ANOTHER METHOD OF DERIVING SIN 2α, sin 3α, AND SO ON.

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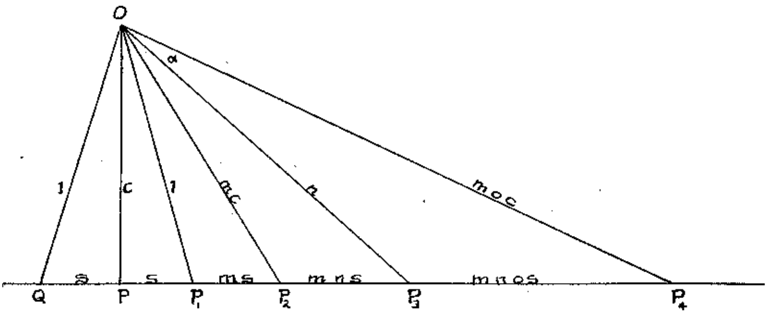
We assume the theorem that the bisector of the vertical angle of a triangle divides the base into segments that are proportional to the adjacent sides.

In the figure, OP is ⊥ QP and angles QOP, POP_1, P_1OP_2, P_2OP_3, and so on are each equal to α.

Choose OQ as unit length.

Then $OP = \cos\alpha \equiv c$,
and $PQ = PP_1 = \sin\alpha \equiv s$, and $c^2+s^2 = 1$.



Since OP_1 bisects angle POP_2, we may, using the above theorem, denote OP_2 and P_1P_2 by mc and ms where m remains to be determined.

Using the same theorem, we may denote,
OP_3 and P_2P_3 by n and mns,
OP_4 and P_3P_4 by moc and $mnos$, and so on.

Case I. The Double Angle.

Since $\triangle POP_2$ is right-angled at P,
$m^2c^2 = c^2+(s+ms)^2$,
$m^2(c^2-s^2)-2ms^2-1 = 0$,
$\{m(c^2-s^2)-1\}\{m+1\} = 0$.
Only the first of these factors is of use to us:

$$m = \frac{1}{c^2-s^2}$$

$$\cos 2\alpha = \frac{c}{mc} = \frac{1}{m} = c^2-s^2 = \cos^2\alpha - \sin^2\alpha.$$

$$\sin 2\alpha = \frac{s+ms}{mc} = \left(1+\frac{1}{m}\right)\frac{s}{c} = (c^2+s^2+c^2-s^2)\frac{s}{c},$$

$$= 2sc = 2\sin\alpha\cos\alpha.$$

CASE II. THE TRIPLE ANGLE.

Since OP_1 bisects angle QOP_3,

$$\frac{n}{1}=\frac{ms+mns}{2s},\quad 2n = m+mn,$$

$$\frac{1}{n}=\frac{2}{m}-1 = 2c^2-2s^2-1,$$

$$= 4c^2-2c^2-2s^2-1 = 4c^2-3.$$

$$\cos 3a = \frac{c}{n} = 4c^3-3c = 4\cos^3 a-3\cos a.$$

$$\sin 3a = \frac{s(1+m+mn)}{n},$$

$$= s\left\{\frac{1}{n}+m\left(1+\frac{1}{n}\right)\right\} = s\left(\frac{1}{n}+m\cdot\frac{2}{m}\right),$$

$$= s\left(\frac{1}{n}+2\right) = s(4c^2-1),$$

$$= s(3-4s^2),$$

$$= 3\sin a-4\sin^3 a.$$

CASE III. HIGHER MULTIPLES.

Since OP_2 bisects angle POP_4,

$$\frac{mo}{1}=\frac{mn+mno}{1+m},$$

$$o(1+m-n) = n,\ \frac{1}{o}=\frac{1}{n}+\frac{m}{n}-1=\frac{1-8c^2s^2}{c^2-s^2},$$

$$\cos 4a = \frac{1}{mo} = 1-8\cos^2 a\sin^2 a.$$

$$\frac{1}{o}=\frac{1}{n}+\frac{m}{n}-1.$$

Similarly, it will be found that

$$\frac{1}{p}=\frac{1}{o}+\frac{n}{o}-1,$$

$$\frac{1}{q}=\frac{1}{p}+\frac{o}{p}-1, \text{ and so on.}$$

This recurring formula enables us to extend the investigation at will. The geometric proof and interpretation, and the proof of the following theorem are left as an exercise. If the vertical angle AOE of a triangle be quadrisected by lines which meet the base in B, C, and D so that A, B, C, D, and E are in alphabetical order, then the segments of the base satisfy the following relation:

$$AB\cdot CD\cdot CE = BC\cdot DE\cdot AC.$$