THE CONVERSE OF ABEL'S THEOREM ON POWER SERIES

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Introduction.

Abel's theorem states that if $\sum_0^\infty a_n$ is convergent, then $\lim \sum_0^\infty a_n x^n$ exists as $x \to 1$ by real values, and is equal to Σa_n. The converse theorem, however, that the existence of $\lim_{x \to 1} \Sigma a_n x^n$ implies the convergence of Σa_n, is very far from being true; for example, either the Cesàro or the Borel summability of Σa_n suffices for the existence of Abel's limit. It is known, however, that the existence of this limit, *combined with certain conditions satisfied by the a's*, does imply the convergence of Σa_n. Three such sets of conditions, for example, are:

(*a*)† the a's are all positive;

(*b*) the order of a_n has a certain upper limit;

(*c*)‡ the function $\Sigma a_n x^n$ is regular at the point $x = 1$ and $a_n \to 0$.

In the present paper we are concerned with the problems arising out of case (*b*), where the only additional restriction on the a's is an upper limit to the order of a_n. The theorem of this case is due to M. Tauber.§ The result is remarkable and apparently paradoxical in view of Abel's theorem, for it may be expressed roughly by saying that if Σa_n is not actually convergent, then the more nearly convergent it is, the more un-

* The paper has been modified considerably from the form in which it was first communicated. It then consisted of Theorems (A), (B), and (C), with the index λ_n limited to the form n^k, and the note in § 10. An introduction was added, and the title changed, at the suggestion of the referees. I am indebted to Mr. G. H. Hardy for several important additions, and also for directing my attention to the subject in the first place. The application of Tauber's theorem to finitely oscillating power series, and the distinction in this respect between Tauber's case and that considered here are due to him, as also is the application to Fourier's series in § 9.

† Pringsheim, *Münchener Sitzungsberichte*, Bd. 30, 1900, p. 37. The result, however, was practically proved, although not explicitly stated, by Abel.

‡ Fatou, *Thèse*, Stockholm, 1906, p. 389.

§ *Monatshefte f. Math. u. Phys.*, Bd. 8, 1897, p. 273. Pringsheim (*Münchener Sitzungsberichte*, Bd. 31, 1901, p. 507) has generalised the theorem by showing that Σa_n converges if Abel's limit exists as $x \to 1$ along any path lying within any pair of straight lines which cut the circle at finite angles, and if $(a_1 + 2a_2 + \dots + na_n)/n \to 0$.

likely is it that $\Sigma a_n x^n$ should exist.* M. Tauber's actual result is that the existence of $\lim\limits_{x\to 1} \Sigma a_n x^n$, and the relation $na_n \to 0$, together involve the convergence of Σa_n. The proof of this theorem is as follows.†

We have
$$\left| \sum_0^{\nu-1} a_n(1-x^n) \right| < (1-x) \sum_0^{\nu-1} n\,|a_n|,$$
because
$$(1-x^n)/(1-x) = 1+x+x^2+\ldots+x^{n-1} \leqslant n;$$
also, if H_ν is the upper limit to $n\,|a_n|$, $(n+1)\,|a_{n+1}|$, ... to ∞, we have
$$\left| \sum_\nu^\infty a_n x^n \right| < H_\nu \nu^{-1} x^\nu (1+x+x^2+\ldots) < H_\nu \nu^{-1} x^\nu/(1-x).$$

Take $x = 1-\nu^{-1}$; we then have
$$\left| \sum_0^{\nu-1} a_n - \sum_0^\infty a_n x^n \right| < \nu^{-1} \sum_0^{\nu-1} n\,|a_n| + H_\nu. \tag{1}$$

As $\nu \to \infty$, each of the terms on the right tends to zero; and so
$$\lim_{\nu\to\infty} \sum_0^{\nu-1} a_n = \lim_{x\to 1} \sum_0^\infty a_n x^n,$$
which proves the theorem.

If the restriction in this theorem on the order of $|a_n|$ is the minimum possible, we ought to be able to find non-convergent series for which $\overline{\lim}\, na_n > 0$, but is finite, and for which Abel's limit exists. Now the easiest means of constructing a non-convergent series for which Abel's limit exists is to take a series summable by Cesàro's first mean. Mr. Hardy‡ has shown, however, that *no* non-convergent series for which $|na_n| < K$ can be summable by any one of Cesàro's means. The method therefore fails to provide an instance of the desired kind, and we are naturally led, as Mr. Hardy himself remarks, to ask whether Tauber's theorem also does not hold under the wider condition $|na_n| < K$. That it does is proved in Theorem (B) of the present paper.

* Cp. Hardy, "On Slowly Oscillating Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 315 (this paper will be referred to as "O.S."), where an explanation of the corresponding paradox relating to summability will be found.

† I have taken the proof in Bromwich's *Infinite Series*, p. 251, adapting it to the case when $x \to 1$ by real values.

‡ "O.S.," p. 308.

The proof of Tauber's theorem given above proves more than is actually stated. It follows immediately from (1) that, if $\Sigma a_n x^n$ oscillates finitely as $x \to 1$, then the limits of oscillation, as $n \to \infty$, of Σa_n are the same as the limits of oscillation of $\Sigma a_n x^n$. Now it is a somewhat remarkable fact that with the condition $|na_n| < K$ this result no longer holds. To see this it is sufficient to consider the series

$$\Sigma \mathfrak{R} n^{-1-ai}, \qquad \Sigma \mathfrak{R} n^{-1-ai} x^n.$$

The well known asymptotic formula

$$\sum_1^n n^{-1-ai} \sim n^{-ai}/ai$$

shows that the limits of oscillation of Σa_n are $\pm a^{-1}$. On the other hand, as $x \to 1$, we have*

$$\sum_1^\infty n^{-1-ai} x^n \sim \Gamma(ai)\,[\log(1/x)]^{ai}.$$

The limits of oscillation of the *function* $\Sigma a_n x^n$ are therefore

$$\pm |\Gamma(ai)| \quad \text{or} \quad \pm a^{-1} \sqrt{(\pi a \operatorname{cosech} \pi a)},$$

and are different from those of the *series*.

It follows from this result that it is *not* always possible to find a relation between x and ν for which the left-hand side of (1) tends to zero. It is to be expected, therefore, that the extended theorem will require a new and more complicated proof.

The series for which $|na_n| < K$ are much more interesting than those for which $na_n \to 0$; the former class includes, for example, all Fourier's series which satisfy Dirichlet's conditions. Mr. Hardy has shown† that, in virtue of his theorem that a series for which $|na_n| < K$ is convergent if it is summable by Cesàro's method, Fourier's theorem with Dirichlet's conditions becomes an immediate corollary of Fejér's theorem that the Fourier's series for any continuous function is summable by Cesàro's first mean. In the same way our extension of Tauber's theorem exhibits Fourier's theorem as a corollary of the result known under the name of "Poisson's Integral," thereby completing Poisson's attempted proof of Fourier's theorem.

It is shown in Theorem (C) of this paper that the condition $|na_n| < K$ is the minimum possible restriction of its kind; *i.e.*, that, given any

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* Lindelöf, *Le Calcul des Résidus*, p. 139.

† "O.S.," p. 308.

function $\phi(n)$ tending to infinity, there exists a non-convergent series for which Abel's limit exists, while $|na_n| < \phi(n)$.

M. Landau has proved an analogue of Tauber's theorem for Dirichlet's series. His result is that, if $\Sigma\mu_n$ is a divergent series of positive terms, the existence of $\lim\limits_{x\to 0} \sum\limits_1^{\infty} a_n e^{-\lambda_n x}$ and $\lambda_n a_n/\mu_n \to 0$, where

$$\lambda_n = \mu_1+\mu_2+\ldots+\mu_n,$$

together imply the convergence of Σa_n. (Tauber's theorem is the special case $\mu_n = 1$.) This theorem may be extended in the same way as Tauber's: the general index λ_n will, in fact, be considered throughout the proof of Theorem (B). No additional complication results from this procedure; indeed, the general problem suggests a proof considerably simpler in some details than my original proof for the case of the index n.

Theorem (C) also holds with the general index, with, however, one very important reservation. When some unimportant conditions are satisfied by the μ's, when $\mu_n/\lambda_n \to 0$ (a restriction roughly equivalent to $\lambda_n < e^{\epsilon n}$, where ϵ is arbitrarily small), and when $\phi(n) \to \infty$, non-convergent series Σa_n exist such that

$$\Sigma a_n e^{-x\lambda_n} \to s \quad \text{and} \quad |\lambda_n a_n/\mu_n| < \phi(n).$$

But when the order of λ_n is comparable with or greater than that of $e^{\rho n}$, the theorem breaks down entirely. I have not completely solved the problem of these high indices, but I feel practically certain that the existence of $\lim \Sigma a_n e^{-x\lambda_n}$, by itself, implies the convergence of Σa_n. For the present, therefore, I have excluded these cases from consideration in connexion with Theorem (C). In Theorem (B) they are not exceptions, but they require a modified proof, and in view of the fact that the relations of that theorem constitute only a small part of the whole truth about them I have omitted them there also.

The paper concludes with some applications of Theorems (B) and (C), including that mentioned above to Fourier's series, and with a note on the converse of Abel's theorem, with summability of Σa_n in the place of convergence.

1. In the proof of Theorem (B) the following theorem is fundamental.

THEOREM (A).—*If, as* $x \to \infty$ *by real values,* $\phi(x) \to s$, *and every derivate of* $\phi(x)$ *is finite, then every derivate of* $\phi(x)$ *tends to zero.*

We may assume, without loss of generality, that $\phi(x)$ is real.

Suppose that $\phi'(x)$ oscillates. Then there exists a positive constant h such that, for arbitrarily large values x_n of x, $|\phi'(x)| > h$. Let $\phi'(x_n)$ be, say, positive. Then, since $|\phi''(x)| < K$,

$$\phi'(x)-\phi'(x_n) = \int_{x_n}^{x} \phi''(x)\,dx > -K\,|x-x_n|,$$

and therefore
$$\phi'(x) > \tfrac{1}{2}h,$$
when
$$|x-x_n| < \tfrac{1}{2}hK^{-1} = c.$$
Then
$$\int_{x_n-c}^{x_n+c} \phi'(x)\,dx > ch.$$

This is incompatible with $\int^{\infty} \phi'(x)\,dx = s$, and it follows that $\phi'(x) \to 0$.

The argument may evidently be repeated with ϕ', ϕ'', ... successively in the place of ϕ, and we have

$$\phi^{(r)}(x) \to 0 \quad (r \geqslant 1).$$

COROLLARY.—*If, as $y \to 0$ by positive values,*

$$\psi(y) \to s \quad \textit{and} \quad |y^r\,\psi^{(r)}(y)| \textit{ is finite} \quad (r \geqslant 1),$$

then
$$y^r\,\psi^{(r)}(y) \to 0 \quad (r \geqslant 1).$$

For, writing
$$y = e^{-x}, \qquad \psi(y) = \phi(x),$$
we have
$$(-y)^r\,\psi^{(r)}(y) = D(D-1)\ldots(D-r+1)\,\phi(x) \quad (D = d/dx). \tag{1}$$

By induction from (1) we have $|\phi^{(r)}(x)| < K$; therefore, by the theorem, $\phi^{(r)}(x) \to 0$, and therefore, from (1),

$$y^r\,\psi^r(y) \to 0.$$

2. THEOREM (B).—*If $\Sigma \mu_n$ is a series of positive terms such that*

$$\lambda_n = \mu_1+\mu_2+\ldots+\mu_n \to \infty, \qquad \mu_n/\lambda_n \to 0,$$

then Σa_n is convergent, provided

$$\sum_1^{\infty} a_n e^{-x\lambda_n} \to s,$$

and
$$|a_n| < K\mu_n/\lambda_n.$$

We may assume without loss of generality that a_n is real.

The first stage in the proof is to establish the following result:—

$$x^{r+1}\sum_1^\infty s_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \to r!\,s \quad (r \geqslant 1), \tag{2}$$

where
$$s_n = a_1+a_2+\ldots+a_n.$$

We observe once for all that the inequality for a_n involves the absolute convergence of $\Sigma a_n \lambda_n^p e^{-x\lambda_n}$, when $x > 0$. For

$$\begin{aligned} |a_n \lambda_n^p e^{-x\lambda_n}| &< K\mu_n \lambda_n^{p-1} e^{-x\lambda_n} \\ &< K\mu_n \lambda_n^{-2} \quad (\text{since } \lambda_n \to \infty) \\ &< K \int_{\lambda_{n-1}}^{\lambda_n} x^{-2}\,dx, \end{aligned}$$

and the absolute convergence of the series follows immediately from the convergence of $\int^\infty x^{-2}\,dx$. From this result, and the inequality $|s_n| < K\lambda_n$, may be deduced without difficulty the legitimacy of the rearrangements and differentiations of the series $\Sigma a_n e^{-x\lambda_n}$ which occur in the sequel.

We note also that $\mu_n/\lambda_n \to 0$ involves

$$\lambda_{n+1}/\lambda_n \to 1 \quad \text{and} \quad \lambda_{n+1} < K\lambda_n.$$

Let $$\psi(x) = \sum_1^\infty a_n e^{-x\lambda_n} = \Sigma s_n (e^{-x\lambda_n} - e^{-x\lambda_{n+1}}) = x\Sigma s_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-xt}\,dt. \tag{3}$$

Then, since $|a_n| < K\mu_n/\lambda_n$,

$$\begin{aligned} |x^r \psi^{(r)}(x)| &= x^r |\Sigma a_n \lambda_n^r e^{-x\lambda_n}| < Kx^r \Sigma \mu_n \lambda_n^{r-1} e^{-x\lambda_n} \\ &< Kx^r \Sigma \mu_n (K\lambda_{n-1})^{r-1} e^{-x\lambda_n} \\ &< Kx^r \Sigma \int_{\lambda_{n-1}}^{\lambda_n} t^{r-1} e^{-xt}\,dt < Kx^r \int_0^\infty t^{r-1} e^{-xt}\,dt \\ &< K. \end{aligned}$$

Hence, by means of the corollary to Theorem (A),

$$x^r \psi^{(r)}(x) \to 0.$$

With the second form of $\psi(x)$ in (3) this gives

$$(-)^r \left[x^{r+1} \Sigma s_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt - rx^r \Sigma s_n \int_{\lambda_n}^{\lambda_{n+1}} t^{r-1} e^{-xt}\,dt \right] \to 0,$$

whence $$V_r \equiv x^{r+1}\Sigma s_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt = rV_{r-1}+\epsilon_x = \dots \quad (r \geqslant 1)$$
$$= r!\,V_0+\epsilon_x \to r!\,s,$$
the desired result (2).

3. We now write $s_n = s+\sigma_n$; then, since
$$x^{r+1}\Sigma \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt = x^{r+1}\int_{\lambda_1}^{\infty} t^r e^{-xt}\,dt = \int_{\lambda_1 x}^{\infty} y^r e^{-y}\,dy \to r!,$$
we have $$x^{r+1}\Sigma\sigma_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \to 0 \quad (r \geqslant 1). \tag{4}$$

The convergence of Σa_n is equivalent to $\sigma_n \to 0$; the proof of our theorem consists in showing that (4) is false if σ_n does not $\to 0$. What we shall actually prove concerning (4) is that, if $|\sigma_n|$ exceeds a positive h for arbitrarily large values of n, then, when r exceeds a certain number depending on h, $$\left| x^{r+1}\Sigma\sigma_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \right| > K^{-1}$$
for a certain set of values of x tending to 0.

4. If σ_n does not $\to 0$, there exists an $h>0$ such that $|\sigma_n| > h$ for an infinite number of values of n. Let m be such a value: then, if $n>m$,
$$|\sigma_n-\sigma_m| \leqslant |a_{m+1}|+|a_{m+2}|+\dots+|a_n| < K\sum_{m+1}^{n} \mu_n\lambda_n^{-1}$$
$$< K\sum_{m+1}^{n}\int_{\lambda_{n-1}}^{\lambda_n} x^{-1}\,dx < K\log(\lambda_n/\lambda_m).$$
Similarly, if $m>n$, $|\sigma_n-\sigma_m| < K\log(\lambda_m/\lambda_n)$.
Consequently
$$|F(x)| \equiv \left| x^{r+1}\sum_1^{\infty}\sigma_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \right|$$
$$\geqslant \left| x^{r+1}\sum_1^{\infty}\sigma_m \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \right| - \left| x^{r+1}\Sigma(\sigma_n-\sigma_m)\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \right|$$
$$> |\sigma_m|\,x^{r+1}\sum_1^{\infty}\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt - Kx^{r+1}\sum_{m+1}^{\infty}\log(\lambda_n/\lambda_m)\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt$$
$$-Kx^{r+1}\sum^{m-1}\log(\lambda_m/\lambda_n)\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \tag{5}$$

We choose $x = r/\lambda_m$, and consider the limits of each of the terms on the right-hand side of (5), as $m \to \infty$, *i.e.*, as $x \to 0$ through a certain set of values. (m, of course, is always an integer of a special kind.)

In the first of these terms

$$\lim_{x \to 0} x^{r+1} \sum_1^\infty \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\, dt = r!,$$

and hence
$$\varliminf_{x \to 0} |\sigma_m|\, x^{r+1} \sum_1^\infty \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\, dt \geqslant hr!. \tag{6a}$$

In the second term,

$$\log(\lambda_n/\lambda_m) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\, dt < \int_{\lambda_n}^{\lambda_{n+1}} \log(t/\lambda_m)\, t^r e^{-xt}\, dt,$$

and therefore

$$\varlimsup_{m \to \infty} x^{r+1} \sum_{m+1}^\infty \log(\lambda_n/\lambda_m) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\, dt$$

$$\leqslant \varlimsup_{m \to \infty} x^{r+1} \int_{\lambda_m}^\infty \log(t/\lambda_m)\, t^r e^{-xt}\, dt \leqslant \int_r^\infty \log(u/r)\, u^r e^{-u}\, du, \tag{6b}$$

where we have written $t = x^{-1}u = \lambda_m u r^{-1}$.

In the third term, since $\lambda_{n+1}/\lambda_n \to 1$,

$$\log(\lambda_m/\lambda_n) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\, dt = \log(\lambda_m/\lambda_{n+1}) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\, dt + \epsilon_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\, dt,$$

where $\epsilon_n \to 0$ as $n \to \infty$,

$$< \int_{\lambda_n}^{\lambda_{n+1}} \log(\lambda_m/t)\, t^r e^{-xt}\, dt + \epsilon_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\, dt,$$

so that

$$\varlimsup_{m \to \infty} x^{r+1} \sum_1^{m-1} \log(\lambda_m/\lambda_n) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\, dt$$

$$\leqslant \varlimsup_{m \to \infty} \int_0^{\lambda_m} \log(\lambda_m/t)\, t^r e^{-xt}\, dt + \varlimsup_{m \to \infty} \sum_1^{m-1} \epsilon_n x^{r+1} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\, dt.$$

If we write, as before, $t = ux^{-1} = \lambda_m u r^{-1}$,

the first term becomes $\int_0^r \log(r/u)\, u^r e^{-u}\, du.$

In the second term let ν be the value of n after which $|\epsilon_n| < \epsilon$. Then

$$\overline{\lim_{m\to\infty}} \sum_1^{m-1} \epsilon_n x^{r+1} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt$$

$$< \overline{\lim_{x\to 1}} \sum_1^{\nu} K x^{r+1} \int_{\lambda_n}^{\lambda_{n+1}} t\, e^{-xt} dt + \overline{\lim_{m\to\infty}} \sum_{\nu+1}^{m-1} \epsilon x^{r+1} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt$$

$$< K \overline{\lim_{x\to 0}} \int_0^{x\lambda_{\nu+1}} u^r e^{-u} du + \epsilon \int_0^{\infty} u^r e^{-u} du \qquad (xt = u)$$

$$< 0 + \epsilon r!,$$

and therefore $\qquad = 0,$

since ϵ is arbitrary. Hence

$$\overline{\lim_{m\to\infty}} x^{r+1} \sum_1^{m-1} \log(\lambda_m/\lambda_n) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt \leqslant \int_0^1 \log(r/u) u^r e^{-u} du. \qquad (6c)$$

From (5) and (6a, b, c), we have

$$\underline{\lim_{x\to 0}} |F(x)| \geqslant hr! - K\left[\int_r^{\infty} \log(u/r)\, u^r e^{-u} du + \int_0^r \log(r/u) u^r e^{-u} du\right], \qquad (7)$$

in which the K is the K of (5), *and is independent of* r.

5. We shall now show that

$$I_1 = \int_0^r \log(r/u)\, u^r e^{-u} du < K r^r e^{-r}, \qquad (8)$$

$$I_2 = \int_r^{\infty} \log(u/r)\, u^r e^{-u} du < K r^r e^{-r}, \qquad (9)$$

where, as also in what follows, K denotes a constant independent of r.

Writing $u = r(1-z)$, we have

$$I_1 = r^{r+1} e^{-r} \int_0^1 [e^z(1-z)]^r \log[(1-z)^{-1}]\, dz. \qquad (10)$$

As z increases from 0 to 1, $e^z(1-z)$ decreases steadily from e to 0. Let c be the value of z for which

$$e^z(1-z) = e^{-1}.$$

Then, if $z \geqslant c$, $\qquad e^z(1-z) \leqslant e^{-1}$,

and if $\quad z \leqslant c, \quad e^z(1-z) < e^{-\frac{1}{2}z^2}, \quad$ and $\quad \log[(1-z)^{-1}] < Kz.$

Therefore $\displaystyle\int_0^1 [e^z(1-z)]^r \log[(1-z)^{-1}]\,dz$

$$< \int_0^c e^{-\frac{1}{2}rz^2} Kz\,dz + e^{-r}\int_c^1 \log[(1-z)^{-1}]\,dz$$

$$< K\int_0^\infty e^{-\frac{1}{2}rz^2} z\,dz + Ke^{-r} < Kr^{-1} + Ke^{-r} < Kr^{-1},$$

and (8) follows from (10).

In the integral I_2 we write $u = r(1+z)$. Then

$$I_2 = r^{r+1}e^{-r}\int_0^\infty [(1+z)e^{-z}]^r \log(1+z)dz < Kr^{r+1}e^{-r}\int_0^\infty [(1+z)e^{-z}]^r z\,dz. \quad (11)$$

Now, if $z \leqslant \frac{1}{2}$, $\qquad (1-z)e^{-z} < e^{-\frac{1}{3}z^2}$,

[since $\qquad \log(1+z) < z - \frac{1}{2}z^2 + \frac{1}{3}z^3 < z - \frac{1}{3}z^2$],

and, if $z \geqslant \frac{1}{2}$, $\qquad (1+z)e^{-z} < e^{-\frac{1}{6}z}$,

[since $-\frac{5}{6}z + \log(1+z)$ decreases as z increases, and is negative when $z = \frac{1}{2}$]. Therefore

$$\int_0^\infty [(1+z)e^{-z}]^r z\,dz < \int_0^{\frac{1}{2}} e^{-\frac{1}{3}rz^2} z\,dz + \int_{\frac{1}{2}}^\infty e^{-\frac{1}{6}rz} z\,dz$$

$$< \int_0^\infty (e^{-\frac{1}{3}rz^2} + e^{-\frac{1}{6}rz})\,z\,dz < Kr^{-1} + Kr^{-2} < Kr^{-1},$$

and (9) follows from (11).

6. From (7), (8), and (9),

$$\overline{\lim_{x\to 1}}\,|F(x)| > hr! - Ke^{-r}r^r,$$

Now, by Stirling's theorem,

$$hr!/(Ke^{-r}r^r) \sim \sqrt{(2\pi)}\,hK^{-1}r^{\frac{1}{2}}.$$

Hence $\overline{\lim\limits_{x\to 1}}\,|F(x)|$ is positive for sufficiently large values of r, and our theorem is proved.

7. We proceed now to the proof that the restriction on a_n is the minimum possible. Mr. Hardy has shown* that $\lim \Sigma a_n e^{-\lambda_n x}$ exists, if

$$\lim_{n\to\infty} s_n e^{-\lambda_n x} = 0,$$

when $x > 0$, and if Σa_n is summable $R(1, \mu_n)$, *i.e.*, if

$$\lim (\mu_1 s_1 + \mu_2 s_2 + \ldots + \mu_n s_n)/\lambda_n$$

exists.† To establish our result it is therefore sufficient to show that, given any function $\phi(n)$ tending to infinity, we can find a non-convergent series Σa_n which is summable $R(1, \mu_n)$, and for which

$$\lim_{n\to\infty} s_n e^{-\lambda_n x} = 0 \quad \text{and} \quad |\lambda_n a_n| < \phi(n)\,\mu_n.$$

This result requires a new condition, of an unimportant nature, to be satisfied by the μ's.

THEOREM (C).—*If*

$$\nu_n = \mu_n/\lambda_n \to 0, \quad \nu_{n-1}/\nu_n < K, \quad \textit{and} \quad \lambda_n = \mu_1 + \mu_2 + \ldots + \mu_n \to \infty,$$

then, given any function $\phi(n)$ tending to infinity, a finitely oscillating series Σa_n exists such that

$$\lim_{n\to\infty} (\mu_1 s_1 + \mu_2 s_2 + \ldots + \mu_n s_n)/\lambda_n \quad \textit{and} \quad \lim_{x\to 0} \Sigma a_n e^{-x\lambda_n}$$

exist, while $$|\lambda_n a_n| < \mu_n \phi(n).$$

Let $$\sigma_n = (\mu_1 s_1 + \mu_2 s_2 + \ldots + \mu_n s_n)/\lambda_n,$$

so that $$s_n = (\lambda_n \sigma_n - \lambda_{n-1}\sigma_{n-1})/\mu_n = \Delta\sigma_n/\nu_n + \sigma_{n-1},$$

where we write $\Delta f(n)$ for $f(n) - f(n-1)$.

We shall define the series Σa_n by means of the equation

$$\Delta\sigma_n/\nu_n = \Delta e^{i\psi_n}/\Delta\psi_n = f_n,$$

where ψ_n is a real function of n which will be chosen later.

Then $$\sigma_n = \sum^n \nu_n f_n, \qquad e^{i\psi_n} = \Sigma \Delta\psi_n f_n.$$

We shall suppose that $\psi_n \to \infty$, $\Delta\psi_n \to 0$. Then $e^{i\psi_n}$ and hence $\sum^n \Delta\psi_n f_n$ oscillate finitely. If now $\Delta\psi_n/\nu_n \to \infty$ steadily, it follows by Abel's test for convergence that $\Sigma\nu_n f_n$ is convergent, *i.e.*, that

$$\sigma_n = \sum^n \nu_n f_n \to \text{a definite limit},$$

so that Σa_n is summable $R(1, \mu_n)$.

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* "O.S.," p. 311.

† This generalised method of summation is due to Riesz, *Comptes Rendus*, July 5th, 1909. Cesàro's first mean has 1 in place of μ_n: if this mean exists we say that Σa_n is summable $(C1)$. The notation (Cr), $R(r, \mu_n)$ was introduced by Hardy.

Again, $$f_n = \Delta e^{\iota\psi_n}/\Delta\psi_n = e^{\iota\psi_n}(1-e^{-\iota\Delta\psi_n})/\Delta\psi_n$$
$$\sim \iota e^{\iota\psi_n} \quad (\text{since } \Delta\psi_n \to 0).$$

Hence $s_n = f_n + \sigma_{n-1}$ oscillates finitely, since f_n does and $\sigma_{n-1} \to$ a limit.

Our series is now finitely oscillating and summable $R(1, \mu_n)$: we proceed to show that, with further assumptions as to ψ_n, we have also

$$|a_n| < K\phi(n)\,\nu_n.$$

Now
$$\begin{aligned} a_n &= \Delta s_n = \Delta f_n + \Delta\sigma_{n-1} \\ &= \Delta\left[e^{\iota\psi_n}(1-e^{-\iota\Delta\psi_n})/\Delta\psi_n\right] + \nu_n f_n \\ &= \Delta\left[\iota e^{\iota\psi_n} + O(\Delta\psi_n)\right] + O(\nu_n)^* \\ &= O(\Delta\psi_n) + O(\Delta\psi_{n-1}) + O(\nu_n). \end{aligned} \tag{12}$$

We can construct a function $\phi_1(n)$ tending steadily to infinity, and such that $\phi_1(n) < \phi(n)$, for example by choosing for $\phi_1(n)$ the minimum value of $\phi(m)$ for $m \geqslant n$. Suppose now that

$$\Delta\psi_n < \phi_1(n)\,\nu_n.$$

Then, from (12), $$|a_n| < K\phi_1(n)\nu_n + K\phi_1(n-1)\,\nu_{n-1} + K\nu_n$$
$$< K_1\phi_1(n)\,\nu_n \quad (\text{since } \nu_{n-1}/\nu_n < K).$$

Collecting our results we see that Σa_n is summable $R(1, \mu_n)$ and finitely oscillating, while $$|a_n| < K_1\phi_1(n)\,\nu_n,$$
provided that (*a*) $\psi_n \to \infty$, (*b*) $\Delta\psi_n \to 0$, (*c*) $\Delta\psi_n/\nu_n \to \infty$ steadily, and (*d*) $\Delta\psi_n/\nu_n < \phi_1(n)$.

We must now show that it is possible to choose ψ_n so as to satisfy these conditions. Since $\Sigma\mu_n$ is divergent, it follows, by a well known result due to Abel, that $\Sigma\nu_n$ is divergent. Hence (*c*) includes (*a*). Again, since $\nu_n \to 0$, $\phi_1(n)$ can be chosen so that $\nu_n\phi_1(n) \to 0$, for example, by choosing for $\phi_1(n)$ the minimum value of $\phi(m)$ for $m \geqslant n$, or the minimum value of $\nu_m^{-\frac{1}{2}}$ for $m \geqslant n$, whichever of these values is least. Then (*d*) includes (*b*), and we have only to choose ψ_n so as to satisfy (*c*) and (*d*). This may be done by choosing $$\Delta\psi_n = \nu_n\left[\phi_1(n)\right]^{\frac{1}{2}}.$$

The series $\Sigma a'_n = \Sigma K_1^{-1}a_n$ is now finitely oscillating and summable $R(1, \mu_n)$, while $$|a'_n| < \phi_1(n)\nu_n < \phi(n)\,\mu_n/\lambda_n.$$
Also the condition that $\lim\limits_{n\to\infty} |s'_n e^{-\lambda_n x}| = 0$ for $x > 0$ is satisfied, since s'_n is finitely oscillating. The theorem is therefore established.

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* The notation $O\,[f(x)]$ for any function $\phi(x)$ such that $|\phi(x)| < K\,|f(x)|$ is due to Landau.

8. The conditions of Theorem (C) are satisfied when λ_n is any function of less order than $e^{\epsilon n}$ for all values of ϵ, which increases in a regular manner. When, however, $\lambda_n \geqslant e^{\rho n}$, the theorem breaks down altogether. I have proved, for example, that if $\lambda_n = e^{\rho n}$, $n^{-1}\log|a_n| \to 0$, and $\rho >$ a certain numerical constant, then the convergence of Σa_n is implied by the existence of the generalised Abel's limit. I do not, however, propose to discuss the proof (which is quite simple) here, for I believe that *no* explicit restriction on the a's is necessary when $\lambda_n \geqslant e^{\rho n}$, and I hope to consider this question in another paper.

9. *Applications.*

(i) Poisson attempted to prove Fourier's theorem by showing that, if $f(x)$ is a continuous function, then

$$\lim_{r\to 1}\left[a_0+\sum_1^\infty (a_n\cos nx+b_n\sin nx)\, r^n\right]=f(x),$$

where $a_0+\sum_1^\infty (a_n\cos nx+b_n\sin nx)$ is the Fourier's series corresponding to $f(x)$.* He supposed this result to involve the convergence of the Fourier series; *i.e.*, he assumed the converse of Abel's theorem. It is, however, possible, by means of Theorem (B), to justify this inference in the case when $f(x)$ satisfies Dirichlet's conditions. It follows immediately from Theorem (B) that if

$$\left|\int_0^{2\pi} {\sin \atop \cos} n\theta f(\theta)\, d\theta\right| < K/n, \tag{13}$$

then the Fourier's series is convergent and has the sum $f(x)$. Hence, in order to prove Fourier's theorem with Dirichlet's conditions, it is sufficient to establish the inequality (13). This may be done without difficulty: for example, if $f(x)$ is monotonic the result follows immediately from the second mean value theorem of the differential calculus.†

(ii) M. Fejér and Mr. Hardy‡ have generalised as follows a well known result due to Frobenius:

If Σa_n is summable (C1) to sum s, then

$$\lim_{x\to 1}\Sigma a_n x^{n^k} = s.$$

Mr. Hardy shows also that this result does *not* necessarily hold when the index n^k is replaced by a^n, or, roughly speaking, any index of still

* *Journal de l'école polyt.*, cah. 19, 1823, p. 404. Poisson's proof of this result does not satisfy modern requirements of rigour. The first complete proof was given by Schwarz (*Math. Abh.*, Vol. II, pp. 144, 175).

† Cp. "O.S.," p. 308. Fatou (*loc. cit.*) has made the corresponding deduction from Tauber's theorem, viz., that the Fourier series converges if the upper and lower derivates of $f(x)$ for $0 \leqslant x \leqslant 2\pi$ are bounded.

‡ Fejér, *Math. Ann.*, Vol. 58, p. 51. Hardy, *Quarterly Journal*, Vol. XXXVIII, p. 269.

higher order. Theorem (C), combined with Tauber's theorem,* enables us to show that the result ceases to be true when n^k is replaced by *any* function of higher order than n^k for every value of k.

More precisely:

If Σa_n *is summable* $(C1)$ *but not convergent, and if* λ_n, μ_n *satisfy the conditions of Theorem* (C), *together with* $n\mu_n/\lambda_n \to \infty$, *then* $\lim\limits_{x \to 1} \Sigma a_n e^{-x\lambda}$ *does not necessarily exist.*

For, since $n\mu_n/\lambda_n \to \infty$, it follows from Theorem (C) that there exists a series Σa_n, non-convergent but summable $(C1)$, such that

$$|na_n| < n\mu_n/\lambda_n, \quad \text{or} \quad |a_n| < \mu_n/\lambda_n.$$

If, now, $\lim \Sigma a_n e^{-x\lambda_n}$ exists, Theorem (B) requires that Σa_n should converge, which is untrue. Hence the theorem.

It is easy to prove generalisations, relating to summability $R(1, \mu_n)$, of both the positive and the negative result.

[*Added May* 18*th*, 1911.—

(iii) The following theorem is an immediate deduction from Theorem (B).

If $\lim \Sigma a_n e^{-xn}$ *exists,*† *and* $|n^{1-a} a_n| < K$, *where* $a > 0$, *then* $\Sigma n^{-a} a_n$ *is convergent.*

For $$\Sigma n^{-a} a_n e^{-xn} = \int_0^\infty \Sigma a_n e^{-(x+t)n} t^{a-1}\, dt.$$

It is not difficult to show that, with our assumptions, the right-hand side tends to a definite limit as $x \to 0$: the result then follows at once.

A similar theorem holds when n^{-a} is replaced by any function $\phi(n)$ which tends to zero and can be expressed in the form

$$\phi(n) = \int_0^\infty e^{-nt} \psi(t)\, dt,$$

where $\int_0^\infty e^{-xt} \psi(t)\, dt$ is absolutely convergent for $x > 0$. For example, a possible form of $\phi(n)$ is $n^{-a} (\log n)^{\rho_1} \ldots (\log_k n)^{\rho_k}$.]

10. We have seen that if $\Sigma a_n x^n \to s$ and $a_n \to 0$, Σa_n is not necessarily convergent. There remains, however, the possibility that Σa_n is necessarily summable by one of Cesàro's means. I will add, in conclusion, a note on this point.

The most obvious types of series for which Abel's limit exists, and for

* We shall use, as a matter of fact, the wider Theorem (B) in the proof, but Tauber's theorem suffices at the expense of a little further complication.

† Or, more generally, if $|\Sigma a_n e^{-xn}| < Kx^{-a+\eta}$, where $\eta > 0$.

which $|a_n| < Kn^p$, are summable, and, so far as I know, no one has constructed a non-summable series with the two former properties. It is, however, quite easy to do this. For example, $\lim_{x\to 1} \sum_2^\infty n^{-1}(ln)^a \exp[\iota ln(l_2 n)^2] x^n$ exists, while $\Sigma n^{-1}(ln)^a \exp[\iota ln(l_2 n)^2]$ is not summable by any one of Cesàro's means, unless $a \leqslant 0$, in which case it is convergent.

The latter part of this statement is easily verified. To establish the existence of Abel's limit we employ the formula, easily established by means of the theory of residues,

$$\sum_2^\infty n^{-1}(ln)^a \exp[\iota ln(l_2 n)^2] x^n = \int_{\frac{3}{2}}^{\infty} y^{-1}(ly)^a \exp[\iota ly(l_2 y)^2] x^y \, dy$$

$$-\int_{\frac{3}{2}}^{\frac{3}{2}+\infty\iota} y^{-1}(ly)^a \exp[\iota ly(l_2 y)^2] x^y \frac{dy\, e^{-2\pi\iota y}}{1-e^{-2\pi\iota y}}$$

$$+\int_{\frac{3}{2}}^{\frac{3}{2}-\infty\iota} y^{-1}(ly)^a \exp[\iota ly(l_2 y)^2] x^y \frac{dy}{e^{2\pi\iota y}-1}. \qquad (14)$$

In the first integral the integration may be taken along a line lying in the first quadrant. It is then easy to show that the limit of the right hand side of (14), as $x \to 1$, is obtained by writing $x = 1$ in the subjects of integration.

It appears, therefore, that $\Sigma a_n x^n \to s$ and $|na_n| < K(\log n)^a$ do not necessarily imply the summability of Σa_n. There is thus a very strong presumption that whatever function $\phi(n)$, tending to infinity, may be chosen, $\Sigma a_n x^n \to s$ and $|na_n| < \phi(n)$ do not necessarily imply the summability of Σa_n. But the discussion of the lacuna seems difficult, and I do not propose to consider the matter further at present.

[*Added May* 18*th*.—The work of §§ 1–6 may be modified so as to establish the following theorem.

The series Σa_n is summable $R(1, \mu_n)$ if it is known to be finitely oscillating, if $\lim_{x\to 0} \Sigma a_n e^{-x\lambda_n}$ exists, and if the index λ_n satisfies the conditions of Theorem (B). In particular, *Σa_n is summable (C1) if it is finitely oscillating and if Abel's limit exists.* This last result may also be generalised as follows :—*If Abel's limit exists, and Σa_n is "finite (Cr)," i.e., if Cesàro's r-th mean is finite, then Σa_n is summable $(C\,\overline{r+1})$.*

In the case of Cesàro summation, we start from the formula

$$x^2 \Sigma \sigma_n e^{-xn} \to s,$$

where $\sigma_n = s_1 + s_2 + \ldots + s_n$. The proof then follows the lines of that of Theorem (B), σ_n and s_n playing the rôles of s_n and a_n.]