i.e., $F_{3,2}(a+2\beta-1, a, a+\beta, 1-\gamma, t, 1-2t)$

$$= \frac{AB}{BC} t^{\beta+\gamma+1} F_{3,2-\beta-\gamma}(2-\beta-\gamma, \beta, \beta+\gamma, 1-\gamma, t, t)$$

$$\begin{aligned} \text{or } \frac{\Gamma(a)}{\Gamma(a+\beta)} \{ & F(2-2\gamma, a, a+\beta, t) \\ & + (1-\gamma) \frac{a}{a+\beta} F(3-2\gamma, a+1, a+\beta+1, t)(1-2t) \\ & + \frac{(1-\gamma)(2-\gamma)}{2!} \frac{a(a+1)}{(a+\beta)(a+\beta+1)} \\ & \quad \times F(4-2\gamma, a+2, a+\beta+2, t)(1-2t)^2 + \dots \} \\ = & \frac{AB}{BC} t^{\beta+\gamma+1} \frac{\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \\ & \times \left\{ F(2-\beta-\gamma, \beta, \beta+\gamma, t) + (1-\gamma) F(3-\beta-\gamma, \beta, \beta+\gamma, t) t \right. \\ & \quad \left. + \frac{(1-\gamma)(2-\gamma)}{2!} F(4-\beta-\gamma, \beta, \beta+\gamma, t) t^2 + \dots \right\}, \end{aligned}$$

an equation for t in the form of a power-series.

If $\beta + \gamma = 1$,

that is, if AB and CD are parallel, the powers of t are all integral.

If the pentagon is regular, the equation becomes

$$\begin{aligned} F\left(\frac{4}{5}, \frac{3}{5}, \frac{6}{5}, t\right) + \frac{2}{5} \cdot \frac{3}{5} \cdot F\left(\frac{9}{5}, \frac{8}{5}, \frac{11}{5}, t\right)(1-2t) + \dots \\ = t^{\frac{1}{5}} \left\{ F\left(\frac{4}{5}, \frac{3}{5}, \frac{6}{5}, t\right) + \frac{2}{5} \cdot F\left(\frac{9}{5}, \frac{8}{5}, \frac{11}{5}, t\right) t + \dots \right\}. \end{aligned}$$

On certain Regular Polygons in Modular Network. By L. J. ROGERS. Received June 7th, 1898; in revised form, August 29th, 1898. Read June 9th, 1898.

1. In connexion with the theory of modular functions, it is proved that a plane may be entirely covered with a network of circles whose interstices are curvilinear triangles equiangular to one another whose angles are all sub-multiples of two right angles. These networks are of three kinds, according as the sum of the angles in each triangle is greater, equal, or less than two right angles. In the

first case, the triangles are limited in number and diverge. They can, moreover, be stereographically projected on to the surface of a sphere, which is thereby divided into equal spherical triangles (see Klein, *Lectures on the Icosahedron*, or Klein-Fricke, *Theorie der elliptischen Modulfunctionen*). In the second case, the network consists of plane triangles, and the circles degenerate into straight lines. In the third kind, the circles all intersect a certain circle orthogonally, and the portions of them internal to this circle form a network which converges to it, while the external portions form a network inverse to the former and re-diverging to infinity (see Klein-Fricke, Vol. I., p. 108). The network of the third kind is the subject of the present paper.

It is easy to see that a property analogous to the stereographic property of the first kind also exists for the third kind. For, if

$$x^2 + y^2 = 1$$

be the orthogonal circle, and

$$a(x^2 + y^2 + 1) + 2gx + 2fy = 0, \quad a'(x^2 + y^2 + 1) + 2g'x + 2f'y = 0$$

be any two circles of the network, then they intersect at an angle

$$\cos^{-1} \frac{gg' + ff' - aa'}{\sqrt{g^2 + f^2 - a^2} \sqrt{g'^2 + f'^2 - a'^2}},$$

so that, if imaginary lines be drawn whose direction-cosines with respect to three rectangular axes in space are proportional to (g, f, ai) and $(g', f', a'i)$, then planes through the origin perpendicular to them will meet a sphere $x^2 + y^2 + z^2 = 1$ in imaginary great circles which intersect in the same real angle as the above circles. The network is thus projected on to an imaginary part of the sphere, and divides it into imaginary spherical triangles. The method, however, has obvious drawbacks, as we cannot re-project any plane section of the sphere back into a real circle unless the normal to the plane has direction cosines of the form $(\lambda, \mu, \nu i)$, where λ, μ, ν are real.

However, the triangles of the network possess a trigonometry with formulæ analogous to those in spherical trigonometry, and which we may easily establish by means of homographic complex transformation, as is indicated by Poincaré in *Act. Math.*, Vol. I., p. 6, &c.

Three portions of network are given in the diagrams accompanying this paper, viz., Fig. 1, in which the angles of every triangle are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$; Fig. 2, in which they are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{9}$; and Fig. 3, in which they are $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{5}$. In general, a network with angles

$\frac{\pi}{\nu_1}, \frac{\pi}{\nu_2}, \frac{\pi}{\nu_3}$ will be alluded to as (ν_1, ν_2, ν_3) , and in the purely geometric treatment of the subject, irrespective of its connexion with modular functions, it is immaterial in what order we write the symbols. We shall, however, always assume that the centre of the orthogonal circle ($x^2 + y^2 = 1$) is a node of the net.

Nodes of the net will be called ν_1 -points, &c., according as the circles meeting there intersect at angles π/ν_1 , &c. The origin we shall take as a ν_3 -point, and the initial line of coordinates will be one of the diameters of the circle $x^2 + y^2 = 1$ belonging to the network.

If we trace along any circle of the net, it is easy to see the law for the order in which the kinds of nodes follow one another, viz., that a node of even order (i.e., where ν is even) lies between two like nodes, but a node of odd order lies between two unlike.

Thus, if ν_1, ν_2, ν_3 are all odd, they follow one another in rotation.

If, however, one is even (say ν_1), and ν_2, ν_3 are odd, the order is $\nu_1 \nu_2 \nu_3 \nu_1 \nu_3 \nu_2$, where dots have the same significance as in circulating decimals. Moreover, this order, or this reversed, is the only possible one.

If two are even (say ν_1, ν_2), while ν_3 is odd, some circles only contain $\nu_1 \nu_3$, while others contain $\nu_3 \nu_1 \nu_2$.

If the ν 's are all even, the orders are $\nu_1 \nu_2, \nu_2 \nu_3, \nu_3 \nu_1$.

2. It will be convenient to write down for reference the formulæ connecting the sides and angles of any one of these curvilinear triangles which compose the modular network, as indicated by Poincaré in the *Act. Math.*, Vol. I., p. 6. The formulæ in the two following sections will therefore be stated without proof.

The principle involved in the modular function theory is that any circles of the network can be conformally transformed into any other on which the nodes lie in the same order as on the original circle, by the homographic relation $\zeta = (za + b)/(zc + d)$, where z is the complex of any point on the first circle and ζ its copy on the second, and where the orthogonal circle $x^2 + y^2 = 1$ is transformed into itself.

The most general form for such a substitution is

$$\zeta = e^{i\theta} \frac{z - re^{i\phi}}{z\bar{r} - e^{-i\phi}}, \quad (1)$$

where r is real.

If a circle cut the unit orthogonal circle in points whose complexes

are x and y , the complex of any point on it is of the form

$$\sqrt{xy} \frac{\kappa \sqrt{x+i\sqrt{y}}}{\kappa \sqrt{y+i\sqrt{x}}}, \quad (2)$$

where κ is some real quantity.

It will be convenient to allude to κ as the *parameter* of the point with respect to the circle.

If κ, λ are the parameters of two points on any circle of the system, then the ratio λ/κ is unaltered for any transformation (1).

The natural logarithm of this ratio will be called the *parametral distance* between the points λ and κ on the circle. It is called by Poincaré the " L " of the arc.

3. The trigonometrical formulæ connecting the angles ψ_1, ψ_2, ψ_3 of a curvilinear triangle with the parametral lengths u_1, u_2, u_3 of the opposite sides can be obtained by supposing the triangle to be made up of two diameters of the orthogonal circle and a circle of the network which lies nearest to the origin.

If CA is the axis of real quantities, the x and y in § 2, (2), are 1

and -1 ; the parameter of C , the origin, is unity, and the parameter of A is given by

$$CA = \frac{\kappa - 1}{\kappa + 1} = \frac{e^{u_2} - 1}{e^{u_2} + 1}$$

By elementary geometry, we have

$$\frac{\sinh u_1}{\sin \psi_1} = \frac{\sinh u_2}{\sin \psi_2} = \frac{\sinh u_3}{\sin \psi_3}. \quad (1)$$

$$\cosh u_1 = \frac{\cos \psi_1 + \cos \psi_2 \cos \psi_3}{\sin \psi_2 \sin \psi_3}, \quad (2)$$

$$\begin{aligned} \sinh u_1 \sin \psi_2 \sin \psi_3 &= \sqrt{2 \cos \psi_1 \cos \psi_2 \cos \psi_3 + \cos^2 \psi_1 + \cos^2 \psi_2 + \cos^2 \psi_3 - 1} \\ &= n \text{ (say)}, \end{aligned} \quad (3)$$

$$\cosh (u_1 + u_2 + u_3) = \frac{\cos \psi_1 + \cos \psi_2 + \cos \psi_3 - 1}{4 \sin \frac{1}{2} \psi_1 \sin \frac{1}{2} \psi_2 \sin \frac{1}{2} \psi_3}. \quad (4)$$

4. The object of the present paper is to investigate under what conditions it is possible that a network should contain a parametrically regular polygon whose angles are all equal to one of the angles $\frac{\pi}{\nu_1}, \frac{\pi}{\nu_2}, \frac{\pi}{\nu_3}$, and whose sides traced in the same order are conformally equivalent, and are composed of the shortest circular arcs connecting conformal nodes (the angular points of the polygon).

The method employed will be to suppose that two adjacent diametrical lines of the network shall be adjacent sides of the polygon, which may afterwards be transformed so as to have its centre at the origin and thus appear in a symmetrical shape.

Such a polygon, having an angle $\frac{\pi}{\nu}$, I propose to call a *primary* polygon; while a polygon having angles equal to $\frac{m\pi}{\nu}$ will be called *m-ary* polygons. The existence of primary polygons is proved in Klein's *Modulfunctionen*, Vol. I., p. 369, where

$$\psi_1, \psi_2, \psi_3 = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7};$$

also by R. Fricke, *Math. Ann.*, Vol. XL., p. 458, for the same values of the angles.

Secondary polygons, moreover, exist, as is shown by R. Fricke in *Act. Math.*, Vol. XVII., p. 373, for the network (2, 4, 5), but the aim of the present paper is to prove the existence, and to find the limit to the number, of primary polygons only.

To develop the theory, we need reference to the general treatment of the substitution § 2, (1).

It is shown in Klein's *Modulfunctionen*, Vol. I., p. 163, and in Forsyth's *Theory of Functions*, p. 512, &c., that the substitution in question may be reduced to the form

$$\frac{\zeta - z_1}{\zeta - z_2} = k \frac{z - z_1}{z - z_2},$$

where z_1, z_2 , the fixed points of the substitution, are

$$\frac{1}{r} e^{i(\theta + \phi)} \left\{ \cos \frac{1}{2} (\theta - \phi) \pm \sqrt{\cos^2 \frac{1}{2} (\theta - \phi) - r^2} \right\}, \quad (1)$$

$$\text{while} \quad k = \frac{1}{1 - r^2} \left\{ \sin \frac{1}{2} (\theta - \phi) + i \sqrt{\cos^2 \frac{1}{2} (\theta - \phi) - r^2} \right\}^2. \quad (2)$$

$$\text{If} \quad r^2 < \cos^2 \frac{1}{2} (\theta - \phi),$$

this value of k is a complex of form $e^{\psi i}$, and the substitution is *elliptic*, while the fixed points are mutually inverse with respect to $x^2 + y^2 = 1$.

If $r^2 > \cos^2 \frac{1}{2}(\theta - \varphi)$,

k is real, while the moduli of the fixed points are unity; that is, the substitution is *hyperbolic*, and the fixed points lie on the orthogonal circle.

Returning to the subject of the polygons, we see that, if $CA \dots C_1$ be the initial line and $CB \dots C_{-1}$ the next diameter in the direction of positive rotation, then the order of nodes from C_{-1} to C must be the same as from C to C_1 .

A substitution can then be found to bring C_{-1} to C and C to C_1 . Now $CC_1 = CC_{-1}$, for the parametral lengths are equal, and such lengths measured along straight lines from the origin imply equality of actual distance also.

If, then, $CC_1 = \lambda$, the substitution required is

$$\zeta = \frac{z - \lambda e^{\psi_3 i}}{z\lambda - e^{\psi_3 i}} \quad (\text{say } S), \quad (3)$$

since it brings C ($z = 0$) to C_1 ($z = \lambda$) and C_{-1} ($z = \lambda e^{\psi_3 i}$) to C .

The same substitution S will convert CC_1 into the nearest circle C_1A_1 , which passes through C_1 , making an angle of ψ_3 with CC_1 towards C , and a continued application of S produces a series of circles forming a parametrically regular polygon having angles all equal to ψ_3 , in fact, a polygon of the type we are seeking.

The only condition for the existence of a closed polygon is that S should be elliptic.

This condition is, by (2), that

$$\lambda^2 < \cos^2 \frac{1}{2}\psi_3,$$

in which case,

$$k = e^{2\psi_3 i} \quad (\text{say})$$

$$= \frac{1}{1 - \lambda^2} (\sin \frac{1}{2}\psi_3 + i \sqrt{\cos^2 \frac{1}{2}\psi_3 - \lambda^2})^2,$$

i.e.,

$$\cos \psi = \frac{\sin \frac{1}{2}\psi_3}{\sqrt{1 - \lambda^2}}. \quad (4)$$

Conversely, if this requirement is fulfilled, S will be elliptic, and must belong to the group of substitutions which leave the network unaltered. Hence ψ must be a multiple of either ψ_1 , ψ_2 , or ψ_3 .

Let us first suppose the numbers $\nu_1, \nu_2 = \frac{\pi}{\psi_1}, \frac{\pi}{\psi_2}$ are both odd.

Then the period OC_1 is of three segments, and its parametral length is $u_1 + u_2 + u_3$, so that

$$\lambda = \tanh \frac{1}{2} (u_1 + u_2 + u_3).$$

The relation (4) then becomes

$$\begin{aligned} \cos \psi &= \sin \frac{1}{2} \psi_3 \cosh \frac{1}{2} (u_1 + u_2 + u_3) \\ &= \frac{\cos \psi_1 + \cos \psi_2 + \cos \psi_3 - 1}{4 \sin \frac{1}{2} \psi_1 \sin \frac{1}{2} \psi_2}, \quad \text{by § 3.} \end{aligned} \quad (5)$$

The condition for a closed polygon is, therefore,

$$\cos \psi_1 + \cos \psi_2 + \cos \psi_3 - 1 < 4 \sin \frac{1}{2} \psi_1 \sin \frac{1}{2} \psi_2,$$

i.e.,

$$\cos^2 \frac{1}{2} \psi_3 < (\sin \frac{1}{2} \psi_1 + \sin \frac{1}{2} \psi_2)^2$$

or

$$\cos \frac{1}{2} \psi_3 < \sin \frac{1}{2} \psi_1 + \sin \frac{1}{2} \psi_2, \quad (6)$$

since all the angles are acute.

The condition gives a very limited number of cases:—

CASE I.—If

$$\psi_1 = \psi_2 = \frac{\pi}{3},$$

(6) is satisfied, and

$$\cos \psi = \cos \psi_3,$$

so that for any network $(3, 3, \nu)$ we have a polygon of the type sought. Moreover, since

$$\psi = \psi_3 = \frac{\pi}{\nu},$$

the period of the substitution is ν , so that the polygon has ν sides.

If

$$\psi_1 = \frac{\pi}{3}, \quad \psi_2 = \frac{\pi}{5},$$

then

$$\cos \frac{1}{2} \psi_3 < \frac{\sqrt{5} + 1}{4} = \cos \frac{\pi}{5},$$

so that

$$\psi_3 > \frac{2\pi}{5}.$$

The only possible value for ψ_3 is therefore $\frac{\pi}{2}$; but, since

$$\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{5} > \pi$$

(the icosahedral case), we exclude this type as not belonging to the network of the third species.

CASE II.—If $\psi_1 = \frac{\pi}{3}$, $\psi_2 = \frac{\pi}{7}$, $\psi_3 = \frac{\pi}{2}$,

then $\cos \psi = \frac{2 \cos \frac{\pi}{7} - 1}{4 \sin \frac{\pi}{14}} = \frac{\cos \frac{3\pi}{14}}{2 \sin \frac{\pi}{7}} = \frac{\sin \frac{2\pi}{7}}{2 \sin \frac{\pi}{7}} = \cos \frac{\pi}{7}$,

so that we here have a rectangular heptagon.

However, if $\psi_1 = \frac{\pi}{3}$, $\psi_2 = \frac{\pi}{9}$,

$$\cos \frac{1}{2}\psi_3 < \frac{1}{2} + \sin 10^\circ = .6736 < .7071 = \cos \frac{\pi}{4},$$

as is seen by tables, so that $\psi_3 > \frac{\pi}{2}$,

which is impossible.

A fortiori, if $\psi_1 = \frac{\pi}{3}$,

ψ_2 cannot be less than $\frac{\pi}{9}$, so that we can exclude all values of ψ_1 and ψ_2 , except $\frac{\pi}{5}$ and $\frac{\pi}{7}$.

Again, when $\psi_1 = \psi_2 = \frac{\pi}{5}$,

$$\cos \frac{1}{2}\psi_3 < 2 \sin 18^\circ = .6180 < \cos \frac{\pi}{4},$$

so that we may reject this case also, and all other cases in which ν_1 and ν_2 are greater than 5.

We have, therefore, investigated all cases in which ν_1 and ν_2 are odd, and have shown that $(3, 3, \nu)$ and $(3, 7, 2)$ are the only networks for odd ν_1 and ν_2 in which primary polygons occur. Also, when ν_3 is even, ν_1 and ν_2 must be odd, since otherwise we cannot have the same set of nodes on the two lines CA and CB . For along CA , if ν is even, they follow the order $\nu_3 \nu_1 \nu_3 \nu_1 \dots$, and along CB , if ν_2 is odd, the order $\nu_3 \nu_2 \nu_1 \nu_2 \nu_3 \dots$.

We will now proceed to the case in which ν_1 is even.

Here there are six segments of the line CC_1 before arriving at C_1 (§ 1), viz., two of each kind, so that the parametral distance CC_1 is $2(u_1 + u_2 + u_3)$ and $\lambda = \tan(u_1 + u_2 + u_3)$.

This gives $\sqrt{1-\lambda_2} = \operatorname{sech} 2u = \frac{1}{2 \cosh^2 u - 1},$

so that $\cos \psi = \sin \frac{1}{2} \psi_3 (2 \cosh^2 u - 1),$

which, by § 3, (4),

$$= \frac{c_1 c_2 c_3 + c_2 c_3 + c_1 c_2 + c_1^2 + c_2^2 + c_3^2 - c_1 - c_2 - c_3}{2 \sin \frac{1}{2} \psi_3 (1 - c_1)(1 - c_2)}, \quad (7)$$

where c_1, c_2, c_3 denote the cosines of ψ_1, ψ_2, ψ_3 , respectively.

Now it is easy to show, by differentiating $\cos \psi$ partially with respect to ψ_1, ψ_2, ψ_3 , that $\cos \psi$ increases when any of these angles decrease, so that we easily obtain a limit as to the smallness of these angles. For instance, when

$$\psi_1 = \psi_3 = \frac{\pi}{3} \quad \text{and} \quad \psi_2 = \frac{\pi}{4},$$

then

$$\cos \psi = \frac{3 + 2\sqrt{2}}{2},$$

which is impossible, and consequently no case is possible unless at least one angle is greater than one of these. Thus the even denominator must be 2. Let, then,

$$\psi_1 = \frac{\pi}{2}.$$

Now, testing for

$$\psi_2 = \psi_3 = \frac{\pi}{5},$$

we find

$$\cos \psi = \frac{3\sqrt{5} + 5}{4},$$

so that one of the odd denominators must be 3.

CASE III.—Let, then, $\psi_2 = \frac{\pi}{3}$ (supposing $\psi_2 > \psi_3$).

We now have

$$\cos \psi = \frac{2 \cos^2 \psi_3 - \cos \psi_3 - \frac{1}{2}}{2 \sin \frac{1}{2} \psi_3} = \frac{\cos \frac{5}{2} \psi_3}{2 \sin \psi_3}.$$

If $\psi_3 = \frac{\pi}{7}$, then $\cos \psi = \frac{1}{2},$

which, since the period of the substitution is 3, gives the triangle mentioned in Klein's *Modulfunctionen*, Vol. I., p. 369.

CASE IV.—If $\psi_3 = \frac{\pi}{9}$, then

$$\cos \psi = \cos \frac{\pi}{9},$$

which gives a nine-sided polygon.

However, if $\psi_3 = \frac{\pi}{10}$, then

$$\cos \psi = \frac{\sqrt{5}+1}{2\sqrt{2}} > 1,$$

so that ψ_3 cannot be less than $\frac{\pi}{10}$.

Again, we may suppose that

$$\psi_2 < \psi_3 \quad \text{and} \quad \psi_3 = \frac{\pi}{3},$$

so that

$$\cos \psi = \frac{\cos \frac{5}{2}\pi_2}{4 \sin \frac{1}{2}\psi_2 \sin \psi_2}.$$

If $\psi_2 = \frac{\pi}{7}$, then
$$\cos \psi = \frac{1}{4 \sin \frac{\pi}{14}};$$

but, since $\sin^{-1} \frac{1}{4} = 14^\circ 29'$,

approximately, it follows that

$$\sin \frac{\pi}{14} < \sin^{-1} \frac{1}{4} \quad \text{and} \quad \cos \psi > 1.$$

Hence ψ_2 cannot be equal to $\frac{\pi}{7}$ or any smaller angle.

We see, then, that the only cases for networks in which only one of the numbers ν_1, ν_2, ν_3 is even, and the corresponding angle is not the angle of the polygon, are (2, 3, 7) and (2, 3, 9).

We now come to the last case in which ν_1, ν_2 are both even.

Here the recurrence of the ν_3 -points on the line CA (§ 3) takes place after four segments of parametral length u_3, u_3, u_1, u_1 , so that

$$\lambda = \tanh(u_3 + u_1);$$

whence $\cos \psi = \sin \frac{1}{2}\psi_3 \cosh(u_1 + u_2).$

$$\text{Now} \quad \cosh(u_1 + u_2) = \cosh u_1 \cosh u_2 + \sinh u_1 \sinh u_2$$

$$= \frac{(c_3 c_3 + c_1)(c_1 c_3 + c_2) + n^2}{\sin \psi_1 \sin \psi_2 \sin^2 \psi_3}$$

(by the formulæ in § 3)

$$= \frac{c_1 c_2 c_3 + c_1 c_2 + c_1^2 + c_2^2 + c_3 - 1}{2 \sin^2 \frac{1}{2}\psi_3 \sin \psi_1 \sin \psi_2}.$$

Therefore
$$\cos \psi = \frac{c_1 c_2 c_3 + c_1 c_2 + c_1^2 + c_2^2 + c_3 - 1}{2 \sin \frac{1}{2}\psi_3 \sin \psi_1 \sin \psi_2}.$$

Again, in this case, by differentiating, we may show that $\cos \psi$ always increases when any of the angles ψ_1, ψ_2, ψ_3 diminish, so that,

on testing for

$$\psi_1 = \psi_2 = \frac{\pi}{4}, \quad \psi_3 = \frac{\pi}{3},$$

which gives

$$\cos \psi = \frac{5}{9},$$

we see that no cases are possible unless one of the angles, say

$$\psi_1 = \frac{\pi}{2} \quad \text{and} \quad \psi_3 = \frac{\pi}{4}.$$

$$\text{Thus} \quad \cos \psi = \frac{c_2^2 + c_3 - 1}{2 \sin \frac{1}{2} \psi_3 \sin \psi_2} = \frac{2 \cos \psi_3 - 1}{2 \sqrt{2} \sin \frac{1}{2} \psi_3} = \frac{\cos \frac{3}{2} \psi_3}{\sqrt{2} \sin \psi_3}.$$

CASE V.—We omit the case $\psi_3 = \frac{\pi}{3}$,

(the octahedral case), since then

$$\psi_1 + \psi_2 + \psi_3 > \pi,$$

and put

$$\psi_3 = \frac{\pi}{5},$$

which gives

$$\cos \psi = \frac{1}{\sqrt{2}} = \cos \frac{\pi}{4}.$$

We have then, for the network (2, 4, 5), a polygon of four sides.

Further than this we cannot go, for, even if $\psi_3 = \frac{\pi}{6}$, $\cos \psi = 1$;

and, hence, if $\psi_3 < \frac{\pi}{6}$, $\cos \psi > 1$.

Recapitulating, we see that the only possible primary polygons occur in the following networks:—

(3, 3, ν), containing a polygon of ν sides for all integral values of ν ;

(3, 7, 2), „ „ heptagon;

(2, 3, 7), „ „ triangle;

(2, 3, 9), „ „ polygon of nine sides;

(2, 4, 5), „ „ quadrilateral.

If ν denote the number of sides in each case, it is evident that, by transforming the network conformally so as to bring a ν -point to the origin, the ν -gon becomes actually symmetrical instead of only parametrically so. However, in the cases (3, 3, ν), (3, 7, 2), and (2, 3, 9) the origin was, in the first instance, a point of the required type, so that the network will contain the actually symmetrical polygon without further transformation.

Subjoined are figures illustrating the triangle (Fig. 1), the nonagon (Fig. 2), and the quadrilateral (Fig. 3) in their symmetrical position. The figure of the heptagon is given in *Math. Ann.*, Vol. xli., p. 458.

FIG. 2.

FIG. 3.

5. As an assistance in describing a network, it may be noticed that the centre of every circle, which contains the same order of nodes as the diameters, will be an external node conformal with the origin. Thus the centre $\frac{2xy}{x+y}$, and therefore the inverse $\frac{x+y}{2}$ of the centre of the circle (x, y) , will be a ψ_3 -point. To prove this, we can first show that, if an elliptic substitution

$$\frac{\zeta - z_1}{\zeta - z_3} = e^{2\pi i \frac{z - z_1}{z - z_3}}, \quad (1)$$

where z_1, z_3 lie on (x, y) , transform (x, y) into (X, Y) , then the square of the substitution will transform $\frac{x+y}{2}$ into $\frac{X+Y}{2}$.

Let κ , K be the parameters of z_1 on (x, y) and (X, Y) , respectively, so that

$$z_1 = \sqrt{xy} \frac{\kappa\sqrt{x+i\sqrt{y}}}{\kappa\sqrt{y+i\sqrt{x}}} = \sqrt{XY} \frac{K\sqrt{X+i\sqrt{Y}}}{K\sqrt{Y+i\sqrt{X}}},$$

and
$$z_2 = \sqrt{xy} \frac{\kappa\sqrt{x-i\sqrt{y}}}{\kappa\sqrt{y-i\sqrt{x}}} = \sqrt{XY} \frac{K\sqrt{X-i\sqrt{Y}}}{K\sqrt{Y-i\sqrt{X}}}.$$

Now (1) changes x into X , and, substituting these values for z and ζ , we obtain

$$\frac{K\sqrt{Y-i\sqrt{X}}}{K\sqrt{Y+i\sqrt{X}}} = e^{2\psi i} \frac{\kappa\sqrt{y-i\sqrt{x}}}{\kappa\sqrt{y+i\sqrt{x}}}. \quad (2)$$

Again, if
$$z = \frac{x+y}{2},$$

we easily get
$$\frac{z-z_1}{z-z_2} = \left(\frac{\kappa\sqrt{y-i\sqrt{x}}}{\kappa\sqrt{y+i\sqrt{x}}} \right)^2;$$

so that, by (2), we see that

$$\frac{\frac{1}{2}(X+Y)-z_1}{\frac{1}{2}(X+Y)-z_2} = e^{4\psi i} \frac{\frac{1}{2}(x+y)-z_1}{\frac{1}{2}(x+y)-z_2},$$

which shows that the square of (1) brings $\frac{x+y}{2}$ to $\frac{X+Y}{2}$.

Now, for a diameter, $\frac{x+y}{2}$ is the origin, and, as we arrive at any circle conformal with a diameter by a series of rotations about nodes, the inverses of the centres are successively derived from the origin by the series of the squares of the same rotations, and are therefore all conformal with the origin.

The following presents were made to the Library during the Recess:—

“Journal of the Institute of Actuaries,” Vol. xxxiv., Pt. 2; July, 1898.

“Sitzungsberichte der Physikalisch-medizinischen Societät in Erlangen,” 1897, Heft 29.

“Monatshefte für Mathematik und Physik,” Jahrgang ix., Pt. 3; Wien, 1898.

“Archives Néerlandaises des Sciences Exactes et Naturelles,” Série 2, Tome II., Livr. 1; La Haye, 1898.

“Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft zu Leipzig,” Pts. 1-4.

“Nieuw Archief voor Wiskunde,” Tweede Reeks, Deel III., St. 4; Amsterdam, 1898.

“Wiskundige Oplossen met de Oplossingen,” Deel VII., St. 5; Amsterdam, 1898.

"Revue Semestrielle des Publications Mathématiques," Tome vi., Pt. 2, Oct. 1897–Av. 1898; Amsterdam.

"Proceedings of the Cambridge Philosophical Society," Vol. ix., Pt. 8; Cambridge, 1898.

"Science Progress," Vol. vii., No. 8; July, 1898.

"Mémoires de la Section Mathématique de la Société des Naturalistes de la Nouvelle-Russie," Tome xviii.; Odessa, 1897.

"Proceedings of the Royal Irish Academy," Vol. iv., No. 5; Dublin, 1898.

"List of Members of the Royal Irish Academy," Dublin, 1898.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," 1898, Heft 1; Zürich, 1898.

"Proceedings of the Edinburgh Mathematical Society," Vol. xvi.; 1898.

"Nyt Tidsskrift for Matematik," A, Aarg. 9, Nr. 1–4; B, Aarg. 9, Nr. 2; Copenhagen, 1898.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. iv., Fasc. 5–7; Napoli, 1898.

"Transactions of the Canadian Institute," Vol. v., Pt. 2, No. 10; Supplement to Vol. v., Pt. 1, No. 9; Toronto, 1898.

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"Prace Matematyczno-Fizyczne," Tom. ix.; Warsaw, 1898.

"Rendiconti del Circolo Matematico di Palermo," Tomo xii., Fasc. 3–5; Annuario, 1898.

Credner, H.—"Die Sächsischen Erdbeben während 1889 bis 1897," 8vo; Leipzig, 1898.

"Annual Report of the Library Syndicate, 1897"; Cambridge.

"Annales de la Faculté des Sciences de Marseilles," Tome viii., Fasc. 5–10.

"Annals of Mathematics," Vol. xii., Nos. 2, 3.

"Annali di Matematica," Serie 3, Tomo i., Fasc. 2, 3; Milano, 1898.

"Annales de la Faculté des Sciences de Toulouse," Tome xii., Fasc. 3; 1898.

"Acta Mathematica," xxii., 1, 2; Stockholm, 1898.

"Journal für die reine und angewandte Mathematik," Bd. cxix., Heft 3, 4; Berlin, 1898.

"Indian Engineering," Vol. xxiii., Nos. 21–26, Vol. xxiv., Nos. 1–12, May 21–Sept. 17.

Schiaparelli, G.—"Origine del Sistema Planetario Eliocentrico presso i Greci," 4to; Milano, 1898.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxii., St. 6, 7; Leipzig, 1898.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. xlii., Pts. 3, 4.

"Jahrbuch über die Fortschritte der Mathematik," Bd. xxvii., Heft 1, 2.

"Proceedings of the American Philosophical Society," Vol. xxxvi., Nos. 156, 157; Philadelphia, 1898.

"Proceedings of the Royal Society," Vol. lxxiii., Nos. 396–403.

"Proceedings of the Physical Society of London," Vol. xvi., Pts. 1, 2; July, 1898.

"Reale Istituto Lombardo—Rendiconti," Serie 2, Vol. xxx.; Milano, 1897.

- "Bulletin of the American Mathematical Society," Vol. rv., Nos. 9, 10; 1898.
 "Bulletin des Sciences Mathématiques," Tome xxx., Juin-Oct. 1898.
 "Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," 24-39.
 "Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Math.-Phys. Klasse, 1898, Pts. 1, 2; Geschäftliche Mittheilungen, 1898, Pt. 1.
 "Tōkyō Sūgaku-Butsurigaku Kwai Kiji," Maki viii., Dai. 1, 2.
 "Bulletin de la Société Mathématique de France," Tome xxvi., Nos. 3-6.
 "Educational Times," July, Aug., Sept., Oct., 1898.
 "Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. vii., Fasc. 1-6; Sem. 1, Vol. vii., Fasc. 11, 12, and Index to Vol.; "Rendiconto dell' Adunanza Solenne del 12 Giugno 1897"; Roma, 1898.

APPENDIX.

The Society's loss, through the deaths of its members, has been again a large one. For the following notice of Signor Francesco Brioschi we are indebted to documents put at our disposal by Prof. Luigi Cremona.

La sera del 13 mese corrente moriva in Milano, sua città natale, il nostro collega Francesco Brioschi, dopo breve malattia, aggravata forse dalla fiducia ispiratagli dalla costante immunità. La sua morte, del tutto impreveduta, ci ha colpiti del più doloroso stupore: sebbene egli fosse alla vigilia del compimento del suo 73^{mo} anno, a vederlo così vivace ed agile, forte e grigio come acciaio, nessuno avrebbe dubitato che la sua operosità non fosse per durare ancora molti anni: tanta era in lui la robustezza della fibra e della mente, che non avevano mai conosciuto stanchezza. Era nato il 22 dicembre 1824, aveva fatto gli studi universitari a Pavia e vi si era laureato ingegnere con onore, nel 1845. Ad innamorarlo vieppiù dell'alta matematica contribuì Gabrio Piola, il dotto commentatore delle opere di Lagrange.*

* From the *Resoconto Sommario* (Senato del Regno), Lunedì, 20 Dicembre 1897, the President, for the time, being Prof. Cremona. The rest of the speech treats rather of Brioschi as a public man than as a mathematician.

The following estimate of Brioschi, as a mathematician, is taken from that furnished by Signor Beltrami to *La Perseveranza* of Thursday, December 23rd, 1897.*

Straordinaria versatilità d'ingegno, singolare potenza di assimilazione rapida e profonda, instancabile tenacia in ogni ricerca come in ogni intrapresa, tali furono le doti onde rifulse la mente, davvero eccelsa, di Francesco Brioschi.

Nelle scienze matematiche poi, in cui egli raggiunse più presto la gloria, e gloria solida e duratura, recò un'altra preziosissima dote, quella che or si direbbe una impareggiabile virtuosità, cioè una agilità elegante di forma e di pensiero, la quale, se sgomentava gli impazienti ed i meno provetti, formava l'ammirazione dei dotti e degli studiosi di lena.

Allevato alla scuola di Bordoni nel culto, forse un po' troppo esclusivo, dei metodi lagrangiani, entrò poco appresso con Piola nell'ambiente meno rigido delle ricerche fisico-matematiche di Fourier, di Poisson e di Cauchy. Ma, per quanto grande e feconda fosse allora la produzione matematica francese, che era la sola cui attingessero i pochi studiosi d'Italia, egli intuì ben presto la necessità d'allargare la cerchia delle fonti, estendendola alla produzione delle altre nazioni colte d'Europa, massimamente della Germania e dell'Inghilterra. Fra noi egli fu indubbiamente il primo a mettersi risolutamente per questa via, e ad indirizzarvi quanti allievi e studiosi poté attirare con sè in quest'opera, che può ben dirsi di risanamento, giacchè soltanto per essa cessò quel tal quale ristagno che da lungo tempo pesava sulla scienza italiana, e incominciò quel sempre più attivo e fecondo ricambio intellettuale colla scienza e cogli scienziati di fuori, che fu certamente favorito e promosso dalla fortunata ricostituzione dell'unità nazionale, ma che sarebbe ingiustizia non revocare a lui, per ciò che spetta alle sue prime origini, ben più modeste, ma ben più laboriose.

È incredibile la quantità di lavori che il Brioschi seppe comporre e produrre in luce, nei più svariati indirizzi, non appena si fu rapidamente orientato nel vastissimo campo delle ricerche che occupavano, alla metà di questo secolo, i più valorosi matematici d'Europa. La recente scomparsa del grande Jacobi, col quale il Brioschi aveva

* The same writer furnishes a "Necrologia" to the *Annali di Matematica*, Tomo xxvi., Fasc. 4, pp. 343-347. Cf. also "Francesco Brioschi, Cenno Necrologico per Alfredo Capelli," in the *Giornale di Matematiche*, Vol. xxxvi., Jan.-Feb., 1898, p. 51.

tanta affinità di temperamento scientifico, richiamava allora la attenzione sul grande problema delle equazioni dinamiche, e fu questo uno dei primi soggetti a cui egli si rivolse, trattovi anche dalla natura dell'insegnamento che impartiva all'Università di Pavia.

Numerosi ed apprezzatissimi sono i suoi lavori, sia sul problema d'integrazione, sia sulle affini teorie delle equazioni a derivate parziali e delle equazioni isoperimetriche. Nè cessò mai, anche a maggior distanza di tempo, di ritornare su quei primi studii con altre geniali pubblicazioni, come, a cagion d'esempio, con quelle relative all'ellissoide fluido di Dirichlet ed al problema dei tre corpi.

Altro argomento di numerose ed interessantissime pubblicazioni fu la teoria analitica delle superficie, rimessa allora in onore da una celebre Memoria di Gauss, che era passata per lungo tempo inosservata, ma di cui i geometri riconoscevano finalmente la fondamentale importanza. Il Brioschi prese parte grandissima allo svolgimento (divenuto poi sempre più largo e più complesso) di questa teoria, e vi arrecò più d'un contributo essenziale, tra altro col concetto di coordinate curvilinee tangenziali, da lui primamente adombrato in una nota sulla superficie delle onde.

Non volle rimanere estraneo agli studi di pura geometria, il cui decisivo risveglio risale a un dipresso alla medesima epoca, benchè l'indole peculiare del suo ingegno lo chiamasse di preferenza alle ricerche di pura analisi; e fu felicissimo nella trattazione di quelle questioni in cui l'una e l'altra disciplina gareggiano nel raggiungere una stessa meta, del che basterà citare l'esempio fornito dai poligoni di Poncelet.

Ma l'indirizzo in cui il Brioschi si lanciò con vera passione e con istraordinario successo fu quello delle ricerche sulle equazioni algebriche e sui nuovi algoritmi che si riassumono nell'uso sistematico dei determinanti, degli invarianti, dei covarianti e delle forme algebriche e simboliche. Basterebbe già il libro dei Determinanti, che risale ai primissimi anni della sua carriera scientifica e che fu tradotto in pressochè tutte le lingue colte, per constatare le eminenti sue doti d'assimilazione e d'invenzione, come il profondo e sicuro possesso d'ogni più disparato dominio dello scibile matematico. Ma sarebbe impossibile analizzare anche sommariamente, senza entrare in particolari troppo disformi dall'indole d'un giornale, l'infinita copia di nuove vedute, di nuove proposizioni, di nuovi procedimenti che si trovano disseminati nelle numerosissime Memorie di lui sulle indicate teorie, fra le quali basterà menzionare quella Monografia sulle forme binarie che doveva riassumere gran parte dei suoi studii e che,

sebbene rimasta incompieta, contiene pur tuttavia un ricco tesoro di materiali preziosi. E, per citare almeno uno dei moltissimi argomenti speciali in cui maggiormente brillarono l'acume e la genialità del Brioschi, giovi ricordare le sue elegantissime ricerche sulle serie analoghe a quella di Sturm.

Per ciò poi che spetta alla dottrina delle equazioni algebriche, basti il dire che, nella memorabile scoperta della risoluzione dell'equazione di 5° grado, il nome di Brioschi è indissolubilmente legato a quelli di Hermite e di Kronecker, con questo di più, ch'egli non ha poi mai cessato di illustrare con nuove ricerche questo campo così irto di difficoltà, preparando il terreno e partecipando attivamente ad altri non meno cospicui progressi.

Un altro larghissimo campo di studii ai quali, non meno che ai precedenti, il Brioschi si trovò spontaneamente attratto dalle sue peculiari attitudini e preferenze scientifiche, e che del resto si collegava necessariamente ed intimamente coll'ultimo dei dianzi accennati, fu quello delle funzioni trascendenti, inaugurato da Legendre e recato d'un tratto a smisurate altezze dai lavori di Abel, di Jacobi e da quelli, allora recentissimi, di Weierstrass. Qui forse, più che altrove, il Brioschi era destinato a raccogliere una messe oltremodo feconda e rigogliosa, la materia prestandosi mirabilmente a quel suo genio, ormai maturo, di analista supremamente classico, e già egli era entrato gloriosamente nell'arringo, ispirandosi ai lavori di Weierstrass, quando, sopravvenuti gli eventi del 1859, si trovò d'un tratto chiamato a spendere in altro modo le forze esuberanti del suo ingegno. Così si chiuse il periodo eroico della sua operosità scientifica, periodo che durò non più d'un decennio, ma che bastò a circondare per sempre il suo nome d'una aureola di gloria purissima così presso di noi, come presso tutte le culte nazioni, di cui in così breve tempo egli aveva saputo assimilare ed eguagliare la poderosa produzione scientifica.

Se tuttavia, in tutto il tempo successivo, egli non potè mai più consacrare alla scienza pura l'intera somma delle sue smisurate energie intellettuali, neppur cessò mai di tener sempre ed amorosamente fiso in essa lo sguardo, tornando ad ogni tratto, e più d'una volta abbastanza intensamente, al culto di essa, così da aggiungere molto al moltissimo già prodotto, e nulla trascurando di ciò che poteva, direttamente ed indirettamente, favorire la diffusione ed il progresso degli alti studii nel nostro paese. In quest'ultimo senso merita principalmente d'esser ricordata l'opera indefessa da lui data nel mantenere in vita dapprima, e nel recare poscia a rigogliosa

fioritura quella pubblicazione periodica che s' intitola *Annali di matematica pura ed applicata*, e che da non breve tempo rappresenta degnamente l' Italia fra le congeneri e più apprezzate pubblicazioni d' Europa e d' America. Già fin dal 1858, quando questo periodico sorse in Roma, in continuazione d' un altro più modesto che lo precedette, egli aveva contribuito moltissimo a dargli alimento e notorietà; ma nel 1867, quando la vita ne era divenuta alquanto languida e stentata, egli ne trasportò la sede da Roma a Milano, e ne assunse la direzione, dapprima insieme col collega Cremona, poi, dopo la partenza di questo, da solo. Nei trent'anni trascorsi dopo questo rinnovamento dell' antico periodico romano, ne sono apparsi in luce ben 26 volumi in-4°, ai quali collaborarono tutti i migliori matematici italiani e non pochi fra gli stranieri d' ogni nazione, attivando così anche fra noi quel ricambio d' ospitalità scientifica che già s' era iniziato altrove ed al quale il Brioschi aveva già tanto e così ampiamente contribuito coll' esempio e col consiglio.

È inutile dire a lungo dell' opera data, in un senso più universale, a pro della scienza italiana durante la lunga presidenza dell' Accademia dei Lincei. Niuno ignora come primo pensiero del Brioschi sia sempre stato quello di far convergere i maggiori mezzi possibili all' ampliamento delle pubblicazioni accademiche, così da potervi accogliere, come può ora ben dirsi che avvenge, ogni degna manifestazione degli studii nazionali.

Senonchè l' analisi delle varie, per non dire infinite forme sotto cui si estrinsecò l' ardore inestinguibile del Brioschi per tutto ciò che s' attiene agli studii ed agli studiosi del nostro paese, ardore che andò facendosi sempre più largo e comprensivo di ogni sana manifestazione del pensiero scientifico, mentre da un lato condurrebbe troppo lontano e sconfinerebbe dal campo prefisso, riuscirebbe forse dall' altro ad offuscare un cotal poco, specialmente agli occhi di chi non ebbe la ventura di conoscere da vicino l' uomo, l' immagine integra e distinta che di lui dobbiamo formarci, e che è bene rimanga scolpita indelebilmente nella storia degli intelletti d' Italia. La quale immagine è quella d' un forte campione della schietta stirpe latina, la cui mente, sovraneamente equilibrata, fu sempre aperta ad ogni più alta aspirazione ideale come ad ogni più intimo bisogno di vertiginosa attività esterna, e che nell' opera sua, di qualunque natura si fosse, recò invariabilmente la benevolenza, la sicurezza, la serenità e, non ultima attrattiva, quell' amabile scioltezza che, nell' esercizio delle discipline e delle cose severe, lascia come spirare un sottile profumo di squisita artisticità.

Signor Brioschi, who was an Honorary Fellow of the Cambridge Philosophical Society, was President of the Reale Accademia dei Lincei, Rome, and Director of the Istituto Tecnico Superiore of Milan. He was elected an Honorary Foreign Member of our Society May 9th, 1878. His only communications to the Society were "Sur une Propriété du Paramètre de la Transformée Canonique des Formes Cubiques Ternaires," Vol. XII., pp. 58-63 (read January 13th, 1881); "Sur la Transformation des Equations Algébriques," Vol. XX., pp. 127-131 (read February 14th, 1889).*

The Council are indebted to Mr. H. M. Taylor for the following notice of Dr. Frost:—

Percival Frost, the second son of Mr. Charles Frost, F.S.A., a solicitor, of Hull, was born in that town on September 1st, 1817. His schooldays were spent at Beverley and Oakham. In October, 1835, he commenced residence at St. John's College, Cambridge. At Oakham he had shown great aptitude for languages, and had competed on equal terms with James Atlay, afterwards Bishop of Hereford, who was ninth in the Classical Tripos of 1840.

At Cambridge, Frost devoted his time chiefly to mathematics, and, for three years, ran an almost neck-and-neck race with B. M. Cowie, the present Dean of Exeter. They were elected Fellows of St. John's College on the same day, March 18th, 1839. In 1841 Frost was ordained, and for some years acted as a curate, but he never held any preferment in the Church. In 1841 he vacated his Fellowship at St. John's on his marriage, and settled down in Cambridge to the work of a private tutor. In this work he was at once popular and successful. Among his pupils, the one who rose to the highest eminence in scientific reputation was the late W. K. Clifford; from him Frost was always ready to admit that he had learnt much. Frost held a Mathematical Lectureship at Jesus College for twelve years, and afterwards at King's College for thirty years. The latter College showed their appreciation of his services as a Lecturer by electing him to a Fellowship in 1882.

Frost examined in the Mathematical Tripos for the first time in January, 1846, and for the last time in the divided Tripos which took place in June, 1882, and January, 1883. Altogether Frost examined

* Cf. also *Proc.*, Vol. XI., p. 157, and Vol. XII., p. 215.

ten times, and his record has only been beaten by Dr. Ferrers, who has examined eleven times.

In 1854 Frost published his first work, a treatise on part of Newton's *Principia*, and, in 1863, in conjunction with the late Joseph Wolstenholme, *A Treatise on Solid Geometry*. Each of these works has passed through several editions, and both are still popular text-books. In 1872 Frost published a third work, *A Treatise on Curve-Tracing*, a work of greater originality, of which Professor Chrystal says, in speaking of the reversion of series and of the expansion in power-series of an algebraic function: "The applications are very numerous, for example, to the finding of curvatures and curves of closest contact, and to curve-tracing generally. A number of beautiful examples will be found in that much-to-be-recommended text-book." He was the author of a considerable number of papers, a list of which will be found in the Royal Society's Catalogue.

On December 9th, 1869, Frost was elected a member of this Society. He never served on the Council or contributed a paper to the Society's *Proceedings*; but he acted as a referee on two or three occasions. In 1882 Frost took the degree of Sc.D. at Cambridge.

It would be an injustice to Frost's memory to omit all reference to his wide interests, his many and varied accomplishments, and to the characteristic simplicity of his generous nature. The playful spirit of the man, who was always ready to see the humorous side of every situation and of every subject, peeps out in the prefaces to his books and occasionally in his mathematical papers.

Frost inherited a splendid constitution. When he completed his eightieth year, there seemed to be a probability of his attaining extreme old age. His mother had reached the age of ninety-two, and her uncle, who held the living of Holy Trinity, Hull, for seventy years, died in his ninety-eighth year. A painful disorder, however, which attacked him, proved fatal after six weeks' illness, and Frost died at Cambridge on Trinity Sunday, June 5th, 1898.*

The following Notice of Dr. Hopkinson has been put at the service of the Council by Mr. R. T. Glazebrook:—

John Hopkinson was the eldest son of Mr. Alderman Hopkinson, of Manchester. He was born in 1849, and educated at Lindon Grove

* An *in memoriam* sketch by Dr. H. A. Morgan was published in the *Cambridge Review* of June 16th, 1898.

School and Queenswood College until the age of fifteen and a-half years. After two and a-half years spent at Owens College, he entered at Trinity College, Cambridge. He graduated as Senior Wrangler in 1871, and was first Smith's Prizeman of his year. The same year he was elected a Fellow of the College. Early in 1872 he became associated with Messrs. Chance Bros. & Co. as engineer at their lighthouse works near Birmingham. After a residence of about six years in that city, he removed to London, and laid the foundations of his large consulting practice as an electrical engineer. He had taken his D.Sc. degree at London University while still at Cambridge. In 1878 he was elected a Fellow of the Royal Society, and in 1890 he received the Royal Medal for his researches in electricity and magnetism. During his work at Messrs. Chance's, Dr. Hopkinson introduced many improvements into lighthouse machinery, and papers dealing with some of these were communicated to the Royal Society; among them may be named that on the group-flashing apparatus described in the *Proceedings* of the Society for 1874. In the year 1877, the first of his papers on the "Residual Charge of the Leyden Jar" was published, and this was followed by others on the "Measurement of Specific Inductive Capacity." The importance in the theory of electro-magnetism of Maxwell's relation connecting the refractive index with the specific inductive capacity of a substance was early apparent to him, and his observations on glass (*Phil. Trans.*, 1878) were made with a view to verifying this. Experiment showed that, for the materials used—various specimens of flint glass—the specific inductive capacity is not equal to the square of the refractive index. These results were contradicted by the work of Gordon (*Phil. Trans.*, 1879, Pt. 1.), and Hopkinson, in 1880 (*Phil. Trans.*, 1881, Pt. 2), repeated his observations, modifying, however, his method so as to use a rapidly alternating electromotive force—in his earlier experiments the electromotive force had been uniform in direction. Differences in the value of the specific inductive capacity were found in some cases; they were, however, small compared with the differences between his own results and those of Gordon, and Hopkinson's experiments showed conclusively that the large difference between them was due to the want of uniformity of the electric field in Gordon's apparatus, which causes errors greater than any one would suspect without actual trial. About the same time, the result of a series of experiments on the specific inductive capacity of liquids was made. These were extended in 1887 (*Proc. Roy. Soc.*, Vol. XLIII., 1887), and it appeared that, while for hydrocarbons and other liquids

the square of the refractive index is equal to the specific inductive capacity, for animal and vegetable oils it is not so. More recent research has fully confirmed the view that the discrepancy arises from the difference in the length of the waves for which the two quantities, inductive capacity and refractive index respectively, were necessarily measured, and Hopkinson's work has been greatly instrumental in establishing the above-mentioned fundamental relation of Maxwell's theory.

In handing the Royal Medal to Hopkinson in 1890, the President (Lord Kelvin) referred at length to his researches on the effect of temperature on the magnetic properties of iron and other metals. These researches were published in the *Phil. Trans.* for 1889. The experiments show that, for small magnetizing forces, the susceptibility of iron increases with rising temperature until a temperature varying from 690° to 870° C. in different specimens is reached. On approaching this critical temperature from below, the rise is very rapid; when the temperature has been passed, the iron ceases almost completely to be magnetic.

But Hopkinson's most important work is connected with the theory of dynamo-electric machinery. In one of the earliest of his papers on this subject (*Proc. Mech. Eng. Instit.*, 1879, p. 246; 1880, p. 266) he defines as the characteristic of a machine the curve which gives the relation between the electromotive force of the armature and the current round the magnets, and he shows how, from this curve, may be deduced "the result in all employments of the machine, whether as a motor, or to produce a current through a resistance, through an electric arc, or in charging accumulators; also the result of varying the winding of the machine, whether in the armature or the magnets."

In 1886 (*Phil. Trans.*, A, p. 331) appeared his paper on "Dynamo-electric Machinery," written in conjunction with his brother, Dr. E. Hopkinson. The paper, to quote from Prof. A. Gray when reviewing a collected edition of Hopkinson's papers on "Dynamo-electric Machinery" (*Nature*, July, 1893, p. 244), "has become justly famous as that in which his enormously useful idea of the magnetic circuit was first employed in a complete and consistent manner to the discussion of the results of experiments on different types of dynamo." One purpose, as the authors state, is to give "an approximately complete construction of the characteristic curve of a dynamo of given form from the ordinary laws of electro-magnetism and the known properties of iron, and to compare the result of such construction with the actual characteristic of the machine." This is done in a

masterly manner, and the results obtained are tested by experiment.

The latter part of the paper contains a method for measuring the efficiency of a dynamo. Two similar machines have their shafts coupled, and are connected electrically; one acts as a generator, the other as a motor. The motor helps to drive the generator which feeds it. If the two machines are similar, the only additional power necessary to drive the combination is the waste in the two dynamos and in the connexions. This is supplied from outside, and, being small compared with the power supplied from the generator, which can be measured electrically, can be determined by a dynamometer with sufficient accuracy. This idea of the Hopkinsons has borne fruit in various methods for testing dynamos and transformers since devised by others. The paper was completed in 1892 by one presented to the Royal Society by Dr. Hopkinson and E. Wilson. Another paper, published in 1884, dealt with the question of running alternating machines in parallel. In 1882 he invented and described the famous three-wire system of electrical distribution.

Hopkinson was the author of numerous other papers of importance, but sufficient has been written to show the fertility of his powers and the magnitude of his work. Moreover, he was known as a practical engineer no less than as a writer. He was connected with some of the most important of electrical undertakings; it may be sufficient to mention the City and South London Railway and the Manchester Electric Supply Station. At the time of his death he was engaged on the tramway systems of Leeds, Liverpool, and other towns. His work is distinguished by its clearness, its brevity, and its precision. Some of his papers are almost too condensed. At meetings of scientific societies, and in public, he did not usually speak at great length, but every word was to the point; he knew his own mind, and expressed his views concisely and clearly. The style in which he describes his experiments may be taken as a model by others; he states briefly the problem to be solved and then the method he intends to follow for its solution. Sufficient detail of each experiment is given to enable the reader to check and verify his results, and all is clearly arranged; there is no unnecessary verbiage, and no repetition. The loss to physical science by his tragic death is difficult for us who knew him to estimate; there are few whose influence was felt in so many fields of activity, few who are devoting themselves more completely to the advancement of their fellows. He was killed, with a son and two daughters, by a fall from the rocks of the Petite Dent de

Veisivi, at Arolla, on August 27 last; how the accident happened no one can say with certainty. Hopkinson himself, his son, and one of his daughters were mountaineers of exceptional skill and experience in rock climbing, and we may feel sure that no ordinary occurrence was the cause of so terrible a catastrophe. The scene in the beautiful cemetery at Territet, where they were buried the following week, will ever live in the minds of those who were present.

Dr. Hopkinson was elected a member of the Society, January 13th, 1873.

The Secretary is indebted to Mr. Joseph Edwards for the following brief notice of Mr. Walter Wren :—

One of the earliest members of our Society has passed away in the person of Mr. Walter Wren, whose election bears date October 16th, 1865.

Wren was the second son of Mr. Richard Wren, of Buntingford, and was born in 1831. He received his early education at Buntingford Grammar School and Elizabeth College, Guernsey, whence he proceeded to Christ's College, Cambridge, where he had for contemporaries a group of distinguished men, amongst whom were C. S. Calverley, Prof. Seeley, Sir Walter Besant, and Prof. Skeat. Wren's college career was blighted by his being attacked, whilst an undergraduate, by spinal disease, which prevented his taking an Honours degree. He, however, graduated in 1859, and proceeded to the M.A. degree in 1862.

His subsequent career has been a remarkable one, the more so in his success having been achieved in spite of a life of continuous suffering. His fame as a private tutor for candidates for the Government services is well known. For more than thirty years he has been engaged in preparing candidates for the Indian and Home Civil Service Examinations, as well as for Colonial Cadetships, the Army, the Woods and Forests Service, the Indian Police Service, &c. So great has been his success in this task that, at the present day, hundreds of the most distinguished members of the State Services owe their education and position to Mr. Wren's establishment.

Mr. Wren was an excellent judge of capacity. He always insisted upon thoroughness of work. He abhorred the acquirement of superficial knowledge in any subject, and would always urge with all the force of his powerful personality the importance of possessing a thorough mastery of all subjects taken up by his pupils.

As though the management of a great institution like his were not sufficient outlet for Wren's energy, we find him standing at the General Election of 1880, for Wallingford, where he was returned by a majority of forty-one. He was subsequently unseated on petition, but made several subsequent attempts to enter Parliament. He was a member of the first London County Council, and one of the founders of the National Liberal Club.

Wren died on August 5th, 1898, from paralysis, with which he had been seized only a week previously.*

Henry Perigal, who died on June 6th, 1898, in the ninety-eighth year of his age, came of a family noted for their longevity. He traced his descent from Sigurd, a Dane, who, after a successful raid into Normandy, in 908, settled there and assumed the name of Perigal. He was in the tenth generation from Gideon Perigal, and his wife Madeline Duval, of Dieppe, who as Huguenots escaped from prison and settled in London. Mr. Perigal's father, who reached the age of ninety-nine, was one of thirteen children, nine of whom reached a great age. Mr. Perigal, who was born April 1st, 1801, was the eldest of six, the youngest of whom, Mr. Frederick Perigal, is now in his eighty-seventh year. As a result of the great age to which his father lived, he was "Henry Perigal, junior," till he was an old man himself. In early life he was a clerk in the Privy Council Office for a short time, and subsequently he held for many years a position in the office of Messrs. Henry Tudor & Son, Stockbrokers, the senior partner of which firm was a personal friend of his. He was a member of the Astronomical, Microscopical, Meteorological, Photographic, and several other scientific Societies, and a very regular attendant at the meetings of all, usually staying to the end, and often being the last to leave; and it may be said that for thirty or forty years, during eight months of the year, he divided his time between daily office work at 29 Threadneedle Street and evening attendance at the meetings of scientific societies, the lectures at the Royal Institution, &c.

His connexion with the Meteorological Society was remarkable: the Society was founded in 1850; in May, 1853, he became Treasurer,

* [A letter from Sir Walter Besant in the *Times*, over the date August 7th, gives an interesting general sketch of Wren's work, founded upon "an almost life-long friendship."]

and retained this office up his death—for forty-five years—signing cheques and attending to the financial business of the Society almost to the last. When our Society was formed he would have liked to join it at once, but feared that his paradoxical views on the subject of astronomical motions would disqualify him; he was much pleased therefore when, some years later, De Morgan offered to propose him as a member and signed his nomination form.

As a young man Mr. Perigal was much interested in all kinds of lathe work, but especially in curve-tracing by means of Ibbetson's geometric chucks. To this subject he brought to bear great patience, perseverance, and delicacy of manipulation. Many of the curves so drawn by him between 1830 and 1840 are exquisite examples of fine geometrical tracings performed by means of the bow pen. He afterwards delineated mechanically a great number of bicircloidal curves, *i.e.*, curves described by two circular motions. These curves were generated by a point moving uniformly round a circle the centre of which also moved uniformly round another circle. The curve traced depended upon the ratio of the angular velocities in these two circles, this ratio being fixed during the description of the same curve. He also devised little apparatuses, which he called kinesiopes, for exhibiting these curves to the eye as they were formed. The curve was traced by a bright point which, by a system of multiplying wheels, was made to move so rapidly as to exhibit a continuous bright line. He also occupied himself with curves generated by more than two circular motions, and as early as about 1835 had delineated the curves now generally known by the name of Lissajous, including the finite or retrogressive parabola, which he had obtained by points some years before he described it mechanically. He also displayed great ingenuity in proving geometrical theorems by means of dissection and transposition of parts. These dissections were mostly discovered between 1830 and 1840. Among the best are his proof of Euclid I. 47, and his decomposition of a square into three squares. Unfortunately nearly all his work was published merely for private distribution among his friends, and, as he had not the pen of a ready writer (as he himself used to say), his diagrams were generally issued without sufficient explanation, and sometimes without any at all. These fugitive pieces, which were being produced by him for sixty years or more, are very numerous. They consist principally of diagrams of bicircloidal curves (some on large sheets), of the "Lissajous" curves, and of dissections. Some of the dissections he made as puzzles, forming

them in card and wood, and gave to his friends. He also engraved them (as well as the finite parabola) upon some of his visiting cards. His proof of Euclid I. 47 was reproduced in Vol. II. of the *Messenger of Mathematics*.

Mr. Perigal's mechanical work in connexion with the lathe led him astray with respect to the motions of the moon and planets. As the Moon always turns the same face to the Earth, he considered that she should be regarded as subject to one simple circular motion, and not (as mathematicians said) to a motion of revolution round the Earth combined with one of rotation round her own axis. The point in dispute was not, as most people thought, merely a question of definitions, for Mr. Perigal assumed as an axiom that a central force necessarily acted upon a body describing a circular orbit round it exactly as if the one was attached to the other by a rigid bar. The "Moon paradox" was afterwards taken up by Mr. Jellinger Symons, and is frequently connected with his name. Mr. Perigal made a great many mechanical devices to illustrate his views, and wrote a number of little tracts and fly-sheets, in prose and verse, on "Revolution and Rotation," the "Fallacy of the Moon's Rotation," &c. These were generally circulated privately, but some appeared in the *Astronomical Register*. He was almost unique as a paradoxer in being a *persona grata* wherever he went, and neither at our Society, nor at the Astronomical, did he ever bring forward his own views aggressively or in a manner that was objectionable to those who differed from him. He is included in the "Budget of Paradoxes"; but De Morgan lets him off very gently, saying that "he has published valuable diagrams in profusion," and that he was "entirely indebted to him for the illustrations of the article on 'Trochoidal Curves,' published in the *Penny Cyclopædia*, without which the article would not have been written." There is no reason to doubt that he always believed that the "Perigalian system of Astronomy" would eventually supersede the Newtonian, but this did not interfere with his personal friendships with those who believed in a system which seemed to him obviously absurd. Mr. Proctor's book on the *Geometry of Cycloids* was chiefly illustrated by blocks cut in the lathe by Mr. Perigal; and Northcote in his work on *The Lathe* gives an illustration of the actual apparatus used, together with Perigal's rules for the use of Ibbetson's chucks. It may be mentioned that about 1868 Mr. Perigal publicly offered a prize of £10 for the discovery of a curve (one of his own curves produced by circular motions) satisfying certain conditions. The prize was won by Clerk

Maxwell, who obtained Mr. Perigal's curve, and also by Mr. W. D. Bushell. Many years ago he wrote [a paper (in the *Journal of the Society of Arts*) on the way in which great monoliths, like those at Stonehenge, could be moved by such appliances as savages could use, and he also wrote upon the probable methods used in the construction of the Great Pyramid.

He was a warm friend, and a pleasant companion, gentle in manner, amiable, kind-hearted, and generous, interested in the topics of the day, and shrewd without being cynical. He rarely left London, and probably never went abroad in the last half of his life. At some time he had made a voyage to Madeira. Twenty years or more ago he let his beard grow, and since then his refined face, slight and fragile form, and long white beard and hair, made him a striking and picturesque figure. He ceased to attend scientific meetings two or three years ago, and for the last year or more had ceased to leave his lodgings; but there was no failure of mind or eye-sight. He was never married. By his own wish he was cremated at Woking.*

Mr. Perigal was elected a member of our Society, January 23rd, 1868, and was a frequent attendant at our meetings. There is a characteristic note by him, entitled "Link Trammels," in Vol. v., p. 144, in which he claims that Jopling had anticipated Peaucellier. The only other contribution he made was a verbal account of some "Geometrical Metamorphoses by Partition and Transformation" (January 8th, 1891). Many of these figures are to be seen in his pamphlet "Graphic Demonstrations of Geometric Problems," selected by H. Perigal, which he presented to the members of the Association for the Improvement of Geometrical Teaching (1891), of which Association he was elected a member in 1874.

Prof. Burnside sends the following note:—

While the paper "On the Continuous Group that is defined by any Given Group of Finite Order," pp. 546–565, was passing through the press, the author had not the leisure to study carefully the memoir by Herr Frobenius "Über die Darstellung der endlichen Gruppen durch lineare Substitutionen," referred to on p. 548. Having done so since, he wishes to acknowledge that the result contained in the concluding paragraph of his paper (p. 565) is anticipated in Herr Frobenius' memoir.

* Dr. Glaisher, in a note which accompanied the above account of his old friend, writes "it was in 1855 that I first saw Perigal."