

A HIERARCHY WITHOUT A GENERAL FACTOR<sup>1</sup>.

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1. *Introduction.*

THE object of this paper is to show that the cases brought forward by Professor Spearman<sup>2</sup> in favour of the existence of General Ability are by no means 'crucial.' They are it is true not inconsistent with the existence of such a common element but neither are they inconsistent with its non-existence. The essential point about Professor Spearman's hypothesis is the existence of this General Factor. Both he and his opponents are agreed that there are Specific Factors peculiar to individual tests, both he and his opponents agree that there are Group Factors which run through some but not all tests. The difference between them is that Professor Spearman says there is a further single factor which runs through all tests, and that by pooling a few tests the Group Factors can soon be eliminated and a point reached where all the correlations are due to the General Factor alone.

The proof advanced for this hypothesis rests upon the possibility of forming a 'hierarchy' with the correlation factors between a number

<sup>1</sup> This paper was prepared in 1914 but I requested that its publication should be delayed as both Major Spearman and myself became engaged in military work. There appears however to be no advantage in further delay and after correspondence with Major Spearman I have decided to put it forward. G. H. T.

<sup>2</sup> This *Journal*, v. (1912), and elsewhere.

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of tests taken in pairs (what is meant by a hierarchy will be explained immediately). If a hierarchy can be formed the existence of a General Factor is said to be proved. Such a statement is however incorrect or at least misleading and it is to this point that my paper is devoted. I propose to show that an excellent hierarchy can be made with Specific and Group Factors only, without a General Factor.

TABLE I.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>k</i>	<i>l</i>
<i>a</i>		867	727	600	372	186	162	138	134	110
		867	730	593	356	174	167	120	116	112
<i>b</i>	867		665	549	340	170	148	126	122	100
	867		650	550	341	143	137	088	084	082
<i>c</i>	727	665		461	286	143	129	106	102	084
	730	650		500	292	143	091	088	084	041
<i>d</i>	600	549	461		235	118	106	087	084	069
	593	550	500		244	095	091	088	042	041
<i>e</i>	372	340	286	235		073	065	054	052	043
	356	341	292	244		093	089	043	041	040
<i>f</i>	186	170	143	118	073		033	027	026	022
	174	143	143	095	093		044	042	040	039
<i>g</i>	162	148	129	106	065	033		024	024	020
	167	137	091	091	089	044		040	039	037
<i>h</i>	138	126	106	087	054	027	024		020	016
	120	088	088	088	043	042	040		037	036
<i>k</i>	134	122	102	084	052	026	024	020		015
	116	084	084	042	041	040	039	037		034
<i>l</i>	110	100	084	069	043	022	020	016	015	
	112	082	041	041	040	039	037	036	034	

The Upper Hierarchy contains a General Factor = 19. The total number of factors in each test is

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>k</i>	<i>l</i>
20	24	34	50	130	520	650	940	1000	1500

The Lower Hierarchy contains no General Factor. The average difference is 0.016.

To bring the average p.e. down to anything like this, over 1000 cases would be required.

In a hierarchy such for example as either of the two given in Table I every coefficient is larger than its neighbour on the right and its neighbour below it. For a perfect hierarchy not only should this be the case but there should exist a constant ratio between all the numbers

in any given pair of columns. In such a perfect hierarchy the coefficient of correlation between any pair of columns will be unity. Professor Spearman has shown that a General Factor will, in the absence of Group Factors, produce such a perfect hierarchy. In practice there will necessarily be sampling errors, and in practice no hierarchy can be expected to be perfect. The question then arises how great a deviation from perfection we can allow, and still believe it probable that a General Factor exists, and no Group Factors.

The answer to this question of course depends on a further consideration, viz., what likelihood is there of obtaining a hierarchical arrangement in the absence of a General Factor? Professor Spearman has considered this point and answered it as follows.

If none but quite Specific Factors are present, the correlations will all be zero, and the pairs of columns will show no correlation with one another. If however correlations exist, but are due to Group Factors alone, then tests which share a Group Factor will correlate highly, but others will not correlate at all. Let there be three such Group Factors; then we shall obtain not a hierarchy but an arrangement like this:

	$S_1$	$S_2$	$S_3$	$A_1$	$A_2$	$A_3$	$D_1$	$D_2$	$D_3$
$S_1$	.	$h$	$h$	$l$	$l$	$l$	$l$	$l$	$l$
$S_2$	$h$	.	$h$	$l$	$l$	$l$	$l$	$l$	$l$
$S_3$	$h$	$h$	.	$l$	$l$	$l$	$l$	$l$	$l$
$A_1$	$l$	$l$	$l$	.	$h$	$h$	$l$	$l$	$l$
$A_2$	$l$	$l$	$l$	$h$	.	$h$	$l$	$l$	$l$
$A_3$	$l$	$l$	$l$	$h$	$h$	.	$l$	$l$	$l$
$D_1$	$l$	$l$	$l$	$l$	$l$	$l$	.	$h$	$h$
$D_2$	$l$	$l$	$l$	$l$	$l$	$l$	$h$	.	$h$
$D_3$	$l$	$l$	$l$	$l$	$l$	$l$	$h$	$h$	.

$h$  = high correlation.  $l$  = low correlation. See this *Journal*, v. (1912), p. 57.

in which the high correlations are concentrated along the diagonal. In this arrangement some columns will correlate positively, namely those in which the high correlations come opposite one another; but these will be in the minority and most pairs of columns will correlate negatively. Professor Spearman concludes therefore that in the absence of a General Factor the average correlation between columns will be either zero or negative, and that only a General Factor will give a very high positive correlation between pairs of columns.

In this consideration of Group Factors however Professor Spearman has tacitly assumed that there is no overlapping of such factors. If this were so then indeed a hierarchy would be impossible. But it is

at any rate a conceivable hypothesis that such overlapping should occur, that for example there might exist a factor common to three tests  $a, b, c$  and another common to  $c, d, e$ , so that  $c$  contains both factors: and on this hypothesis an excellent hierarchy can be obtained without any General Factor, and the average column correlation can even approach unity.

In order now to avoid all psychological controversy I propose to deal with entirely non-psychological material and to apply Professor Spearman's formulae to dice throwing. To do so it is first necessary for me to consider an extension of Weldon's well-known experiment.

2. *An Extension of Weldon's Experiment.*

A description of this experiment is easily accessible in Dr Brown's *Mental Measurement*. If a number of dice be thrown and the score taken, and then half of them (which for convenience are red) be left lying and the remainder rethrown, then clearly the red dice are common to both throws, and in fact if the correlation between a number of such pairs be found it proves to be one-half. More generally, as Dr Brown shows, if  $n$  dice be thrown,  $l$  left lying and the remainder rethrown then the correlation is

$$r = \frac{l}{n}.$$

A still more general case not given by Dr Brown is as follows. Let  $l + m$  dice be thrown for the first throw,  $l$  left lying, and  $k$  thrown to form with  $l$  the second throw of  $l + k$  dice. Here  $l$  dice are common 'factors' and the correlation  $r$  will lie between  $\frac{l}{l+m}$  and  $\frac{l}{l+k}$ . Its exact value can be found as follows. Let the successive scores be

$$\begin{array}{ll} a_1 + \boxed{c_1, c_1} + b_1, & \\ a_2 + c_2, & c_2 + b_2, \\ a_3 + c_3, & c_3 + b_3, \\ \dots\dots\dots & \dots\dots\dots \\ \dots\dots\dots & \dots\dots\dots \\ a_N + c_N & c_N + b_N, \end{array}$$

then if  $x = a + c$  and  $y = c + b$ ,

we have 
$$r_{xy} = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}.$$

But

$$\sigma_{x-y}^2 = \sigma_a^2 + \sigma_b^2,$$

$$\sigma_x^2 = \sigma_a^2 + \sigma_c^2,$$

$$\sigma_y^2 = \sigma_b^2 + \sigma_c^2,$$

whence

$$r_{xy} = \frac{2\sigma_c^2}{\sqrt{(\sigma_a^2 + \sigma_c^2)(\sigma_b^2 + \sigma_c^2)}}.$$

Now

$$\sigma_c^2 : \sigma_a^2 : \sigma_b^2 :: l : m : k,$$

so that

$$r_{xy} = \frac{l}{\sqrt{(l+m)(l+k)}}.$$

This result<sup>1</sup>, which I have tested by experiment, seems to be of considerable interest in connection with the significance of the correlation coefficient.

Here however I cannot dwell on this point. At present I only need the above formula for the purposes of the next paragraph.

### 3. *Dice-throwing Experiments to imitate Specific, Group, and General Factors.*

In psychological tests a boy John (say) is tested in perhaps ten ways and gets a numerical score for each test. Let us imitate this with dice. Note that I do not for one moment suggest that psychological 'factors,' if they exist, can be added together like dice: I merely intend to apply Professor Spearman's formulae to dice throwing. If we throw an entirely fresh set of dice to represent each of John's ten tests, then clearly the factors are entirely specific, there will be no correlation whatever between the scores in the ten tests.

If some of the dice are red, and these red dice are left lying and counted in to every score, there will be a General Factor. The red dice will of course be rethrown when we are finding the scores of the next boy, in whom the General Factor may be greater or less. The number of white dice thrown will vary from test to test to represent the Specific Factors in each test. The correlations in this case will form a perfect hierarchy. For example the upper hierarchy in Table I could be obtained in this way. The numbers there given are the theoretical values calculated by

$$r = \frac{l}{\sqrt{(l+m)(l+k)}}.$$

The number of dice common to all the tests is 19, and the total number of dice in each test is shown below the Table.

<sup>1</sup> I am pleased to see from Professor Spearman's comments which follow this paper, that this result is deducible from a previous formula of his.

Notice however the lower numbers. They also form a hierarchy: not, it is true, a perfect hierarchy, for the relationship  $\frac{r_{ap}}{r_{bp}} = \frac{r_{aq}}{r_{bq}}$  is not everywhere satisfied; but nevertheless a very good hierarchy with no reversals, that is the numbers steadily decrease from the N.W. corner to the S.E. Even the few cases where the successive numbers are equal instead of decreasing could have been avoided if I had cared to face a little more labour in the construction and calculations. This hierarchy however, which apes so closely the former, contains no General Factor whatever. Its construction is described in the next paragraph.

#### 4. *A Hierarchy without a General Factor.*

Fig. 1 shows an arrangement in which ten tests depend upon 145 factors, of which 109 are quite specific and only occur in one test each, while the remaining 36 are group factors which run through more than one test each, but never through all. The distribution of these 36 group factors is shown in the figure. Most of them run through only two, three, or four tests, three of them run through five tests, but not one runs through more than five tests out of the ten. There is therefore nothing approaching a General Factor. The number of purely Specific Factors in each test is also given.

In obtaining actual scores with dice thirty-six dice, marked so that each was recognisable, would be thrown, and the score of each placed in the proper place in Figure 1. These are the Group Factors. Then in addition the dice representing the Specific Factors would be thrown. These would be entirely separate for each test—for example fourteen dice would be thrown to complete test *f*. The scores of the various tests could then be added up and the totals would be analogous to the scores of a single boy in the ten tests. The whole would then be repeated for each other boy.

To make this quite clear consider two tests in detail, say tests *d* and *e*. For test *d*, twenty dice in all have been thrown. Of these five are left lying and sixteen other dice thrown to complete test *e*. The correlation between test *e* and test *d* will therefore be by the formula found above

$$r = \frac{l}{\sqrt{(m+l)(l+k)}} = \frac{5}{\sqrt{20 \times 21}} = 0.244.$$

By the same formula the correlations between all the tests in pairs can be found. They are the lower numbers already shown in Table I

and form as has been said an excellent hierarchy. So excellent indeed, as to be indistinguishable from a perfect hierarchy in any actual experiment unless many more subjects were examined than has ever yet been the case. This will be shown in the two following sections of the paper.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
<i>a</i>	x	x	x	x	x	—	x	x	x	x	x	x	x	x	x	—	—	—
<i>b</i>	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	—	—	—
<i>c</i>	x	x	—	x	x	x	x	—	—	x	x	—	—	x	—	—	—	
<i>d</i>	x	—	x	—	x	—	—	x	—	—	—	x	—	—	x	x	—	
<i>e</i>	—	x	—	—	x	—	—	—	x	—	—	—	x	—	—	—	x	
<i>f</i>	—	—	x	—	—	—	x	—	—	x	—	—	—	—	—	—	x	
<i>g</i>	—	—	—	x	—	—	—	x	—	—	—	—	x	—	—	—	—	
<i>h</i>	—	—	—	—	—	x	—	—	—	—	—	x	—	—	—	—	x	
<i>k</i>	—	—	—	—	—	—	—	—	x	—	—	—	—	x	—	x	—	
<i>l</i>	—	—	—	—	—	—	—	—	—	—	x	—	—	—	x	—	x	

  

	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	<i>S</i>
<i>a</i>	—	—	—	—	x	—	—	x	—	x	x	x	—	x	x	x	x	x	0
<i>b</i>	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
<i>c</i>	—	—	—	—	—	x	x	x	—	x	x	x	—	x	x	x	x	x	1
<i>d</i>	—	—	—	—	—	x	x	x	x	x	—	—	—	x	x	x	x	x	3
<i>e</i>	—	—	—	—	—	x	—	—	—	x	—	—	x	x	x	x	—	—	9
<i>f</i>	x	x	—	—	—	—	—	—	x	—	x	—	—	—	—	—	—	—	14
<i>g</i>	x	—	x	x	—	—	—	x	—	—	—	—	x	—	—	—	—	—	16
<i>h</i>	—	—	x	—	x	—	—	—	—	x	—	—	—	—	—	—	—	—	20
<i>k</i>	—	x	x	—	—	—	—	—	—	—	—	x	—	—	—	—	—	—	22
<i>l</i>	—	x	—	x	x	—	—	—	—	—	—	—	—	—	—	—	—	—	24

Fig. 1. Distribution of Group Factors.

*a, b, c, etc.* are the names of 'tests.'

1, 2, 3, etc. are 'factors.'

For example factor number 15 (perchance 'visual memory') runs through tests *a, b, d, l*.

In addition to the Group Factors there are Specific Factors, the number of which in each test is indicated under *S*.

5. *Application of Professor Spearman's Criterion.*

Consider first the average difference between the coefficients of the two hierarchies. It is only 0.016. Mr Burt in his paper<sup>1</sup> compared the average probable errors of his coefficients with the average difference between his experimental numbers and the theoretical numbers of a perfect hierarchy. But to reduce the average probable error of an experimental determination in our case down to anything like 0.016

<sup>1</sup> This *Journal*, III. (1909), p. 159 ff.

more than one thousand subjects would be needed. That is, until this number had been examined we could not even begin to guess from which of these hierarchies our experimental numbers were derived.

Clearly too the correlations between the columns of the imperfect hierarchy are very high. The uncorrected correlation between the earlier columns is more than 0.99 in a large number of cases and even the worst correlation, between columns  $k$  and  $l$ , is 0.88. The average is therefore undoubtedly very high and positive, and there is no doubt that an experimental determination of this hierarchy would furnish apparent proof, according to Professor Spearman's formulae, of the existence of a General Factor, especially as in such an actual case the lower columns would be unlikely to reach his 'correctional standard,' and only the upper columns would be used. Such an experimental determination I have carried out, assisted in the dice-throwing by a number of my students, whom I here sincerely thank for the trouble taken and time spent in that monotonous occupation.

#### 6. *Experimental Values.*

We threw in all 5220 dice, in 36 groups of 145 each, to represent ten tests in a class of 36 boys. The numbers were entered on forms according to Figure 1, the totals obtained, and the marks then treated exactly as marks in tests. The correlational coefficients actually obtained are shown, with their probable errors, in Table II. To this hierarchy I have applied Professor Spearman's criterion in its entirety. His corrected formula for column correlation is

$$R_{ab}' = \frac{S(\overline{\rho_{xa}\rho_{xb}}) - (n-1)\overline{r_{ab}}\overline{\sigma_{xa}\sigma_{xb}}}{\sqrt{S(\overline{\rho_{xa}^2}) - (n-1)\overline{\sigma_{xa}^2}} \sqrt{S(\overline{\rho_{xb}^2}) - (n-1)\overline{\sigma_{xb}^2}}},$$

in which the  $\rho$ 's are the coefficients  $r$  measured from the mean of the column, and the  $\sigma$ 's are the probable errors of the  $r$ 's, divided by .6745. The bar indicates mean values.

It is laid down further that only those pairs of columns may be used in each of which  $S(\overline{\rho^2})$  is at least twice its correction  $(n-1)\overline{\sigma^2}$ . In our hierarchy four pairs of columns reach this standard, and for these we have

for	$ab$	$R' = 1.04$
	$ac$	$= 1.00$
	$bc$	$= 1.01$
	$cd$	$= 1.11$
	Mean	<u>1.04</u>



Our hierarchy therefore triumphantly passes the test, and would be added, by anyone ignorant of its real composition, to the list of those already considered by Professor Spearman to contain a General Factor.

TABLE II.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>k</i>	<i>l</i>
<i>a</i>		823 036	703 057	708 056	367 097	337 100	281 103	371 097	112 111	133 110
<i>b</i>	823 036		732 052	679 060	287 103	338 100	209 108	254 105	183 109	222 107
<i>c</i>	703 057	732 052		560 072	317 101	117 111	222 107	180 109	043 112	286 103
<i>d</i>	708 056	679 060	560 072		293 103	187 109	344 099	406 094	205 108	181 109
<i>e</i>	367 097	287 103	317 101	293 103		-059 112	080 111	-062 112	012 112	090 111
<i>f</i>	337 100	338 100	117 111	187 109	-059 112		-031 112	330 100	-054 112	314 101
<i>g</i>	281 103	209 108	222 107	344 099	080 111	-031 112		431 092	228 107	031 112
<i>h</i>	371 097	254 105	180 109	406 094	-062 112	330 100	431 092		135 110	166 110
<i>k</i>	112 111	183 109	043 112	205 108	012 112	-054 112	228 107	135 110		-039 112
<i>l</i>	133 110	222 107	286 103	181 109	090 111	314 101	031 112	166 110	-039 112	

Experimental values obtained by Dice-throwing. 145 dice were thrown 36 times. This is comparable with tests applied to a class of 36 boys.

The lower number in each square is the probable error.

Moreover, it passes equally well when tested by the more stringent *C'* criterion. This criterion was introduced by Professor Spearman because the ordinary correlation between columns may rise to unity even when the equation

$$\frac{r_{ap}}{r_{bp}} = \frac{r_{aq}}{r_{bq}}$$

is not exactly fulfilled, whereas *C'* (it is said) only becomes unity when this latter condition is fulfilled. The corrected value is

$$C' = \frac{S(r_{xa}r_{xb}) - (n-1)r_{ab}\sigma_{xa}\sigma_{xb} - \frac{1}{n}S(r_{xa})S(r_{xb})}{\sqrt{S(r_{xa}^2) - (n-1)\sigma_{xa}^2} \sqrt{S(r_{xb}^2) - (n-1)\sigma_{xb}^2} - \frac{1}{n}S(r_{xa})S(r_{xb})}$$

and we obtain

for	$ab$	$C' = 1.01$
	$ac$	1.00
	$bc$	1.01
	$cd$	1.03
	Mean	<u>1.01</u>

It may perhaps be urged by someone that this result is only possible because the number of cases, thirty-six, is small. But in the first place this number is as large as in most<sup>1</sup> of the cases in Professor Spearman's list. The only case of very large numbers is that of Bonser<sup>2</sup> and he only applied five tests, and one of his columns failed to reach correctional standard. And in the second place I have already shown theoretically that even with a huge number of cases  $R$  uncorrected would be well over 0.9, and that even with one thousand cases there would not yet be the slightest hope of distinguishing between the two hierarchies in Table I.

#### 7. Conclusion.

It has here been shown that a certain set of correlation coefficients, which we *know* to contain no General Factor, would be claimed by Professor Spearman as giving further support to the existence of such a factor. *There is therefore nothing to show whether the many cases brought forward by him really contain a General Factor or not.*

It must not be hastily and illogically concluded by anyone that therefore General Ability is a fiction. Its existence or non-existence is, as far as the mathematical argument goes, an entirely open question, which will not be answered mathematically until someone successfully carries out a very much more extensive set of experiments than has yet been attempted. What the work on correlation and hierarchies has shown is as follows:

1. Since correlation does actually exist between tests, there must be either Group Factors or a General Factor present, or both.

2. If there is no General Factor, then it is probable that the Group Factors *overlap* in a complicated fashion; for otherwise there would be no hierarchy. But even this is by no means certain for as a rule

<sup>1</sup> I gather from Professor Spearman's comments which follow this paper that this is not now the case. It was so in the list here referred to.

<sup>2</sup> See Hart and Spearman, this *Journal*, v. (1912), p. 60 ff.

very few columns reach the correctional standard which Professor Spearman has laid down and these few may be among the minority which do correlate highly even on Thorndike's theory of non-overlapping Group Factors<sup>1</sup>.

3. There is not the slightest mathematical evidence so far forthcoming which will enable us to distinguish between *overlapping* Group Factors and a General Factor.

Of course there may be reasons other than mathematical which may help us to decide; but with these I do not wish to deal in this paper in order not to confuse the issue.

Let me therefore reiterate that all I have shown is that Professor Spearman's calculations are incapable of discriminating between a General Factor and overlapping Group Factors.

*Note.* I regret to see from Professor Spearman's comments on this paper (see next page) that I have apparently not made the argument perfectly clear. The arrangement of overlapping factors defined by Figure 1 is *of course* not a random arrangement. The point is that it, and numbers of other arrangements which could be made, would be erroneously accepted by Professor Spearman's tests as containing a General Factor, so that if a hierarchy is presented to us we are unable by these tests to say whether it contains a General Factor or some special arrangement of overlapping Group Factors. Professor Spearman's remarks about prearranged cards and coins are irrelevant. Figure 1 is part of the 'definitely stated conditions' and the variation comes in when the dice are thrown.

<sup>1</sup> Hart and Spearman, *loc. cit.* p. 57.

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