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On Waves Propagated along the Plane Surface of an Elastic Solid. By Lord Rayleigh, D.C.L., F.R.S.

[*Read November 12th,* 1885.]

It is proposed to investigate the behaviour of waves upon the plane free surface of an infinite homogeneous isotropic elastic solid, their character being such that the disturbance is confined to a superficial region, of thickness comparable with the wave-length. The case is thus analogous to that of deep-water waves, only that the potential energy here depends upon elastic resilience instead of upon gravity.*

Denoting the displacements by α, β, γ, and the dilatation by θ, we have the usual equations

$$\rho\frac{d^2\alpha}{dt^2} = (\lambda+\mu)\frac{d\theta}{dx} + \mu\nabla^2\alpha, \text{ \&c.} \qquad\qquad (1),$$

in which
$$\theta = \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} \qquad\qquad (2).$$

If α, β, γ all vary as e^{ipt}, equations (1) become

$$(\lambda+\mu)\frac{d\theta}{dx} + \mu\nabla^2\alpha + \rho p^2\alpha = 0, \text{ \&c.} \qquad\qquad (3).$$

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* The statical problem of the deformation of an elastic solid by a harmonic application of pressure to its surface has been treated by Prof. G. Darwin, *Phil. Mag.*, Dec., 1882. [Jan. 1886.—See also *Camb. Math. Trip. Ex.*, Jan. 20, 1875, Question IV.]

Differentiating equations (3) in order with respect to x, y, z, and adding, we get

$$(\nabla^2+h^2)\,\theta = 0 \qquad \text{(4)},$$

in which $$h^2 = \rho p^2/(\lambda+2\mu) \qquad \text{(5)}.$$

Again, if we put $$k^2 = \rho p^2/\mu \qquad \text{(6)},$$

equations (3) take the form

$$(\nabla^2+k^2)\,\alpha = \left(1-\frac{k^2}{h^2}\right)\frac{d\theta}{dx},\ \&\text{c.} \qquad \text{(7)}.$$

A particular solution of (7) is*

$$\alpha = -\frac{1}{h^2}\frac{d\theta}{dx},\quad \beta = -\frac{1}{h^2}\frac{d\theta}{dy},\quad \gamma = -\frac{1}{h^2}\frac{d\theta}{dz} \qquad \text{(8)};$$

in order to complete which it is only necessary to add complementary terms u, v, w satisfying the system of equations

$$(\nabla^2+k^2)\,u = 0,\quad (\nabla^2+k^2)\,v = 0,\quad (\nabla^2+k^2)\,w = 0 \qquad \text{(9)},$$

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} = 0 \qquad \text{(10)}.$$

For the purposes of the present problem we take the free surface as the plane $z = 0$, and assume that, as functions of x and y, the displacements are proportional to e^{ifx}, e^{igy}. Thus (4) takes the form

$$\left(\frac{d^2}{dz^2}+h^2-f^2-g^2\right)\theta = 0;$$

so that $$\theta = Pe^{-rz}+Qe^{+rz} \qquad \text{(11)},$$

where $$r^2 = f^2+g^2-h^2 \qquad \text{(12)}.$$

In (11), r is supposed to be real; otherwise the dilatation would penetrate to an indefinite depth. For the same reason, we must retain only that term (say the first) for which the exponent is negative within the solid.† Thus $Q = 0$, and we will write for brevity $P = 1$, or rather $P = e^{ipt}e^{ifx}e^{igy}$, but the exponential factors may often be omitted without risk of confusion, so that we may take

$$\theta = e^{-rz} \qquad \text{(13)}.$$

* Lamb on the Vibrations of an Elastic Sphere, *Math. Soc. Proc.*, May, 1882.

† By discarding these restrictions we may deduce the complete solution applicable to a plate, bounded by parallel plane free surfaces; but I have not obtained any results which seem worthy of quotation.

At the same time the particular solution becomes

$$\alpha = -\frac{if}{h^2}e^{-rz}, \quad \beta = -\frac{ig}{h^2}e^{-rz}, \quad \gamma = \frac{r}{h^2}e^{-rz} \quad\text{.........(14).}$$

For the complementary terms, which must also contain e^{ifx}, e^{igy} as factors, equations (9) become

$$\left(\frac{d^2}{dz^2}+k^2-f^2-g^2\right)u = 0, \text{ \&c.} \quad\text{....................(15);}$$

whence, as before, on the assumption that the disturbance is limited to a superficial stratum,

$$u = Ae^{-sz}, \quad v = Be^{-sz}, \quad w = Ce^{-sz} \quad\text{..................(16),}$$

where
$$s^2 = f^2+g^2-k^2 \quad\text{...........................(17).}$$

In order to satisfy (10), the coefficients in (16) must be subject to the relation

$$ifA+igB-sC = 0 \quad\text{.........................(18).}$$

The complete values of α, β, γ may now be written

$$\alpha = -\frac{if}{h^2}e^{-rz}+Ae^{-sz}, \quad \beta = -\frac{ig}{h^2}e^{-rz}+Be^{-sz}, \quad \gamma = \frac{r}{h^2}e^{-rz}+Ce^{-sz} \quad\text{...(19),}$$

in which A, B, C are subject to (18); and the next step is to express the boundary conditions for the free surface. The two components of tangential stress must vanish, when $z = 0$, and these are proportional to

$$\frac{d\beta}{dz}+\frac{d\gamma}{dy}, \quad \frac{d\gamma}{dx}+\frac{d\alpha}{dz},$$

respectively. Hence

$$sB = \frac{2igr}{h^2}+igC, \quad sA = \frac{2ifr}{h^2}+ifC \quad\text{..............(20).}$$

Substituting from (20) in (18), we find

$$C\,(s^2+f^2+g^2)\,h^2+2r\,(f^2+g^2) = 0 \quad\text{..............(21).}$$

We have still to introduce the condition that the normal traction is zero at the surface. We have, in general,

$$N_3 = \lambda\theta+2\mu\frac{d\gamma}{dz};$$

or, if we express λ in terms of μ, h, k,

$$N_3 = \mu \left\{ \left(\frac{k^2}{h^2} - 2 \right) \theta + 2 \frac{d\gamma}{dz} \right\};$$

so that the condition is

$$k^2 - 2h^2 - 2\,(+r^2 + h^2 sC) = 0,$$

or, on substitution for r^2 of its value from (12),

$$k^2 - 2\,(f^2+g^2) - 2h^2 sC = 0 \quad \text{..................(22).}$$

By eliminating C between (21) and (22), we obtain the equation by which the time of vibration is determined as a function of the wave-lengths and of the properties of the solid. It is

$$\{k^2 - 2\,(f^2+g^2)\}\,\{s^2+f^2+g^2\} + 4rs\,(f^2+g^2) = 0,$$

or, by (17), $$\{2\,(f^2+g^2) - k^2\}^2 = 4rs\,(f^2+g^2) \quad \text{..................(23).}$$

If we square (23), and introduce the values of r^2 and s^2 from (12), (17), we get

$$\{2\,(f^2+g^2) - k^2\}^4 = 16\,(f^2+g^2)^2\,(f^2+g^2-h^2)(f^2+g^2-k^2).$$

As f and g occur here only in the combination (f^2+g^2), a quantity homogeneous with h^2 and k^2, we may conveniently replace (f^2+g^2) by unity. Thus

$$k^8 - 8k^6 + 24k^4 - 16k^2 - 16h^2k^2 + 16h^2 = 0 \quad \text{............. (24).}$$

Since the ratio $h^2 : k^2$ is known, this equation reduces to a cubic and determines the value of either quantity.

If the solid be incompressible ($\lambda = \infty$), $h^2 = 0$, and the equation becomes $$k^6 - 8k^4 + 24k^2 - 16 = 0 \quad \text{........................(25).}$$

The real root of (25) is found to be $\cdot 91275$, and the equation may be written $$(k^2 - \cdot 91275)(k^4 - 7\cdot 08725k^2 + 17\cdot 5311) = 0.$$

The general theory of vibrations of stable systems forbids us to look for complex values of k^2, as solutions of our problem, though it would at first sight appear possible with them to satisfy the prescribed conditions by taking such roots of (12), (17), as would make the *real* parts of the exponents in e^{-rz}, e^{-sz} negative. But, referring back to (23), which we write in the form

$$(2-k^2)^2 = 4rs,$$

or, in the present case of incompressibility, by putting $r = 1$,

$$(2-k^2)_2 = 4s,$$

we see that we are not really free to choose the sign of s. In fact, from the complex values of k^2, viz., $3\cdot5436 \pm 2\cdot2301i$, we find

$$4s = -\,2\cdot7431 \pm 6\cdot8846i\,;$$

so that the real part of s is of the opposite sign to r, and therefore e^{-rz}, e^{-sz} do not both diminish without limit as we penetrate further and further into the solid.

Dismissing then the complex values, we have, in the case of incompressibility, the single solution

$$k^2 = \frac{\rho p^2}{\mu} = \cdot91275\,(f^2+g^2) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26).$$

From (19), (20), (21), we get in general

$$h^2\alpha = f\left\{-e^{-rz} + \frac{2rs}{s^2+f^2+g^2}\,e^{-sz}\right\} \;\ldots\ldots\ldots\ldots\ldots(27),$$

$$h^2\beta = ig\left\{-e^{-rz} + \frac{2rs}{s^2+f^2+g^2}\,e^{-sz}\right\} \;\ldots\ldots\ldots\ldots\ldots(28),$$

$$h^2\gamma = r\left\{\quad e^{-rz} - \frac{2\,(f^2+g^2)}{s^2+f^2+g^2}\,e^{-sz}\right\} \;\ldots\ldots\ldots\ldots\ldots(29).$$

In the case of incompressibility, we have k^2 given by (26), and

$$r^2 = f^2+g^2, \quad s^2 = \cdot08725\,(f^2+g^2).$$

Hence

$$\left.\begin{aligned} h^2\alpha &= if\left\{-e^{-rz}+\cdot5433e^{-sz}\right\} e^{ipt}\,e^{ifx}\,e^{igy} \\ h^2\beta &= ig\left\{-e^{-rz}+\cdot5433e^{-sz}\right\} e^{ipt}\,e^{ifx}\,e^{igy} \\ h^2\gamma &= \sqrt{(f^2+g^2)}\left\{e^{-rz}-1\cdot840e^{-sz}\right\} e^{ipt}\,e^{ifx}\,e^{igy} \end{aligned}\right\} \;\ldots\ldots\ldots\ldots(30).$$

If we suppose the motion to be in two dimensions only, we may put $g = 0$; so that $\beta = 0$, and

$$\left.\begin{aligned} h^2\alpha\,/\,f &= i\left\{-e^{-fz}+\cdot5433e^{-sz}\right\} e^{ipt}\,e^{ifx} \\ h^2\gamma\,/\,f &= \;\;\left\{\quad e^{-fz}-1\cdot840e^{-sz}\right\} e^{ipt}\,e^{ifx} \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots(31),$$

in which

$$k = \cdot9554f, \quad s = \cdot2954f \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

For a progressive wave we may take simply the real parts of (31).

Thus

$$\left.\begin{aligned} h^2\alpha\,/\,f &= (e^{-fz}-\cdot5433e^{-sz})\sin(pt+fx) \\ h^2\gamma\,/\,f &= (e^{-fz}-1\cdot840e^{-sz})\cos(pt+fx) \end{aligned}\right\} \;\ldots\ldots\ldots\ldots(33).$$

The velocity of propagation is p/f, or $\cdot 9554\sqrt{(\mu/\rho)}$, in which $\sqrt{(\mu/\rho)}$ is the velocity of purely transverse plane waves. The surface waves now under consideration move, therefore, rather more slowly than these.

From (32), (33), we see that α vanishes for all values of x and t when $e^{(s-f)z} = \cdot 5433$, *i.e.*, when $fz = \cdot 8659$. Thus, if λ' be the wave-length $(2\pi/f)$, the horizontal motion vanishes at a depth equal to $\cdot 1378\lambda'$. On the other hand, there is no finite depth at which the vertical motion vanishes.

To find the motion at the surface itself, we have only to put $z = 0$ in (33). We may drop at the same time the constant multiplier (h^2/f) which has no present significance. Accordingly,

$$\left.\begin{aligned} \alpha &= \cdot 4567 \sin(pt+fx) \\ \gamma &= -\cdot 840 \cos(pt+fx) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34),$$

showing that the motion takes place in elliptic orbits, whose vertical axis is nearly the double of the horizontal axis.

The expressions for stationary vibrations may be obtained from (30) by addition to the similar equations obtained by changing the sign of p, and similar operations with respect to f and g. Dropping an arbitrary multiplier, we may write

$$\left.\begin{aligned} \alpha &= -f\left\{-e^{-rz}+\cdot 5433e^{-sz}\right\}\cos pt \sin fx \cos gy \\ \beta &= -g\left\{-e^{-rz}+\cdot 5433e^{-sz}\right\}\cos pt \cos fx \sin gy \\ \gamma &= \quad r\left\{\ \ e^{-rz}-1\cdot 840e^{-sz}\right\}\cos pt \cos fx \cos gy \end{aligned}\right\} \quad \ldots\ldots\ldots (35),$$

in which $\quad r = \sqrt{(f^2+g^2)}, \quad s = \cdot 2954\sqrt{(f^2+g^2)}$ ....(36).

As before, the horizontal motion vanishes at a depth such that

$$\sqrt{(f^2+g^2)}\, z = \cdot 8659.$$

We will now examine how far the numerical results are affected when we take into account the finite compressibility of all natural bodies. The ratio of the elastic constants is often stated by means of the number expressing the ratio of lateral contraction to longitudinal extension when a bar of the material is strained by forces applied to its ends. According to a theory now generally discarded, this ratio (σ) would be $\frac{1}{4}$; a number which, however, is not far from the truth for a variety of materials, including the principal metals. In the extreme case of incompressibility σ is $\frac{1}{2}$, and there seems to be no theoretical reason why σ should not have any value between this and -1.*

* Prof. Lamb, in his able paper, seems to regard all negative values of σ as exclu-

The accompanying table will give an idea of the progress of the values of $k^2/(f^2+g^2)$ as dependent upon λ/μ, or upon σ. It will be observed that the value diminishes continuously with λ, in accordance with a general principle.*

λ	σ	h^2/k^2	$k^2/(f^2+g^2)$	$k/\sqrt{(f^2+g^2)}$
∞	$\frac{1}{2}$	0	·9127	·9554
μ	$\frac{1}{4}$	$\frac{1}{3}$	·8453	·9194
0	0	$\frac{1}{2}$	·7640	·8741
$-\frac{2}{3}\mu$	-1	$\frac{3}{4}$	·4746	·6896

As an example of finite compressibility, we will consider further the second case of the table. From (12), (17),

$$r^2 = \cdot 7182\,(f^2+g^2), \quad r = \cdot 8475\sqrt{(f^2+g^2)},$$

$$s^2 = \cdot 1547\,(f^2+g^2), \quad s = \cdot 3933\sqrt{(f^2+g^2)}.$$

Hence, from (27), (28), (29), in correspondence with (30), we have

$$\left.\begin{aligned} h^2\alpha &= if\left\{-e^{-rz}+\cdot 5773e^{-sz}\right\} e^{ipt}e^{ifx}e^{igy} \\ h^2\beta &= ig\left\{-e^{-rz}+\cdot 5773e^{-sz}\right\} e^{ipt}e^{ifx}e^{igy} \\ h^2\gamma &= \cdot 8475\sqrt{(f^2+g^2)}\left\{e^{-rz}-1\cdot 7320e^{-sz}\right\} e^{ipt}e^{ifx}e^{igy} \end{aligned}\right\} \quad \ldots\ldots(37).$$

For a progressive wave in two dimensions, we shall have

$$\left.\begin{aligned} h^2\alpha/f &= (e^{-rz}-\cdot 5773e^{-sz})\sin(pt+fx) \\ h^2\gamma/f &= (\cdot 8475e^{-rz}-1\cdot 4679e^{-sz})\cos(pt+fx) \end{aligned}\right\} \quad \ldots\ldots\ldots(38).$$

At the surface,

$$\left.\begin{aligned} h^2\alpha/f &= \cdot 4227\sin(pt+fx) \\ h^2\gamma/f &= -\cdot 6204\cos(pt+fx) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(39),$$

so that the vertical axes of the elliptic orbits are about half as great again as the horizontal axes.

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ded *a priori*. But the necessary and sufficient conditions of stability are merely that the resistance to compression ($\lambda+\frac{2}{3}\mu$) and the resistance to shearing (μ) should be positive. In the second extreme case of a medium which resists shear, but does not resist compression, $\lambda = -\frac{2}{3}\mu$, and $\sigma = -1$. The velocity of a dilatational wave is then $\frac{4}{3}$ of that of a distortional plane wave. (Green, *Camb. Trans.*, 1838.) The general value of σ is $\lambda/(2\lambda+2\mu)$.

* *Math. Soc. Proc.*, June, 1873, Vol. IV., p. 359. "Theory of Sound," t. 1, p. 85. Lamb, *loc. cit.*, p. 202.

It is proper to remark that the vibrations here considered are covered by the general theory of spherical vibrations given by Lamb in the paper referred to. But it would probably be as difficult, if not more difficult, to deduce the conclusions of the present paper from the analytical expressions of the general theory, as to obtain them independently. It is not improbable that the surface waves here investigated play an important part in earthquakes, and in the collision of elastic solids. Diverging in two dimensions only, they must acquire at a great distance from the source a continually increasing preponderance.

On some Consequences of the Transformation Formula $y = \sin(L+A+B+C+\ldots)$.

By JOHN GRIFFITHS, M.A.

[*Read Nov.* 12*th*, 1885.]

CONTENTS OF PAPER.

Notation.

In order to avoid repetitions, it is convenient, for the purposes of