THE UNIFORM CONVERGENCE OF A CERTAIN CLASS OF TRIGONOMETRICAL SERIES

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1. It is well-known that the series $\Sigma a_n \sin n\theta$, where (a_n) is a sequence of positive numbers decreasing steadily to zero, is convergent for all real values of θ. Since, moreover, by Abel's lemma,

$$|a_n \sin n\theta + \ldots + a_p \sin p\theta| < |a_n \operatorname{cosec} \tfrac{1}{2}\theta|,$$

the series is uniformly convergent throughout any interval which does not include zero or a multiple of 2π.

We shall show that the condition $na_n \to 0$ is both necessary and sufficient to secure the uniform convergence of the series throughout any interval whatever.

Since the series is unaltered, if we substitute $2k\pi + \theta$ for θ, and changed in sign only, if we substitute $2k\pi - \theta$ for θ, k being any integer, we can without loss of generality suppose θ to lie in the interval $(0, \pi)$.

2. We shall prove the two following propositions:—

(1) If p is taken sufficiently great, we can find a value of θ in the interval $(0, \pi)$ such that

$$|a_n \sin n\theta + \ldots + a_p \sin p\theta| > pa_p/\pi.$$

(2) If ra_r, where r is greater than or equal to n, is never greater than M, then

$$|a_n \sin n\theta + \ldots + a_p \sin p\theta| < (\pi + 1)M,$$

for all values of p and θ.

The condition that the series $\Sigma a_n \sin n\theta$ should be uniformly convergent throughout any interval is that, for any ϵ, we can find an integer N_ϵ, such that

$$|a_n \sin n\theta + \ldots + a_p \sin p\theta| < \epsilon,$$

for all values of θ in the interval, and for all values of p greater than n, provided only that $n \geqslant N_\epsilon$.

The first of the above propositions shows that, unless $na_n \to 0$, the interval of uniform convergence cannot include zero, or consequently any multiple of 2π.

The second shows that, if $na_n \to 0$, then there is uniform convergence throughout any interval whatever.

3. To prove the first proposition :—

We have

$$\sin n\theta + \ldots + \sin p\theta = \tfrac{1}{2}\{\cos(n-\tfrac{1}{2})\theta - \cos(p+\tfrac{1}{2})\theta\} \operatorname{cosec} \tfrac{1}{2}\theta.$$

If we give θ the value $\pi/(2p+1)$, this sum becomes $\frac{1}{2}\cos(n-\frac{1}{2})\theta \operatorname{cosec}\frac{1}{2}\theta$, which is greater than $\{1-\frac{1}{2}(n-\frac{1}{2})^2\theta^2\}\,\theta$, which for this value of θ is certainly greater than p/π, if p is greater than $2n-1$.

Since $\sin n\theta, \ldots, \sin p\theta$ are all positive, and $a_n \geqslant a_{n+1} \ldots \geqslant a_p$,

$$a_n \sin n\theta + \ldots + a_p \sin p\theta > a_p(\sin n\theta + \ldots + \sin p\theta) > pa_p/\pi.$$

Hence the first proposition is proved.

4. To prove the second proposition :—

Since $(\sin\phi)/\phi$ decreases steadily as ϕ increases from 0 to $\pi/2$, it follows that, if $0 < \theta < \pi$, $\operatorname{cosec}\frac{1}{2}\theta < \pi/\theta$.

By Abel's lemma,

$$|a_n \sin n\theta + \ldots + a_p \sin p\theta| < a_n \operatorname{cosec}\tfrac{1}{2}\theta < \pi a_n/\theta < na_n,$$

if $\theta \geqslant \pi/n$.

If $\theta < \pi/n$, let $\theta = \pi/(m+a)$, where m is an integer $\geqslant n$, and $0 < a \leqslant 1$.

If $p \leqslant m$, then no one of $n\theta, (n+1)\theta, \ldots, p\theta$ exceeds π, so that

$$\begin{aligned} a_n \sin n\theta + \ldots + a_p \sin p\theta &< \theta(na_n + \ldots + pa_p) \\ &< (p-n+1)M\pi/(m+a) < \pi M. \end{aligned}$$

If $p > m$,

$$
\begin{aligned}
&| a_n \sin n\theta + \ldots + a_p \sin p\theta | \\
&\quad \leqslant | a_n \sin n\theta + \ldots + a_m \sin m\theta | + | a_{m+1} \sin (m+1)\theta + \ldots + a_p \sin p\theta | \\
&\quad < \pi M + a_{m+1} \operatorname{cosec} \tfrac{1}{2}\theta < \pi M + (m+a)\, a_{m+1} < (\pi+1) M.
\end{aligned}
$$

Hence the second proposition is proved, and the result stated is established.

Note.—This paper replaces one communicated to the Society (January 13th, 1916) by Mr. T. W. Chaundy: he then showed that $na_n \to 0$ is a necessary condition for uniform convergence, and that $na_n \to 0$ *steadily* is a sufficient condition. But, as will be seen from the foregoing investigation, the condition $na_n \to 0$ is sufficient by itself.