}{f_{11}}+\dots\right\};$$

from p. 338, Ex. 1, we have

$$1+\Sigma\frac{f_r!\,q^{r^2}}{q_{2r}!}=\frac{P(q^{\frac{1}{2}},2\phi)}{(q)^2}\left\{\frac{q_1}{f_1}-q^2\frac{q_5}{f_5}+q^4\frac{q_7}{f_7}-q^{10}\frac{q_{11}}{f_{11}}+\dots\right\};$$

from p. 342, we have

$$1+\Sigma\frac{f_r!\,q^r}{q_r!}=\frac{P(q^{\frac{1}{2}},2\phi)}{(q)^2}\left\{\frac{q_1}{f_1}-q^3\frac{q_3}{f_3}+q^9\frac{q_5}{f_5}-q^{18}\frac{q_7}{f_7}+\dots\right\}.$$

In a similar way we may establish the relations

$$\frac{A_1(\theta)}{q_1}+\frac{(u-q)(v-q)}{q_3!}A_3(\theta)+\frac{(u-q)(u-q^2)(v-q)(v-q^2)}{q_5!}A_5(\theta)+\dots$$

$$=\frac{(uq)(vq)}{(q)(uv)}\left\{2\cos\theta+\frac{(u-q)(v-q)}{(1-uq)(1-vq)}2\cos 3\theta\right.$$

$$\left.+\frac{(u-q)(u-q^2)(v-q)(v-q^2)}{(1-uq)(1-uq^2)(1-vq)(1-vq^2)}2\cos 5\theta+\dots\right\}\quad\dots(8),$$

and deduce many identities from the examples given in the second memoir which connect series of a's and b's with odd suffixes.

4. It will be necessary, in order to establish further properties of $A(\theta)$, to expand the quotient $\phi\{a, b, c, q, x\}/\left(\frac{abx}{c}\right)$ in powers of x, which may be thrown into a very simple form.

If $$\phi\{a, b, c, q, x\}\equiv 1+a_1x+a_2x^2+\dots,$$

then we know that

$$(1-q^r)(1-cq^{r-1})\,a_n=(1-aq^{r-1})(1-bq^{r-1})\,a_{n-1},$$

whence, denoting the Heinean series by $\phi(x)$, we have

$$(1-x)\phi(x)-\left(1-ax-bx+\frac{c}{q}\right)\phi(xq)+\left(\frac{c}{q}-abx\right)\phi(xq^2)=0.$$

Let $$\phi(x)/\left(\frac{abx}{c}\right)\equiv F(x)\equiv 1+\beta_1x+\beta_2x^2+\dots,$$

so that

$$(1-x)\left(1-\frac{abx}{c}\right)F(x)-\left(1-ax-bx+\frac{c}{q}\right)F(xq)+\frac{cF}{q}(xq^2)=0,$$

and by equating the coefficient of x^r to zero

$$(1-q^r)(1-cq^{r-1})\,\beta_r-\left(1-aq^{r-1}-bq^{r-1}+\frac{ab}{c}\right)\beta_{r-1}+\frac{ab}{c}\beta_{r-2}=0.$$

Let $$\beta_r=\gamma_r/(1-c)(1-cq)\dots(1-cq^{r-1}).$$

Then $$(1-q^r)\,\gamma_r-\left(1-aq^{r-1}-bq^{r-1}+\frac{ab}{c}\right)\gamma_{r-1}+\frac{ab}{c}(1-cq^{r-2})\,\gamma_{r-2}=0,$$

so that $$1+\gamma_1 x+\gamma_2 x^2+\dots$$

is easily seen to $$=\frac{(ax)(bx)}{(x)\left(\frac{abx}{c}\right)}.$$

Hence

$$\phi\left\{a, b, c, q, x\right\}/\left(\frac{abx}{c}\right)=1+\Sigma\Pi_r\left(a, b/1, \frac{ab}{c}\right)\frac{x^r}{(1-cq)^{r-1}!}.$$

5. We have seen by § 3 in the first memoir, Vol. XXIV., p. 341, that $1+\Sigma\Pi_r(x, \lambda e^{\theta i}, \lambda e^{-\theta i})A_r(\theta)$

$$=\frac{(\lambda^2)}{(\lambda)^2 P(\lambda, 2\theta)}\frac{1}{(xe^{\theta i})}\phi\left\{\lambda e^{2\theta i}, \lambda, \lambda^2, q, xe^{-\theta i}\right. \text{ [which, by § 4]}$$

$$=\frac{(\lambda^2)}{(\lambda)^2 P(\lambda, 2\theta)}\left\{1+\Sigma\frac{L_r(\theta)}{q_r!\,(1-\lambda q^{r-1})!}x^r\right\} \quad \dots\dots\dots\dots\dots(1).$$

But $$1+\Sigma\Pi_r(x, \lambda e^{\theta i}, \lambda e^{-\theta i})\,y^r=1\div(xy)\,P(\lambda y),$$

by the definition of Π,

$$=\left(1+\frac{xy}{1-q}+\frac{x^2y^2}{q_1q_2}+\dots\right)\left\{1+\frac{A_1(\theta)}{1-q}\lambda y+\dots\right\}.$$

Substituting for $\Pi_r(x, \lambda e^{\theta}, \lambda e^{-\theta i})$, from this identity the left side of (1) becomes

$$1+\frac{A_1(\theta)^2}{1-q}\lambda+\frac{A_2(\theta)^2}{q_1q_2}\lambda^2+\dots$$
$$+\frac{x}{1-q}\left\{A_1(\theta)+\frac{A_2(\theta)\,A_1(\theta)}{1-q}\lambda+\frac{A_3(\theta)\,A_2(\theta)}{q_2!}\lambda^2+\dots\right\}$$
$$+\dots \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2),$$

the coefficient of $\dfrac{x^r}{q_r!}$ being

$$A_r(\theta)+\frac{A_{r+1}(\theta)\,A_1(\theta)}{1-q}\lambda+\frac{A_{r+2}(\theta)\,A_2(\theta)}{q_2!}\lambda^2+\dots \quad \dots(3).$$

This series may be expanded according to ascending orders of A's by means of Vol. XXVI., p. 343, which states that the coefficient of $A_{r+2n}(\theta)$ in $\dfrac{A_{r+s}(\theta)\,A_s(\theta)}{q_s!}$, where $n \nless 0$, is

$$q_{r+s}!/q_{s-n}!\,q_{r+n}!\,q_n!.$$

Hence the coefficient of $A_{r+2n}(\theta)$ in the whole series (3) is

$$\Sigma\,\frac{\lambda^s q_{r+s}!}{q_{s-n}!\,q_{r+n}!\,q_n!}.$$

Taking r and n as constant and giving s all integral values from n upwards, this becomes

$$\frac{1}{q_n!\,q_{r+n}!}\left\{\lambda^n q_{n+r}!+\lambda^{n+1}\frac{q_{n+r+1}!}{q_1}+\lambda^{n+2}\frac{q_{n+r+2}!}{q_2!}+\dots\right\}$$
$$=\lambda^n/q_n!\,(1-\lambda)(1-\lambda q)\dots(1-\lambda q^{n+r}).$$

Hence the series (3) becomes (with the notation at the beginning of this memoir)

$$\frac{A_r}{\lambda_{r+1}!}+\frac{A_{r+2}}{q_1\lambda_{r+2}!}\lambda+\frac{A_{r+4}}{q_2!\,\lambda_{r+3}!}\lambda^2+\dots,$$

which, by (1), $$=\frac{(\lambda^2)}{(\lambda)^2 P(\lambda,2\theta)}\,\frac{L_r(\theta)}{(1-\lambda^2 q^{r-1})!}\quad\dots\dots\dots\dots(3).$$

We may notice, as a particular case, that

$$\frac{(\lambda^2)}{(\lambda)^2 P(\lambda,2\theta)}$$
$$=\frac{1}{1-\lambda}+\frac{A_2(\theta)\lambda}{(1-q)(1-\lambda)(1-\lambda q)}+\frac{A_4(\theta)\lambda^2}{q_2!\,(1-\lambda)(1-\lambda q)(1-\lambda q^2)}+\dots.$$

6. These results will now enable us to find an L-expansion of the series

$$1+\frac{A_1(\theta)\,A_1(\phi)}{1-q}x+\frac{A_2(\theta)\,A_2(\phi)}{q_2!}x^2+\dots\quad\dots\dots\dots(1).$$

For, by § 2 (1), putting $\lambda=0$, and writing λ for μ, we have

$$\frac{A_r(\theta)}{q_r!}=\frac{L_r\lambda_{r+1}}{q_r!}\,\frac{1}{\lambda_{r+1}!}+\frac{L_{r-2}\lambda_{r-1}}{q_{r-2}!}\,\frac{\lambda}{q_1\lambda_r!}+\frac{L_{r-4}\lambda_{r-3}}{q_{r-4}!}\,\frac{\lambda^2}{q_2!\,\lambda_{r-1}!}+\dots.$$

Substituting in (1) and rearranging according to ascending orders of $L(\theta)$'s, we have

$$\lambda_1\left\{\frac{1}{\lambda_1}+\frac{A_2(\phi)}{q_1\lambda_2!}\lambda x^2+\frac{A_4(\phi)}{q_2!\,\lambda_3!}\lambda^2x^4+\dots\right\}$$
$$+\frac{\lambda_2 L_1(\theta)}{q_1!}x\left\{\frac{A_1(\phi)}{\lambda_2!}+\frac{A_3(\theta)}{q_1\lambda_3!}\lambda x^2+\frac{A_5(\theta)}{q_2!\,\lambda_4!}\lambda^2x^4+\dots\right\}$$
$$+\dots,$$

the coefficient of $\frac{\lambda_r L_r(\theta)}{q_r!}x^r$ being

$$\frac{A_r(\phi)}{\lambda_{r+1}!}+\frac{A_{r+1}(\phi)}{q_1\lambda_{r+2}!}\lambda x^2+\frac{A_{r+4}(\phi)}{q_2!\,\lambda_{r+3}!}\lambda^2x^4+\dots \qquad \dots\dots\dots\dots(2).$$

Now, if x were equal to unity, this series would be equal to that on the left side of § 5 (3), but we should obtain a nugatory result, since in this case (1) would be divergent.

We see, however, that (2) will be equal to

$$\frac{(\lambda^2)}{(\lambda)^2P(\lambda,2\phi)}\,\frac{L_r(\phi)}{(1-\lambda^2q^{r-1})!}+\text{a multiple of }(x^2-1),$$

so that we may say that

$$1+\Sigma\frac{A_r(\theta)\,A_r(\phi)}{q_r!}x^r=\text{some multiple of }(x^2-1)$$

$$+\frac{(\lambda^2)}{(\lambda)^2P(\lambda,2\phi)}\left\{\lambda_1+\frac{\lambda_2L_1(\theta)\,L_1(\phi)}{q_1(1-\lambda^2)}x+\frac{\lambda_3L_2(\theta)\,L_2(\phi)}{q_2!\,(1-\lambda^2)(1-\lambda^2q)}x^2+\dots\right\}$$

$$\dots\dots\dots\dots\dots(3).$$

Since (1) is independent of λ, we get a new form for the right side of (3) by putting $\lambda=q$, which side then becomes, by § 1,

$$\frac{1}{(q)\sin\phi\,.\,P(q,2\phi)}\,\frac{1}{\sin\theta}(\sin\theta\sin\phi+x\sin2\theta\sin2\phi+x^2\sin3\theta\sin3\phi+\dots)$$

$$+\text{some multiple of }(x^2-1).$$

Although the latter terms are not determined, it is easy to see that, if we apply such examples from the second memoir as were used in § 3 to the relation just obtained, we shall obtain results which are convergent even when $x=1$, and into which these unknown terms will not enter.

We may virtually say then, for the purpose of utilizing those results of the second memoir which are applicable as in § 3 to a given relation

$$a_0+a_2A_2(\theta)+\dots=\frac{1}{\sin\theta}(\beta_0\sin\theta+\beta_2\sin3\theta+\dots),$$

or $$a_1A_1(\theta)+a_3A_3(\theta)+\dots=\frac{1}{\sin\theta}(\beta_1\sin2\theta+\beta_3\sin4\theta+\dots),$$

that we have such a relation in the equation

$$1+\frac{A_1(\theta)\,A_1(\phi)}{q_1}+\frac{A_2(\theta)\,A_2(\phi)}{q_2!}+\dots$$

$$=\frac{1}{(q)\sin\phi\,.\,P(q,\,2\phi)}\,\frac{1}{\sin\theta}\,(\sin\theta\sin\phi+\sin 2\theta\sin 2\phi+\dots)$$

$$=\frac{1}{\sin\theta}\,\frac{\sin\theta\sin\phi+\sin 2\theta\sin 2\phi+\dots}{\sin\phi-q\sin 3\phi+q^8\sin 5\phi-\dots}\quad\dots\dots\dots\dots(4),$$

since $$2q^{\frac{1}{4}}\,(q)\sin\phi\,.\,P(q,\,2\phi)=\vartheta_1(\phi,\,q^{\frac{1}{2}}).$$

From p. 338, Ex. 1, we have

$$1+\frac{qA_2(\phi)}{q_2!}+\frac{q^4A_4(\phi)}{q_4!}+\dots$$

$$=\frac{\sin\phi-q^2\sin 5\phi+q^4\sin 7\phi-q^{10}\sin 11\phi+\dots}{\sin\phi-q\sin 3\phi+q^8\sin 5\phi-\dots}\quad\dots(5)$$

But we have already seen in § 3 that this A-series

$$=\frac{1}{(q)}\,\vartheta_3(\phi).$$

Hence $$2q^{\frac{1}{4}}\,(\sin\phi-q^2\sin 5\phi+q^4\sin 7\phi-q^{10}\sin 11\phi+\dots)$$

$$=\frac{1}{(q)}\,\vartheta_3(\phi)\,\vartheta_1(\phi,\,q^{\frac{1}{2}}).$$

From the alternate terms in the same identity, we have

$$\frac{A_1(\phi)}{q_1}+\frac{q^2A_3(\phi)}{q_3!}+\frac{q^6A_5(\phi)}{q_5!}+\dots=\frac{\sin 2\phi-q\sin 4\phi+q^5\sin 6\phi-\dots}{\sin\phi-q\sin 3\phi+q^8\sin 5\phi-\dots}.$$

But, by putting $u=v=0$ in § 3 (8), this left side is seen to be

$$\frac{1}{(q)}\,\{2\cos\phi+2q^2\cos 3\phi+2q^6\cos 5\phi+\dots\};$$

therefore

$$2q^{\frac{3}{4}}(\sin 2\phi-q\sin 4\phi+q^5\sin 6\phi-\dots)=\frac{1}{(q)}\,\vartheta_2(\phi)\,\vartheta_1(\phi,\,q^{\frac{1}{2}}).$$

From p. 337, Ex. 4,

$$1+\frac{qA_2(\phi)}{q_2}+\frac{q^2A_4(\phi)}{q_2q_4}+\dots=\frac{\sin\phi-q^2\sin 3\phi+q^6\sin 5\phi-\dots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\dots}.$$

From p. 342,

$$1+\frac{qA_2(\phi)}{q_1}+\frac{q^2A_4(\phi)}{q_2!}+\ldots=\frac{\sin\phi-q^3\sin 3\phi+q^9\sin 5\phi-\ldots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots},$$

and
$$1+\frac{q(1-q^{\frac{1}{2}})A_2(\phi)}{q_2!}+\frac{q^2(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})}{q_4!}A_4(\phi)+\ldots$$

$$=\frac{\sin\phi-q^{\frac{3}{2}}\sin 3\phi+q^{\frac{9}{2}}\sin 5\phi-\ldots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots}.$$

From p. 339, Ex. 2,

$$1-\frac{q^{\frac{1}{2}}(1-q^{\frac{1}{2}})}{q_2!}A_2(\phi)+\frac{q^2(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})}{q_4!}A_4(\phi)-\ldots$$

$$=\frac{\sin\phi-q^{\frac{1}{2}}\sin 3\phi+q^{\frac{3}{2}}\sin 5\phi-\ldots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots}.$$

7. The formula § 2 (2) yields, on putting $x=q^{n+\frac{1}{2}}$, and changing θ into 2ϕ,

$$f_1f_3\ldots f_{2n-1}=\frac{q_{2n}!}{q_n!\,q_n!}\left\{1-\frac{q_n}{q_{n+1}}2q^{\frac{1}{2}}\cos 2\phi+\frac{q_nq_{n-1}}{q_{n+1}q_{n+2}}2q^2\cos 4\phi-\ldots\right\}.$$

But, since

$$A_{2n}(\theta)=\frac{q_{2n}!}{q_n!\,q_n!}\left\{1+\frac{q_n}{q_{n+1}}2\cos 2\theta+\frac{q_nq_{n-1}}{q_{n+1}q_{n+2}}2\cos 4\theta+\ldots\right\},$$

we see that the above product of the n f's is obtained from $A_{2n}(\theta)$, by writing $2q^{\frac{1}{2}r^2}(-1)^r\cos 2r\phi$ instead of $2\cos 2r\theta$. Hence, if

$$a_0+a_2A_2(\theta)+\ldots=b_0+2b_2\cos 2\theta+\ldots,$$

we have $a_0+a_2f_1+a_4f_1f_3+\ldots=b_0-2b_2q^{\frac{1}{2}}\cos 2\phi+2b_4q^2\cos 4\phi-\ldots$;

...............(1),

which is a formula of the type investigated in the second memoir, and from which many there obtained may be easily derived by giving ϕ special values.

For instance, if $\cos 2\phi=0$, we have Ex. 4 on p. 341 ;

if $\phi=0$, „ Ex. 3 on p. 339 ;

if $e^{2\phi i}=q^{\frac{1}{2}}$, „ (12) on p. 322.

Moreover, from §§ 3 and 6 in the present memoir, we have from formulæ in the second memoir altered as in § 3 to the form

$$a_0+m_2a_2+\ldots=\beta_0+\mu_2\beta_2+\mu_4\beta_4+\ldots,$$

obtained relations of the type

$$1+\frac{m_2f_1}{q_2!}+\frac{m_4f_1f_3}{q_4!}+\ldots=\frac{P(q^{\frac12},\phi)}{(q)^2}\left\{\frac{q_1}{f_1}+\mu_2\frac{q_3}{f_3}+\ldots\right\}$$

and $$1+\frac{m_2A_2(\phi)}{q_2!}+\frac{m_4A_4(\phi)}{q_4!}+\ldots=\frac{\sin\phi+\mu_2\sin3\phi+\ldots}{\sin\phi-q\sin3\phi+q^3\sin5\phi-\ldots}.$$

Hence, by (1), we see that, if

$$(\sin\phi-q\sin3\phi+q^3\sin5\phi-\ldots)(1+2c_2\cos2\phi+2c_4\cos4\phi+\ldots)$$
$$=\sin\phi+\mu_2\sin3\phi+\mu_4\sin5\phi+\ldots,$$

then the coefficients c and μ are so related that we also have

$$1-2c_2q^{\frac12}\cos2\phi+2c_4q^2\cos4\phi-\ldots$$
$$=\frac{(q)^2}{P(q^{\frac12},2\phi)}\left\{\frac{q_1}{f_1}+\mu_2\frac{q_3}{f_3}+\mu_4\frac{q_5}{f_5}+\ldots\right\}.$$

We may note, moreover, that, by § 3 (3), and by (1) in the present section,

$$1+\frac{(u-q^{\frac12})(v-q^{\frac12})(1-2q^{\frac12}\cos2\phi+q)}{q_2!}+\ldots$$
$$=\frac{(q^{\frac12}u)(q^{\frac12}v)}{(q)(uv)}\left\{1-2q^{\frac12}\cos2\phi\frac{(u-q^{\frac12})(v-q^{\frac12})}{(1-uq^{\frac12})(1-vq^{\frac12})}+\ldots\right\}.$$

If $u=0$, $v=1$, we have

$$1-\frac{q^{\frac12}(1-q^{\frac12})f_1}{q_2!}+\frac{q^2(1-q^{\frac12})f_1f_3}{q_4!}-\ldots$$
$$=\frac{(q^{\frac32})}{(q)}(1-2q\cos2\phi+2q^4\cos4\phi-\ldots).$$

In addition to the above formulæ, it is worth while mentioning that we may obtain a very similar set of identities by considering terms which contain odd suffixes in a, A, m, b, &c., but it is scarcely necessary to enter fully into these relations.

8. If $l_0+l_1L_1(\theta)+l_2L_2(\theta)+\ldots=m_0+m_1M_1(\theta)+m_2M_2(\theta)+\ldots$, we can easily obtain m_0 in terms of the l's, by substituting for the

L's by § 2, and equating coefficients of the various M's. Thus

$$\frac{m_0}{\mu_1} = \frac{l_0}{\mu_1} + q_2! \frac{(\mu-\lambda)\,\lambda_1}{\mu_2!\,q_1} l_2 + q_4!\,\frac{(\mu-\lambda)(\mu-\lambda q)\,\lambda_2!}{\mu_3!\,q_2!} l_4 + \dots .$$

Applying this to § 3 (2), which is, of course, equal to the same function of λ's and L's, we get

$$\frac{(q^{\frac{1}{2}}\mu u)(q^{\frac{1}{2}}\mu v)(\lambda)(\lambda uv)}{(q^{\frac{1}{2}}\lambda u)(q^{\frac{1}{2}}\lambda v)(\mu)(\mu uv)} = \frac{\lambda_1}{\mu_1} + \frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{(1-uq^{\frac{1}{2}})(v-q^{\frac{1}{2}})}\,\frac{(\mu-\lambda)\,\lambda_1}{\mu_2!\,q_1}\lambda_3$$

$$+\frac{(u-q^{\frac{1}{2}})(u-q^{\frac{3}{2}})(v-q^{\frac{1}{2}})(v-q^{\frac{3}{2}})}{(1-uq^{\frac{1}{2}})(1-uq^{\frac{3}{2}})(1-vq^{\frac{1}{2}})(1-vq^{\frac{3}{2}})}\,\frac{(\mu-\lambda)(\mu-\lambda q)\,\lambda_2!}{\mu_3!\,q_2!}\lambda_5 + \dots,$$

while, from § 6 (3), we get

$$\frac{(\mu^2)(\lambda)^2 P(\lambda, 2\phi)}{(\mu)^2(\lambda^2) P(\mu, 2\phi)} = \frac{\lambda_1}{\mu_1} + \frac{\lambda_3 L_2(\phi)}{(1-\lambda^2)(1-\lambda^2 q)}\,\frac{(\mu-\lambda)\,\lambda_1!}{\mu_2!\,q_1}$$

$$+\frac{\lambda_5 L_4(\phi)}{(1-\lambda^2 q^3)!}\,\frac{(\mu-\lambda)(\mu-\lambda q)\,\lambda_2!}{\mu_3!\,q_2!} + \dots .$$

9. In conclusion, it is interesting to show that there is a formula giving an expression for the product of $L_r(\theta)$ and $L_s(\theta)$, corresponding to and including that given in Vol. XXIV., p. 343, for the product of $A_r(\theta)\,A_s(\theta)$. This formula is

$$\frac{L_s}{q_s!}\,\frac{L_r}{q_r!} = \frac{\lambda_{r+s+1} L_{r+s}}{\lambda^2_{r+s}!}\,\frac{\lambda_r!\,\lambda^2_{r+s}!}{q_r!\,\lambda_{r+s+1}!} + \frac{\lambda_{r+s-1} L_{r+s-2}}{\lambda^2_{r+s-2}!}\,\frac{\lambda_1!\,\lambda_{s-1}!}{q_1!\,q_{s-1}!}\,\frac{\lambda_{r-1}!\,\lambda^2_{r+s-1}!}{q_{r-1}!\,\lambda_{r+s}!} + \dots,$$

the general term being

$$\frac{\lambda_{r+s-2t+1} L_{r+s-2t}}{\lambda^2_{r+s-2t}!}\,\frac{\lambda_{s-t}!\,\lambda_t!\,\lambda_{r-t}!\,\lambda^2_{r+s-t}!}{q_{s-t}!\,q_t!\,q_{r-t}!\,\lambda_{r+s-t+1}!} \qquad \dots\dots\dots\dots (1),$$

where $\quad t = 0, 1, 2, \dots s, \quad$ and $\quad s \not> r.$

This may be proved by induction, though the process is somewhat tedious. The formula is easily seen to be true when $s = 1$, reducing in fact merely to the relation (3) in § 1.

Assuming (1) above, we may deduce the corresponding formula for $L_{s+1} L_r$ by multiplying each side of (1) by L_1, and replacing $L_1 L_s$ on the left side, and $L_1 L_{r+s-2t}$ on the right, by terms containing single L's. In this way the products $L_{s+1} L_r$ and $L_{s-1} L_r$ remain on the left, and, when the latter has been replaced by the formula similar to (1), we get after considerable reduction a new similar formula for $L_{s+1} L_r$.

In this way, from the known formula for $L_0 L_r$ and $L_1 L_r$, we may obtain all the required formulæ up to L_r^2, and thus obtain the general result. This will, of course, give A-product and B-product formulæ by putting $\lambda = 0$ and $\lambda = \infty$. The purely trigonometrical formulæ got from $\lambda = 1$ or q require no special notice. The most interesting application of the formulæ, however, is to zonal harmonics, since it gives a general formula for expressing the product of two zonal harmonics as a linear function of the same.

Let $\lambda = q^l$, and afterwards put $q = 1$. Then

$$\frac{P(\lambda x)}{P(x)} = (1-2x\cos\theta+x^2)^{-l},$$

where primarily n is a positive integer,

$$\frac{\lambda_r!}{q_r!} = \frac{l^{(r)}}{r!},$$

where $$l^{(r)} \equiv l(l+1)\dots(l+r-1),$$

and $$\frac{\lambda_r^2!}{q_r!} = \frac{(2l)^{(r)}}{r!}.$$

By putting $\mu = q^m$, $P(\mu x)/P(x)$ becomes $(1-2x\cos\theta+x^2)^{-n}$, where m is a positive integer, and the formula § 2 (1) connects the coefficients of x^r in $(1-2x\cos\theta+x^2)^{-l}$ with those of $x^r, x^{r-2}, \dots$ in $(1-2x\cos\theta+x^2)^{-m}$. Just as in the binomial theorem, however, it is not necessary to restrict the values of l and m to positive integers, and we may therefore suppose the resulting formula to hold good for all real values of l and m.

In the case where $m = \frac{1}{2}$, the latter expansion is

$$1+P_1(\theta)x+P_2(\theta)x^2+\dots,$$

where $P_r(\theta)$ is the zonal harmonic of order r, so that we have hereby a means of expanding $(1-2x\cos\theta+x^2)^{-l}$ in zonal harmonics. Since, when $q = 1$, the Heinean series $\phi\{a, b, c, q, x\}$ becomes the hypergeometric series $F\{\alpha, \beta, \gamma, x\}$, by writing $a = q^\alpha$, $b = q^\beta$, $c = q^\gamma$, and evaluating, we see that the coefficient of $P_r(\theta)$ in the expansion of $(1-2x\cos\theta+x^2)^{-l}$ is

$$\frac{l(l+1)\dots(l+r-1)}{1.3.5\dots(2r-1)}\,2^r F\left\{l-\tfrac{1}{2},\ l+r,\ \frac{2r+3}{2},\ x^2\right\}.$$

10. The formula § 9 (1), will also afford a means of expanding the product $P(\lambda y)\,P(\lambda z)\,/\,P(y)\,P(z)$ according to ascending orders of L's.

We have, in fact, to find the coefficient of $L_n(\theta)$ in the product of

$$\left(1+\frac{L_1}{q_1}y+\dots\right)\left(1+\frac{L_1}{q_1}z+\dots\right),$$

after reducing any term $\dfrac{L_r L_s}{q_r!\,q_s!}y^r z^s$ by § 9 (1).

Now, $L_n(\theta)$ can only be derived from such products as

$$y^m z^m \frac{L_{m+n}L_m}{q_{m+n}!\,q_m!}y^n,\quad y^m z^m \frac{L_{m+n-1}L_{m+1}}{q_{m+n-1}!\,q_{m+1}!}y^{n-1}z,\ \dots,$$

and in general $$y^m z^m \frac{L_{m+n+p}L_{m+p}}{q_{m+n-p}!\,q_{m+p}!}y^{n-p}z^p,$$

where $$p=0,\ 1,\ \dots\ n;$$

it occurring as the $(p+1)^{\text{th}}$ term from the end in the expansion of this general expression by § 9 (1).

We will first keep m constant while p runs through all its values.

The coefficient of L_n in $\dfrac{L_{m+n-p}L_{m+p}}{q_{m+n-p}!\,q_{m+p}!}$ is got by writing

$$s=m+n-p,$$

$$r=m+p,$$

$$r+s-2t=n,$$

so that $$r+s-t=m+n,$$

$$t=m;$$

while the coefficient takes the form

$$\frac{\lambda_{n+1}}{\lambda_n^2!}\,\frac{\lambda_{n-p}!\,\lambda_m!\,\lambda_p!\,\lambda_{m+n}^2!}{q_{n-p}!\,q_m!\,q_p!\,\lambda_{m+n+1}!}.$$

Hence, giving p its $(n+1)$ values, we see that this group of terms containing L_n may be written

$$y^m z^m L_n(\theta)\,\frac{\lambda_{n+1}\lambda_m!\,\lambda_{m+n}^2!}{\lambda_n^2!\,\lambda_{m+n+1}!\,q_m!}\left\{\frac{\lambda_n!}{q_n!}y^n+\frac{\lambda_{n-1}!\,\lambda_1}{q_{n-1}!\,q_1}y^{n-1}z+\dots\right\}$$

The bracketed series may be simplified by writing

$$y = xe^{\phi i}, \quad z = xe^{-\phi i},$$

in which case it evidently becomes

$$x^n \frac{L_n(\phi)}{q_n!}.$$

We now have to give m all values from 0 to ∞, and we easily se that the whole term containing $L_n(\theta)$ is

$$x^n \frac{L_n(\theta)\, L_n(\phi)}{q_n!} \sum_{\infty}^{m=0} \left\{ \frac{\lambda_{n+1}\, \lambda_m!\, \lambda^2_{m+n}!}{\lambda_n^2!\, \lambda_{m+n+1}!\, q_m!} x^{2m} \right\}$$

$$= x^n \frac{L_n(\theta)\, L_n(\phi)}{q_n!\, \lambda_n!} \phi \left\{\lambda, \lambda^2 q^n, \lambda q^{n+1}, q, x^2\right\}.$$

Thus $\qquad P(\lambda x e^{\phi i})\, P(\lambda x e^{-\phi i}) \,/\, P(xe^{\phi i})\, P(xe^{-\phi i})$

$$= \phi\left\{\lambda, \lambda^2, \lambda q, q, x^2\right\} + x \frac{L_1(\theta)\, L_1(\phi)}{q_1 \lambda_1} \phi\left\{\lambda, \lambda^2 q, \lambda q^2, q, x^2\right\}$$

$$+ x^2 \frac{L_2(\theta)\, L_2(\phi)}{q_2!\, \lambda_2!} \phi\left\{\lambda, \lambda^2 q^2, \lambda q^3, q, x^2\right\}$$

$$+ \ldots .$$

By the methods of the last section, we get the following zona harmonic expansion:—

$$\left\{1 - 2x \cos(\theta - \phi) + x^2\right\}^{-\frac{1}{2}} \left\{1 - 2x \cos(\theta + \phi) + x^2\right\}^{-\frac{1}{2}}$$

$$= F\left\{\tfrac{1}{2}, 1, \tfrac{3}{2}, x^2\right\} + xP_1(\theta)\, P_1(\phi)\, \tfrac{2}{1} F\left\{\tfrac{1}{2}, 2, \tfrac{5}{2}, x^2\right\}$$

$$+ x^2 P_2(\theta)\, P_2(\phi) \frac{2.4}{1.3} F\left\{\tfrac{1}{2}, 3, \tfrac{7}{2}, x^2\right\}$$

$$+ \ldots .$$