IV. *A second Letter from Mr.* Colin Mc Laurin, *Professor of Mathematicks in the University of* Edinburgh *and F. R. S. to* Martin Folkes, *Esq; concerning the Roots of Equations, with the Demonstration of other Rules in Algebra; being the Continuation of the Letter published in the* Philosophical Transactions, N° 394.

Edinburgh, April 19th, 1729.

S I R,

IN the Y ar 1725, I wrote to you that I had a Method of demonstrating Sir *Isaac Newton*'s Rule concerning the impossible Roots of Equations, deduced from this obvious Principle, that the Squares of the Differences of real Quantities must always be positive; and some time after, I sent you the first Principles of that Method, which were published in the *Philosophical Transactions* for the Month of *May*, 1726. The Design I have for some Time had of publishing a Treatise of Algebra, where I proposed to treat this and several other Subjects in a new Manner, made me think it unnecessary to send you the remaining Part of that Paper. But some Reasons have now determined me to send you with the Continuation of my former Method, a short Account of two other Methods in which I have treated the same Subject, and some Observations on Equations that I take to be new, and which will, perhaps, be more acceptable to you than what relates to the imaginary Roots themselves. Besides Sir *Isaac Newton*'s Rule, there arises from the following general

ral Propositions, a great Variety of new Rules, different from his, and from any other hitherto published, for discovering when an Equation has imaginary Roots. I shall particularly explain one that is more useful for that Purpose, than any that have been hitherto published.

Suppose there is an Equation of (n) Dimensions of this Form,

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + Fx^{n-6} - Gx^{n-7} + Hx^{n-8} - Ix^{n-9} + Kx^{n-10} \text{ \&c.} = 0.$$

And that the Roots of this Equation are, $a, b, c, d, e, f, g, h, i, k, l$, &c. then shall $A = a + b + c + d + e + f$ &c. and therefore I call a, b, c, d, e, f, &c. *Parts* or *Terms* of the Coefficient A. For the same Reason I call $ab, ac, ad, ae, bc, bd, cd$, &c. Parts or Terms of the Coefficient B; abc, abd, abe, acd, bcd, &c. Parts or Terms of C; $abcd, abce, abcf$, Parts or Terms of the Coefficient D, and so on. By the *Dimensions* of any Coefficient; I mean the Number of Roots or Factors that are multiplied into each other in its Parts, which is always equal to the Number of Terms in the Equation that preceed that Coefficient. Thus A is a Coefficient of one Dimension, B of two, C of three, and so of the rest. I call a Part or Term of a Coefficient C *similar* to a Part or Term of any Coefficient G, when the Part of G involves all the Factors of the Part of C: Thus abc, $abcdefg$ are similar Parts of C and G; after the same manner $abcd$, $abcdef$ are similar Parts of D and F, the Part of F involving all the Factors of the Part of D. Those I call *dissimilar* Parts that involve no common Root or Factor: Thus abc, and $defgh$ are dissimilar Parts of the Coefficients C and F. The Sum of all the

the Products that can be made by multiplying the Parts of any Coefficient C by all the similar Parts of G, I express by $C'G'$ placing a small Line over each Coefficient: After the same manner $D'F'$ expresses the Sum of all the Products that can be made by multiplying the similar Parts of D and F by each other; and $C'\times C'$ expresses the Sum of the Squares of the Parts of the Coefficient C, but $C'\times C$; expresses the Sum of the Products that can be made by multiplying any two Parts of C by one another. These Expressions being understood, and the five Propositions in *Phil. Transf.* N° 394, being premised, next follows

PROP. VI.

If the Difference of the Dimensions of any two Coefficients C *and* G *be called* (m) *then shall the Product of these Coefficients multiplied by one another be equal to*

$$C'G' + \overline{m+2}\times B'H' + \frac{m+3}{1}\times\frac{m+4}{2}A'I' + \frac{m+4}{1}\times\frac{m+5}{2}\times\frac{m+6}{3}\times I\times K.$$

Where B and H are the Coefficients adjacent to the Coefficients C and G, A and I the Coefficients adjacent to B and H, I and K the Coefficients adjacent to B and H.

It is known that $C = abc + abd + abe + abf + abg$,&c. and $G = abcdefg + abcdefh + abcdefi + bcdefgh$, &c. and it is manifest,

1. That in the Product CG each Term of $C'G'$ will arise once as $a^2b^2c^2defg$. But

2. Any Term of $B'H'$ as $a^2b^2cdefgh$ may be the Product of abc, and $abdefgh$, or of abd and $abcefgh$, or of abe and $abcdfgh$, or of abf and $abcdegh$,

abcdegh, or of *abg* and *abcdefh*, or laſtly of *abh* and *abcdefg*; ſo that it may be the Product of any Term of C that involves with *ab* one of the Roots, *c*,*d*,*e*,*f*,*g*,*h*, multiplied by that Term of G, which involves *ab* and the other five; that is, it may ariſe in the Product CG as often as there are Roots in $a^2b^2cdefgh$ beſides *a* and *b*, or in general, as often as there are Units in the Difference of the Dimenſions of B and H, that is, $m+2$ times; becauſe *m* expreſſes the Difference of the Dimenſions of C and G, and conſequently in expreſſing the Value of CG the Coefficient of the ſecond Term B′H′ muſt be $m+2$.

3. Any Term of AI, as $a^2bcdefghi$, may be the Product of any Part of C that involves the Root *a* with any two of the reſt *b*,*c*,*d*,*e*,*f*,*g*,*h*,*i* (the Number of which is the Difference of the Dimenſions of A and I, which is in general equal to $m+4$) multiplied by the Part of G that involves *a* and the other ſix; and therefore $a^2bcdefghi$ or any other Term of A′I′ muſt ariſe as often as different Products of two Quantities can be taken from Quantities whoſe Number is $m+4$, that is $\overline{m+4} \times \frac{m+4-1}{2}$ times or $\frac{m+3}{1} \times \frac{m+4}{2}$ times; and conſequently in expreſſing the Value of CG the Coefficient of the third Term A′I′ muſt be $\frac{m+3}{1} \times \frac{m+4}{3}$.

4. Any Term of $1 \times K$ as *abcdefghik*, may be the Product of any Part of C that involves three of its Factors, and of the Part of G that involves the reſt, and therefore may ariſe in the Product CG as often as different

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Products of three Quantities can be taken out of Quantities whose Number is $m+6$ that is, $\overline{m+6} \times \frac{m+5}{2} \times \frac{m+4}{3}$ times, and therefore the Coefficient of the fourth Term in the Value of C G must be $\frac{m+4}{1} \times \frac{m+5}{2} \times \frac{m+6}{3}$.

In general, in expressing the Value of the Product of any two Coefficients C and G, if x express the Order of any Term of this Value as $A'I'$, that is, the Number of Terms that precede it, the Coefficient of that Term must be $\frac{2x+m}{1} \times \frac{2x+m-1}{2} \times \frac{2x+m-2}{3}$ &c. taking as many Factors as there are Units in x.

Cor. I. If it is required to find by this Proposition the Square of any Coefficient E, then suppose $m=0$, the Difference of the Dimensions of the Coefficients in this Case vanishing, and we shall have $E^2 = E' \times E' + 2\,D'F' + 3 \times \frac{4}{2} \times C'G' + 4 \times \frac{5}{2} \times \frac{6}{3} \times B'H'$ &c. $= E' \times E' + 2\,D'F' + 6\,C'G' + 20\,B'H' + 70\,A'I' + 252\,K$. Therefore if $E' \times E_{,}$ express the Sum of the Products of any two parts of E multiplied by each other, we shall have $E^2 = E' \times E' + 2\,E' \times E_{,}$, and therefore $E' \times E_{,} = D'F' + 3\,C'G' + 10\,B'H' + 35\,A'I' + 126\,K$.

COR. II. It follows from this Propofition that

$$\begin{aligned}
E^2 &= E' \times E' + 2D'F' + 6C'G' + 20B'H' + 70A'I' + 252K. \\
DF &= \cdots\cdots D'F' + 4C'G' + 15B'H' + 56A'I' + 210K. \\
CG &= \cdots\cdots\cdots\cdots C'G' + 6B'H' + 28A'I' + 120K \\
BH &= \cdots\cdots\cdots\cdots\cdots\cdots B'H' + 8A'I' + 45K \\
AI &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots A'I' + 10K \\
K &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots K.
\end{aligned}$$

COR. III. It eafily appears by comparing the Theorems given in the laft Corollary, that

$$\begin{aligned}
E'E' &= \cdot\; E^2 - 2DF + 2CG - 2BH + 2AI - 2K. \\
D'F' &= \cdots\cdots DF - 4CG + 9BH - 16AI + 25K \\
C'G' &= \cdots\cdots\cdots\cdots CG - 6BH + 20AI - 50K \\
B'H' &= \cdots\cdots\cdots\cdots\cdots\cdots BH - 8AI + 35K \\
A'I' &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots AI - 10K.
\end{aligned}$$

PROP. VII.

Let $l = n \times \frac{n-1}{2} \times \frac{n-2}{3}$ &c. *taking as many Factors as the Coefficient* E *has Dimenfions and* $\frac{l-1}{2l} \times E^2$ *fhall always exceed* $DF - CG + BH - AI + K$ *when the Roots of the Equation are all real Quantities.*

For it is manifeft that l expreffes the Number of Parts or Terms in the Coefficient E, and it is plain from Propofition V (*See Phil. Tranf.* N° 394) that $\frac{l-1}{2l} \times E^2$ muft always be greater than the Sum of the Products that can be made by multiplying any two of of the Parts of E by each other, that is, than $E' \times E_{,}$; but $2 E' \times E_{,} = E^2 - E' E' =$ (by the first Theorem in the last Corollary) $2 DF - 2 CG + 2 BH - 2 A' I + 2 K$, and therefore since $\frac{l-1}{2l} \times E^2$ must always exceed $E' E_{,}$, it follows that $\frac{l-1}{2l} E^2$ must always be greater than $DF - CG + BH - AI + K$ when the Roots of the Equation are real Quantities.

Schol. In following my Method this was the first general Proposition presented it self. For having first observed that if l expresses the Number of any Quantities, the Square of their Sum multiplied by $\frac{l-1}{2l}$ must always exceed the Sum of the Products made by multiplying any two of them by each other; and that the Excess was the Sum of the Squares of the Differences of the Quantities divided by $2l$, it was easy to see in the Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4}$ &c. $= 0$. Since B is the Sum of the Products of any two of the Parts of A, that if l expresses the Number of the Roots of the Equation, $\frac{l-1}{2l} \times A^2$ must always exceed B; and this is one Part of the 5th Proposition. In the next Place, I compared the Sum of the Products of any two Parts of B with AC, and found that it was not equal to AC but to $AC - D$ from which I inferred, that if l expresses

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the Number of the Parts of B then $\frac{l - 1}{2l} \times B^2$ must always exceed $AC - D$; and these easily suggested this general Proposition.

PROP. VIII.

Let r *express the Dimensions of the Coefficient* C, *and* s *the Difference of the Dimensions of the Coefficients* C *and* G, *then* B *and* H *being Coefficients adjacent to* C *and* G, $\overline{n - r - s} \times r\, C^l G^l$ *shall always be greater than* $\overline{s + 1} \times \overline{s + 2} \times B^l H^l$ *when the Roots of the Equation are all real Quantities affected with the same Sign.*

For taking the Differences of all those Parts of the Coefficient C that are similar in all their Factors but one, as abc, abh, abi, &c. and multiplying the Square of each Difference by such Parts of the Coefficient D (which is of s Dimensions) as are dissimilar to both the Parts of C in that Difference, the Sum of all those Squares thus multiplied, will consist of Terms of $C^l G^l$ taken positively, and of Terms of $B^l H^l$ taken negatively. By multiplying in this manner $\overline{abc - abh}|^2 + \overline{abc - abi}|^2 + \overline{abc - abk}|^2$ &c. $+ \overline{abc - ach}|^2 + \overline{abc - aci}|^2 + \overline{abc - ack}|^2$ &c. $+ \overline{abc - bch}|^2 + \overline{abc - bce}|^2 + \overline{abc - bck}|^2$ &c. by $defg$ the Term of D, that is dissimilar to all those Parts of C, you will find that $a^2b^2c^2defg$ will arise in the Sum of the Products $r \times \overline{n - r - s}$ times: For those Products may be also expressed thus $defga^2b^2 \times \overline{c - h}|^2 + \overline{c - i}|^2 + \overline{c - k}|^2$ &c. $+ defga^2c^2 \times \overline{b - h}|^2 + \overline{b - i}|^2 + \overline{b - k}|^2$ &c. $+$

$defgb^2c^2$

$defgb^2c^2 \times \overline{a-h}|^2 + \overline{a-i}|^2 + \overline{a-k}|^2$ &c. where the Number of the Differences $c-h$, $c-i$, $c-k$, &c. whose Squares are multiplied by $defga^2b^2$ is manifestly equal to the Number of the Roots of the Equation that do not enter $a^2b^2c^2defg$ or $abcdefg$, that is, to the Excess of the Number of the Roots of the Equation above the Dimensions of $abcdefg$, a Term of G, that is to $n-r-s$. But in collecting all the said Products, $\overline{n-r-s} \times a^2b^2c^2defg$ must arise as often as there are Units in r: Because the Terms which are subtracted from abc may differ from it in the Root c, as abh, abi, abk, &c. or in the Root b, as ach, aci, ack, &c. or in the Root a as bch, bci, bck; that is, $\overline{n-r-s} \times a^2b^2c^2defg$ must arise as often as there are Dimensions in abc, a Term of C, or as often in general as there are units in r, which expresses the Dimensions of C: Therefore the Term $a^2b^2c^2defg$ will arise in the Sum of the above-mentioned Products $r \times \overline{n-r-s}$ times.

The Negative Part must consist of the Terms of $B'H'$ doubled; each of which, as $2a^2b^2cdefg$ may arise as often as there can be Differences $c-d$, $c-e$, $c-f$, $c-g$, $d-e$, &c. assumed amongst the Terms c,d,e,f,g whose Number is equal to $s+2$ that is, $\overline{s+2} \times \frac{s+1}{2}$ times; and therefore a^2b^2cdefg or any other Part of $B'H'$ must arise in the negative Part $\overline{s+1} \times \overline{s+2}$ times; and since the whole aggregate must be positive it follows $\overline{n-r-s} \times rC'G'$ must always exceed $\overline{s+1} \times \overline{s+2} \times B'H'$.

Cor.

Cor. I. Suppose we are to compare $E'E'$ the Sum of the Squares of the Parts of E with $D'F'$ the Sum of the Products of the similar Parts of D and F; in this Case s vanishes, and therefore $\overline{n-r}\times r\,E'E'$ must exceed $2\,D'F'$. Let $\overline{n-r}\times r = m$ and consequently $\overline{n-r-1}\times\overline{r-1} = m-n+1$; $\overline{n-r-2}\times\overline{r-2} = m-2n+4$; $\overline{n-r-3}\times\overline{r-3} = m-3n+9$; $\overline{n-r-4}\times\overline{r-4} = m-4n+16$. Since it is plain that $\overline{n-r-q}\times\overline{r-q} = \overline{n-r}\times r - qn + q^2$. Then by this Proposition, supposing

$$\begin{array}{l}
m\times E'E' - 2\,D'F' = a' \\
\overline{m-n+1}\times D'F' - 12\,C'G' = b' \\
\overline{m-2n+4}\times C'G' - 30\,B'H' = c' \\
\overline{m-3n+9}\times B'H' - 56\,A'I' = d' \\
\overline{m-4n+16}\times A'I' - 90\,K' = e'
\end{array}$$

The Quantities a', b', c', d', e', must be always positive when the Roots of the Equation are real Quantities affected with the same Sign. The Coefficients prefixed to the negative Parts are the Numbers 2, 12, 30, 56, 90, whose Differences equally increase by the same Number 8.

Cor. II. Supposing as before, that $\overline{n-r}\times r = m$; and also that $m\times\overline{m-n+1} = m'$; $m'\times\overline{m-2n+4} = m''$; $m''\times\overline{m-3n+9} = m'''$ &c. it may be demonstrated after the manner of this Proposition, that if

$$\begin{array}{l}
m\,E'E' - 2\,D'F' = a' \\
m'E'E' - 2\times 12\,C'G' = a'' \\
m''\,E'E' - 2\times 12\times 30\,B'H' = a''' \\
m'''\,E'E' - 2\times 12\times 30\times 56\,A'I' = a'''' \text{ \&c.}
\end{array}$$

Then

Then shall a', a'', a''', &c. be always positive when the Roots are real Quantities, whether they be affected with the same, or with different Signs. The Negative Coefficients arise by multiplying those in the preceeding Corollary, 2, 12, 30, 56, 90, by one another.

PROP. IX.

Let a', b', c', d', e', and m express the same Quantities as in the Corollaries of the last Proposition, and $mE^2 - \overline{m+n+1} \times DF = a' + b' + 2c' + 5d' + 14e'$.

For by Cor. ii Prop. vi.

$$E^2 = E'E' + 2D'F' + 6C'G' + 20B'H' + 70A'I' + 252K,$$

and by the same

$$DF = \text{- - -}\ D'F' + 4C'G' + 15B'H' + 56A'I' + 210K;$$

therefore $mE^2 - \overline{m+n+1} \times DF = mE'E' + \overline{m-n-1} \times D'F' + \overline{m-2n-2} \times 2C'G' + \overline{m-3n-3} \times 5B'H' + \overline{m-4n-4} \times 14A'I' + \overline{m-5n-5} \times 42K =$ (by substituting successively for $mE'E'$, $\overline{m-n+1} \times D'F'$, $\overline{m-2n+4} \times C'G'$, $\overline{m-3n+9} \times B'H'$, $\overline{m-4n+16} \times A'I'$ their Values deduced from the first Corollary of the last Proposition) $= a' + b' + 2c' + 5d' + 14e'$, where the Coefficients prefixed to a', b', c', d', e', are the Differences of the Coefficients of $E'E'$, $D'F'$, $C'G'$, $B'H'$, $A'I'$ and K in the Values of E^2 and DF taken from Cor. ii. Prop. vi. being $1-0$, $2-1$, $6-4$, $20-15$, $70-56$ and $252-210$.

COR. Since $m = \overline{n-r} \times r$ therefore $m + n + 1 = \overline{n-r+1} \times \overline{r+1}$; and consequently $\frac{r}{r+1} \times \frac{n-r}{n-r+1} \times E^2$ must always be greater than D F the Product of the Coefficients adjacent to E; and hence the Fractions are deduced, that in Sir *Isaac Newton's* Rule are placed over the Terms of the Equation, which multiplied by the Square of the Terms under them, must always exceed the Products of the adjacent Terms of the Equation, when the Roots are real Quantities: For it is manifest that the Fraction to be placed over the Term Ex^{n-r} according to that Rule is the Quotient of $\frac{n-r}{r+1}$ divided by $\frac{n-r+1}{r}$.

PROP. X.

The same Expressions being allowed as in the preceeding Propositions, it will be found in the same manner that as $mE^2 - \overline{m+n+1} \times DF = a' + b' + 2c' + 5d' + 14e'$ so

$$\overline{m-n+1} \times DF - \overline{m+2n+4} \times CG = b' + 3c' + 9d' + 28e'$$

$$\overline{m-2n+4} \times CG - \overline{m+3n+9} \times BH = c' + 5d' + 20e'$$

$$\overline{m-3n+9} \times BH - \overline{m+4n+16} \times AI = d' + 7e'$$

$$\overline{m-4n+16} \times AI - \overline{m+5n+25} \times K = e.$$

These Theorems are easily deduced from the Theorems given in the second Corollary of Prop. vi. and the first Corollary of the viii[th] Proposition; and the Coefficients prefixed to a', b', c', d', e', are the Differences of the Coefficients of the corresponding Terms in the Values of E^2, D F, C G, B H, A I and K in Cor. ii. Prop. vi.

COR.

Cor. Hence the Products of any two Coefficients, as D F and A I may be compared together when the Sum of the Dimensions of D and F is equal to the Sum of the Dimensions of A and I. Let the Dimensions of A and F be equal to s and m respectively, and let $p = \frac{n-s}{s+1} \times \frac{n-s-1}{s+2} \times \frac{n-s-2}{s+3}$ &c. taking as many Factors as there are Units in the Difference of the Dimensions of D and A. Let $q = \frac{n-m}{m+1} \times \frac{n-m-1}{m+2} \times \frac{n-m-2}{m+3}$ &c. taking as many Factors as you took in the Value of p. Then shall $\frac{q}{p} \times$ D F always exceed A I when the Roots of the Equation are real Quantities affected with the same Sign; and this Rule obtains, though the Roots are affected with different Signs when the Coefficients D and F are equal.

PROP. XI.

The same Things being supposed as in the preceeding Propositions.

$$1.\ \left.\begin{array}{l} mE^2 - \overline{m+1} \times 2DF + \overline{m+4} \times 2CG - \overline{m+9} \times \\ 2BH + \overline{m+16} \times 2AI - \overline{m+25} \times 2K \ \ \text{- -} \end{array}\right\} = a'.$$

$$2.\ \left.\begin{array}{l} \overline{m-n+1} \times DF - \overline{m-n+4} \times 4CG + \overline{m-n+9} \times \\ 9BH - \overline{m-n+16} \times 16AI + \overline{m-n+25} \times 25K \end{array}\right\} = b'.$$

$$3.\ \left.\begin{array}{l} \overline{m-2n+4} \times CG - \overline{m-2n+9} \times 6BH + \overline{m-2n+16} \times \\ 20AI + \overline{m-2n+25} \times 50K \ \ \text{- - - - - -} \end{array}\right\} = c'.$$

$$4.\ \overline{m-3n+9} \times BH - \overline{m-3n+16} \times 8AI + \overline{m-3n+25} \times 35K = d'.$$

$$5.\ \text{- -}\ \ \overline{m-4n+16} \times AI - \overline{m-4n+25} \times 10K = e'.$$

These Theorems follow easily from the third Corollary of the vi^th Proposition. The first easily appears thus, $a' = mE'E' - 2D'F' =$ (by that Corollary)

$$\begin{array}{l} mE^2 - 2mDF + 2mCG - 2mBH + 2mAI - 2mK. \\ \qquad - 2DF + 8CG - 18BH + 32AI - 50K. \end{array}$$

$$= mE^2 - \overline{m+1} \times 2DF + \overline{m+4} \times 2CG - \overline{m+9} \times 2BH + \overline{m+16} \times 2AI - \overline{m+25} \times 2K.$$

The other Theorems are deduced from the same Corollary compared with Cor. i. Prop. viii.

PROP. XII.

The same Things being supposed as in the second Corollary of the viii^th *Proposition.*

1. $\left.\begin{array}{l} mE^2 - \overline{m+1} \times 2DF + \overline{m+4} \times 2CG - \overline{m+9} \times \\ 2BH + \overline{m+16} \times 2AI - \overline{m+25} \times 2K \; - - - \end{array}\right\} = a'.$

2. $\left.\begin{array}{l} m'E^2 - 2m'DF + \overline{m' - 12} \times 2CG - \overline{m' - 72} \times \\ 2BH + \overline{m' - 240} \times 2AI - \overline{m' - 600} \times 2K \; - - - \end{array}\right\} = a''.$

3. $\left.\begin{array}{l} m''E^2 - 2m''DF + 2m''CG - \overline{m'' + 360} \times \\ 2BH + \overline{m'' + 360 \times 8} \times 2AI - \overline{m'' + 360 \times 35} \times 2K \end{array}\right\} = a'''.$

4. $\left.\begin{array}{l} m''' \times \overline{E^2 - 2DF + 2CG - 2BH} + \overline{m'' - 750 \times 28} \times \\ AI - \overline{m'''' - 7200} \times 28 \times 2K \; - - - - - - - \end{array}\right\} = a''''.$

&c.

These Theorems follow from the third Corollary of the vi^th Proposition compared with the second Corollary of the eighth Proposition. The first is the same with the first of the last Proposition. The second is demonstrated by substituting in $m'E'E' - 24C'G' = a''$. The Values of $E'E'$ and $C'G'$ given in the third

Cor. of the vith Propoſition. The third is found by ſubſtituting in $m'' E'E' - 720 B'H' = a'''$ the Values of $E'E'$ and $B'H'$; and by a like Subſtitution theſe Theorems may be continued.

A General COROLLARY.

From theſe Propoſitions a great Variety of Rules may be deduced for diſcovering when an Equation has imaginary Roots. The Foundation of Sir *Iſaac Newton*'s Rule is demonſtrated in the ninth Propoſition, and its Corollary. The ſeventh Propoſition ſhews that if $\frac{l-1}{2l} \times E^2$ does not exceed $DF - CG + BH - AI + K$, ſome of the Roots of the Equation muſt be imaginary; and ſometimes this Rule will diſcover impoſſible Roots in an Equation, that do not appear by Sir *Iſaac Newton*'s Rule. Theſe are the only two Rules that have been hitherto publiſhed. But the Rules that ariſe from the Theorems in the eleventh and twelfth Propoſitions, are preferable to both; becauſe any imaginary Roots that can be diſcovered by the viith or ixth always appear from the xith and xiith Propoſitions; and impoſſible Roots will often be diſcovered by the xith and xiith Propoſitions in an Equation, that do not appear in that Equation when examined by the viith and ixth Propoſitions. The Advantage which the Rules deduced from the xith Propoſition, have above thoſe deduced from the preceeding Propoſitions, will be manifeſt by conſidering that in the xith Propoſition we have the Values of the Quantities a', b', c', d', e', ſeparately; whereas in the preceeding Propoſitions, we have only the Values of certain Aggregates of theſe Quantities

 joined

joined with the fame Signs. Now it is obvious that if thefe Quantities be feparately found pofitive, any fuch Aggregates of them muft be pofitive; but thefe Aggregates may be pofitive, and yet fome of the Quantities a', b', c', d', e', themfelves may be found negative: From which it follows, that if the Roots of the Equation are all affected with the fame Sign, and no impoffible Roots appear by Propofition xi[th], none will appear by the preceeding Propofitions; but that fome imaginary Roots may be difcovered by Propofition xi[th], when none appear in the Equation examined by the Propofitions that preceed the xi[th]. If fome of the Roots of the Equation are pofitive, and fome negative (which always eafily appears by confidering the Signs of the Terms of the Equation) then the xii[th] Propofition will be in many Cafes more apt to difcover imaginary Roots in an Equation than thofe that preceed it.

The Rule that flows from the firft Theorem of the xi[th] Propofition, obtains when the Roots of the Equation are affected with different Signs, as well as when they all have the fame Sign, and it is this; Multiply the Number of the Terms in an Equation that preceeds any Term, as Ex^{n-r} by the Number of Terms that follow it in the fame Equation, and call the Product m. Suppofe that $+D, -C, +B, -A, +I$ are the Coefficients preceeding the Term Ex^{n-r}, and that $+F$, $-G$, $+H$, $-I$, $+K$ are the Coefficients that follow it; then if $\frac{1}{2}mE^2$ does not exceed $\overline{m+1} \times DF - \overline{m+4} \times CG + \overline{m+9} \times BH - \overline{m+16} \times AI + \overline{m+25} \times K$ the Equation muft have fome imaginary Roots; where the Coefficients $m+1$, $m+4$, $m+9$, &c.

&c. are found by adding to m the Squares of the Numbers 1, 2, 3, 4, &c. which ſhew the Diſtances of the Coefficients to which they are prefixed, from the Coefficient E. The ſecond Theorem of the xiith Propoſition ſhews, that if $\frac{1}{2} m' E^2$ does not exceed $m' DF - \overline{m' - 12} \times CG + \overline{m - 72} \times BH - \overline{m' - 240} \times AI + \overline{m' - 600} \times K$, the Equation muſt have ſome Roots imaginary.

For an Example, If the four Roots of the Biquadratick Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ are real Quantities, it will follow equally from the vth, viith, ixth, and xith Propoſitions, that $\frac{3}{8}A^2$ muſt be greater than B, and that $\frac{3}{8}C^2$ muſt exceed BD. The viith further ſhews that $\frac{5}{12}B^2$ muſt exceed $AC - D$; the ixth demonſtrates that $\frac{4}{9}B^2$ muſt exceed AC; but our Rule deduced from Prop xi. ſhews that $2B^2$ muſt exceed $5AC - 8D$, the exceſs being $\frac{1}{2}a'$, and the Rule deduced from the ſecond Theorem of the xiith Propoſition ſhews that B^2 muſt always exceed $2AC + 4D$, the Exceſs being $\frac{1}{4}a''$. It appears from ſeveral preceeding Propoſitions, that if the Roots of the Equation have all the ſame Sign, then AC muſt exceed $16D$: Let the Exceſſes $5B^2 - 12AC + 12D = p$, $4B^2 - 9AC = q$, $AC - 16D = s$; and it is plain that a' $(= 4B^2 - 10AC + 16D) = q - s$

$-s = \frac{2}{5} \times \overline{2p - s}$; and that $a'' = q + s = \frac{2}{5} \times \overline{2p + 4s}$. Let us ſuppoſe,

1. That s is poſitive, then it is manifeſt that if either p or q be negative, a' muſt alſo be found negative, and conſequently that when the vii[th] or ix[th] Propoſitions ſhew any Roots to be imaginary, the xi[th] Propoſition muſt diſcover them at the ſame time. But as a' $\left(= q - s = \frac{2}{5} \times \overline{2p - s}\right)$ may be found negative when p and q are both poſitive, it follows that the Rule we have deduced from the xi[th] Propoſition may diſcover imaginary Roots in an Equation, that do not appear by the preceeding Propoſitions: Thus if you examine the Equation $x^4 - 6x^3 + 10x^2 - 7x + 1$ by Sir *Iſaac Newton*'s Rule, or by our vii[th] Propoſition, no imaginary Roots appear in it from either. But ſince $2B^2 - 5AC + 8D \left(= \frac{1}{2}a'\right) = 200 - 210 + 8 = -2$ is in this Equation negative, it is manifeſt that two Roots of the Equation muſt be imaginary. Let us ſuppoſe

2 That s is negative, and that from the Signs of the Terms of the Equation, it appears that ſome Roots are poſitive and ſome negative; then in Order to ſee if the Equation has any imaginary Roots, the moſt uſeful Rule is that we deduced from the ſecond Theorem of Prop. xii. *viz.* that if B^2 does not exceed $2AC + 4D$ ſome of the Roots of the Equation muſt be imaginary: For the Exceſs of B^2 above $2AC + 4D$ being $\frac{1}{4}a'' = \frac{1}{4} \times \overline{q + s} = \frac{1}{10} \times \overline{2p + 4s}$, and s

being

being negative, it is manifeſt, that if q or p be negative $\frac{1}{4}a''$ muſt be negative; and that $\frac{1}{4}a''$ may be negative when q and p are both poſitive; that is, This Rule muſt always diſcover ſome Roots to be imaginary when the vii[th] or ix[th] Propoſitions diſcover any impoſſible Roots in an Equation; and will very often diſcover ſuch Roots in an Equation when theſe Propoſitions diſcover none. For Example, if you examine the Equation $x^4 + 5x^3 + 6x^2 - x - 12 = 0$, you will diſcover no imaginary Roots in it by the vii[th] or ix[th] Propoſitions; and though $AC - 16D$ ($= s$) be negative, it does not follow, that the Equation has any impoſſible Roots, becauſe it appears from the Signs of the Terms, that the Equation has Roots affected with different Signs. But ſince $B^2 - 2AC - 4D$ ($= 36 + 10 - 48 = -2$) is negative, it appears from our Rule, that the Equation muſt have ſome imaginary Roots.

I might ſhew in the next Place, how the Rules deduced from the xi[th] and xii[th] Propoſitions may be extended ſo as to diſcover when more than two Roots of an Equation are imaginary, and in general to determine the Number of imaginary Roots in any Equation; but as it would require a long Diſcuſſion, and ſome *Lemmata* to demonſtrate this ſtrictly, I ſhall only obſerve that theſe xi[th] and xii[th] Propoſitions will be found to be ſtill the moſt uſeful of all thoſe we have given for that Purpoſe. To give one Example of this; If we are to examine the Equation

$$\underset{+}{x^4} - \underset{-}{4ax^3} + \underset{+}{6a^2x^2} - \underset{-}{4ab^2x} + \underset{+}{b^4} = 0$$

by Sir *Iſaac Newton*'s Rule, it is found to have four impoſſible Roots when a is greater than b; for though the Square of the ſecond Term multipli-

ed

ed by $\frac{3}{8}$ be equal to the Product of the firſt and third Terms, yet in that Caſe, in applying Sir *Iſaac Newton's* Rule, the Sign — ought to be placed under the ſecond Term, and the ſame is to be ſaid of the Square of the fourth Term. The Rule deduced from the viith Propoſition ſhews four Roots imaginary, when a is greater than b, and alſo when b^2 is greater than $15\ a^2$; but a Rule founded on the xi th Propoſition, ſhews the four Roots to be imaginary always when a exceeds b, or when b^2 exceeds $9\ a^2$; from which the Excellency of this Rule above theſe two is manifeſt. I have ſaid ſo much of Biquadratick Equations, that I muſt leave it to thoſe that are willing to take the Trouble, to make like Remarks on the higher Sorts of Equations.

In inveſtigating the preceeding Propoſitions, when I found my ſelf obliged to go through ſo intricate Calculations, I often attempted to find ſome more eaſy Way of treating this Subject. The following was of conſiderable Uſe to me,and may perhaps be entertaining to you. By it, I inveſtigate ſome *maxima* in a very eaſy Manner, that could not be demonſtrated in the common Way with ſo little Trouble.

LEMMA V. Let the given Line AB be divided any where in P and the Rectangle of the Parts AP and PB will be a *maximum* when theſe Parts are equal.

This is manifeſt from the Elements of *Euclid.*

LEMMA VI. If the Line AB is divided into any Number of Parts AB, CD, DE, EB, the Product of all thoſe Parts multiplied into one another will be a

max-

maximum when the Parts are equal amongſt themſelves. For let the Point D be where you will, it is manifeſt that if DB be biſſected in E, the Product $AC \times CD \times DE \times EB$ will be greater than $AC \times CD \times De \times eB$

becauſe by the laſt Lemma $DE \times EB$ is greater than $De \times eB$; and for the ſame reaſon AD and CE muſt be biſſected in C and D; and conſequently all the Parts AC, CD, DE, EB muſt be equal amongſt themſelves, that their Product may be a *maximum*.

Lemma VII. The Sum of the Products that can be made by multiplying any two Parts of AB by one another is a *maximum* when the Parts are equal. The Sum of theſe Products is $AC \times CB + CD \times DB + DE \times EB$: Now that $DE \times EB$ may be a *maximum*, DB muſt be biſſected in E by the v[th] Lemma, and for the ſame reaſon AD and CE muſt be biſſected in C and D, that is all the Parts, AC, CD, DE, EB muſt be equal, that the Sum of all theſe Products may be a *maximum*.

Lemma VIII. The Sum of the Products of any three Parts of the Line AB is a *maximum*, when all the Parts are equal. For that Sum is $AC \times CD \times DE + EB \times \overline{AC \times CD + AC \times DE + CD \times DE}$; and ſuppoſing the Point E given, it is manifeſt that AE muſt be equally triſected in C and D that $AC \times CD \times DE$ may be a *maximum* by Lemma vi. and that $AC \times CD + AC \times DE + CD \times DE$ may be a *maximum* by Lemma vii[th]. From which it is manifeſt that all the Parts AC, CD, DE, EB muſt be equal, that the Sum of the Products of any three of them may be a *maximum*.

Lemma IX. It is manifeſt that this way of reaſoning is general, and that the Sum of any Quantities being given, the Sum of all the Products that can be

made

made by multiplying any given Number of them by one another, muſt be a *maximum* when theſe Quantities are equal. But the Sum of the Squares, or of any pure Powers of theſe Quantities, is a *minimum*, when the Quantities are equal.

THEOREM.

Suppoſe $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5}$ &c. $= 0$, *to be an Equation that has not all its Roots equal to one another: Let* r *expreſs the Dimenſions of any Coefficient* D, *and let* $l = n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$ &c. *taking as many Factors as there are Units in* r; *then ſhall* $\frac{l}{n^r} \times A^r$ *be always greater than* D, *if the Roots of the Equation are real Quantities affected with the ſame Sign.*

This may be demonſtrated from the preceeding Propoſitions: But to demonſtrate it from the laſt Lemmata, let us aſſume an Equation that has all its Roots equal to one another, and the Sum of all its Roots equal to A, the Sum of the Roots of the propoſed Equation. This Equation will be $\overline{x - \frac{1}{n}A}\big|^n = 0$, or $x^n - Ax^{n-1} + n \times \frac{n-1}{2} \times \frac{A^2}{n^2} x^{n-2} - n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{A^3}{n^3} x^{n-3}$ &c. $= 0$ and if r expreſs the Dimenſions of the Coefficient of any Term of this Equation (or the Number of Terms which

pre-

preceed it) it is manifeſt that the Term it ſelf will be $l \times \frac{A^r}{n^r} x^{n-r}$: But by the Suppoſition Dx^{n-r} is the Correſponding Term in the propoſed Equation, and D muſt be the Sum of all the Products that can be made by multiplying as many Roots of that Equation by one another, as there are Units in r; and $\frac{l A^r}{n^r}$ muſt be the Sum of the like Products of the Roots of the other Equation; which muſt be the greater Quantity by the preceeding Lemmata, becauſe its Roots are equal amongſt themſelves, and their Sum is equal to the Sum of the Roots of the propoſed Equation; and the Sum of ſuch Products is a *maximum* when the Roots are equal amongſt themſelves. By purſuing this Method, it may be demonſtrated that $\overline{\frac{2B}{n \times \overline{n-1}}}\Big|^{\frac{r}{2}} \times l$ muſt always exceed the Coefficient prefixed to the Term x^{n-r} in an Equation whoſe Roots are all real Quantities affected with the ſame Sign; providing that r be a Number greater than 2; and alſo that $\overline{\frac{2 \times 3C}{n \times \overline{n-1} \times \overline{n-2}}}\Big|^{\frac{r}{3}} \times l$ muſt exceed the ſame Coefficient, if r be any Number greater than 3.

It is eaſy to continue theſe Theorems.

The third Method which I mentioned in the Beginning of this Letter, is deduced from the Conſideration of the Limits of the Roots of Equations; and though it is explained by ſome Authors already, yet as I demonſtrate

monstrate and apply it to this Subject in a different Manner, I shall add a short Account of it.

LEMMA X. If you transform the Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ into one that shall have each of its Roots less than the respective Values of x by a given Difference e; suppose $y = x - e$ or $x = e + y$ and the transformed Equation, the Order of the Terms being inverted, will have this Form.

$$\begin{array}{lllll} e^4 & + 4e^3 y & + 6e^2 y^2 & + 4ey^3 & + y^4 = 0. \\ - Ae^3 & - 3Ae^2 y & - 3Aey^2 & - Ay^3 & \\ + Be^2 & + 2Bey & + By^2 & & \\ - Ce & - cy & & & \\ + D & & & & \end{array}$$

Where it is manifest,

1. That the first Term $e^4 - Ae^3 + Be^2 - Ce + D$ is the Quantity that arises by substituting e in Place of x in the proposed Equation $x^4 - Ax^3 + Bx^2 - Cx + D$.

2. That the Coefficient of the second Term $4e^3 - 3Ae^2 + 2Be - C$ is the Quantity that arises by multiplying each Part of the first $e^4 - Ae^3 + Be^2 - Ce + D$ by the Index of e in that Part, and dividing the Product by e.

3. That the Coefficient of the third Term $6e^2 - 3Ae + B$ is the Quantity that arises from the preceeding Coefficient $4e^3 - 3Ae^2 + 2Be - C$ by multiplying each Part by the Index of e in it, and dividing the Product by $2e$.

4. That the Coefficient of the fourth Term arises in like Manner from the preceeding, only you now divide by $3e$; and in general, the Coefficient of any Term may be deduced from the Coefficient of that Term which preceeds it, by multiplying each Part of

the preceeding Coefficient by the Index of e in that Part, and dividing the Product by e and by the Index of y, in the Term whose Coefficient is required.

LEMMA XI. If any Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$ be transformed in the same Manner, by supposing $x = y - e$ or $x = e + y$, and consequently $x^n = \overline{e+y}|^n$, $Ax^{n-1} = A \times \overline{e+y}|^{n-1}$, $Bx^{n-2} = B \times \overline{e+y}|^{n-2}$ &c. The transformed Equation will have this Form, the Order of the Terms being inverted,

$$
\begin{array}{llll}
e^n & + ne^{n-1}y & + n \times \frac{n-1}{2} \times e^{n-2}y^2 \text{ \&c.} & = 0 \\
- Ae^{n-1} & - \overline{n-1} \times Ae^{n-2}y & - \overline{n-1} \times \frac{n-2}{2} \times Ae^{n-3}y^2 \text{ \&c.} & \\
+ Be^{n-2} & + \overline{n-2} \times Be^{n-3}y & + \overline{n-2} \times \frac{n-3}{2} \times Be^{n-4}y^2 \text{ \&c.} & \\
- Ce^{n-3} & - \overline{n-3} \times Ce^{n-4}y & - \overline{n-3} \times \frac{n-4}{2} \times Ce^{n-5}y^2 \text{ \&c.} & \\
\text{\&c.} & \text{\&c.} & \text{\&c.} &
\end{array}
$$

Where it is manifest,

1. That the first Term $e^n - Ae^{n-1} + Be^{n-2} - Ce^{n-3}$ &c. is the Quantity that arises by substituting e in the Place of x in the proposed Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c.

2. That the Coefficient of the second Term $ne^{n-1} - \overline{n-1} \times Ae^{n-2} + \overline{n-2} \times Be^{n-3} - \overline{n-3} \times Ce^{n-4}$ &c. is deduced from the preceeding $e^n - Ae^{n-1} + Be^{n-2} - Ce^{n-3}$ &c. by multiplying each of its Parts by the Index of e in that Part, and dividing by e.

3. That

3. That the Coefficient of the third Term is deduced from the Coefficient of the second Term, by multiplying after the same manner, each of its Parts by the Index of e and dividing by $2e$. In general, the Coefficient of any Term y^r is deduced from the Coefficient of the preceeding Term, that is of y^{r-1} by multiplying every Part of that Coefficient by the Index of e in it, and dividing the Product by re.

LEMMA XII. If you substitute any two Quantities K and L in the Place of x in $x^4 - Ax^3 + Bx^2 - Cx + D$, and the Quantities that result from these Substitutions be affected with contrary Signs, the Quantities K and L must be *Limits* of one or more real Roots of the Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$. That is, one of these Quantities must be greater, and the other less than one or more Roots of that Equation.

For if you suppose that $a, b, c, d,$ are the Roots of that Equation, then it is plain from the *Genesis* of Equations, that $x^4 - Ax^3 + Bx^2 - Cx + D = \overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$; and therefore K and L being substituted for x in $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$, the Product becomes in the one Case positive, and in the other negative; so that one of the Factors $x-a$, $x-b$, $x-c$, $x-d$ must have a Sign when K is substituted for x in it, contrary to the Sign which it is affected with when L is substituted in in it for x, suppose that Factor to be $x-b$; and since $K-b$ and $L-b$ are Quantities whereof the one is positive, and the other negative, it is manifest that b one of the Roots of the Equation must be less than one, and greater than the other of the two Quantities

tities K and L : So that K and L muſt be the *Limits* of the Root b.

I ſay further, that the Root whereof K and L are *Limits*, muſt be a real Root of the Equation ; for the Product of the Factors that involve impoſſible Roots in an Equation can never have its Signs changed by ſubſtituting any real Quantity whatſoever in place of x; becauſe the Number of ſuch Roots is always an even Number, and the Product of any two of theſe Roots ſuch as $x - m - \sqrt{-n}$, and $x - m + \sqrt{-n}$ is $\overline{x - m}|^2 + n^2$ which muſt be always poſitive, whatever Quantity be ſubſtituted for x while n remains poſitive, that is, while theſe two Roots are impoſſible.

LEMMA XIII. If you ſubſtitute K and L for x in $x^n - Ax^{n-1} + Bx^{n-2}$ &c. and the Quantities that reſult be affected with contrary Signs, then ſhall K and L be the *Limits* of one or more real Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$. This may be demonſtrated after the ſame Manner as the laſt Lemma.

THEOREM I. If a, b, c, d are the Roots of the Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, they ſhall be the *Limits* of the Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$.

Suppoſe a to be the leaſt Root of the biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, b the ſecond Root, c the third, and d the fourth, and the Values of y in the Equation in the x[th] Lemma, will be $a - e$, $b - e$, $c - e$, $d - e$; then by ſubſtituting ſucceſſively a, b, c, d for e in that Equation of y, one of the Values of y will vaniſh in every Subſtitution, and the firſt Term of the Equation of y, *viz.* $e^4 - Ae^3 + Be^2 - Ce + D$ vaniſhing, the Equation will be reduced to a Cubick of this Form.

$$\begin{array}{llll} 4e^3 & +\,6e^2y & +\,4ey^2 & +\,y^3 = 0 \\ -\,3Ae^2 & -\,3Aey & -\,Ay^2 & \\ +\,2Be & +\,By & & \\ -\,C & & & \end{array}$$

And consequently $4e^3 - 3Ae^2 + 2Be - C$ muſt be the Product of the three remaining Values of y having its Sign changed; that is, it muſt be equal to $-\overline{b-a}\times\overline{c-a}\times\overline{d-a}$ when e is ſuppoſed equal to a, it muſt be $-\overline{a-b}\times\overline{c-b}\times\overline{d-b}$ when $e=b$; it muſt be $-\overline{a-c}\times\overline{b-c}\times\overline{d-c}$ when $e=c$; and it muſt be $-\overline{a-d}\times\overline{b-d}\times\overline{c-d}$ when $c=d$. Now it is manifeſt that theſe Products $\overline{b-a}\times\overline{c-a}\times\overline{d-a}$, $\overline{a-b}\times\overline{c-b}\times\overline{d-b}$, $\overline{a-c}\times\overline{b-c}\times\overline{d-c}$, $\overline{a-d}\times\overline{b-d}\times\overline{c-d}$ muſt be affected with the Signs $+$, $-$, $+$, $-$ reſpectively; the firſt being the Product of three poſitive Quantities, the ſecond the Product of one negative and two poſitives, the third the Product of two negatives and one poſitive, and the fourth the Product of three negatives. Therefore ſince by ſubſtituting a, b, c, d for e in the Quantity $4e^3 - 3Ae^2 + 2Be - C$, it becomes alternately a poſitive and a negative Quantity, it follows from the laſt Lemma that a, b, c, d muſt be the *Limits* of the Roots of the Equation $4e^3 - 3Ae^2 + 2Be - C = 0$, or of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$.

Cor. It follows from this Theorem, that if $a'\,b'$ and c' are the three Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$, they muſt be *Limits* betwixt a, b, c, d the Roots of the Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$: For if a, b, c, d are *Limits* of the Roots

Roots a', b', and c'; theſe Roots converſely muſt be *Limits* betwixt a, b, c and d.

THEOREM II. Multiply the Terms of any Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ by any Arithmetical Series of Quantities $l + 4m, l + 3m, l + 2m, l + m, l$, and the Roots of the Biquadratick a, b, c, d will be the *Limits* of the Roots of the Equation that reſults from that Multiplication that is of the Equation.

$$\begin{array}{l} lx^4 - lAx^3 + lBx^2 - lCx + lD = 0 \\ +4mx^4 - 3mAx^3 + 2mBx^2 - mCx \end{array}$$

Suppoſe that ſubſtituting the Roots a, b, c, d of the biquadratick Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ ſucceſſively, for x in $4x^3 - 3Ax^2 + 2Bx - C$, the Quantities that reſult are $-R$, $+S$, $-T$, $+Z$; while $x^4 - Ax^3 + Bx^2 - Cx + D$ is in every Subſtitution equal to nothing; and it is manifeſt that the Quantity

$$\begin{array}{l} +lx^4 - lAx^3 + lBx^2 - lCx + lD \\ +4mx^4 - 3mAx^3 + 2mBx^2 - mCx \end{array}$$

will become (when a, b, c, d are ſubſtituted ſucceſſively in it for x) equal to $-mRx$, $+mSx$, $-mTx$, $+mZx$; where the Signs of theſe Quantities being alternately negative and poſitive, it follows that a, b, c, d muſt be *Limits* of that Equation by Lemma xii.

COR. Hence it follows, that a, b, c and d are *Limits* of the Roots of the Cubick Equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$, and converſely, that the Roots of this Cubick are *Limits* of the Roots of the biquadratick Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, for multiplying the Terms of this biquadratick Equation by the Arithmetical Progreſſion $0, -1, -2, -3, -4$, the Cubick $Ax^3 - 2Bx^2 + 3Cx - AD = 0$ ariſes.

THEOREM III. *In general, the Roots of the Equation* $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$, *are the* Limits *of the Roots of the Equation* $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$, *or of any Equation that is deduced from it by multiplying its Terms by any Arithmetical Progression* $l \mp d, l \mp 2d, l \mp 3d$ &c. *and conversely the Roots of this new Equation will be the* Limits *of the Roots of the proposed Equation* $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$.

This Theorem is demonstrated from the xi[th] and xiii[th] Lemmata in the same manner as the preceeding Theorems were demonstrated from the x[th] and xii[th]. From these Theorems it is easy to infer all that is delivered by the Writers of Algebra on this Subject.

THEOREM IV. *The Equation* $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$ *will have as many imaginary Roots as the Equation* $nx^{n-1} - \overline{n-1} \times Ax^{n-2} - \overline{n-2} \times Bx^{n-3}$ &c. $= 0$, *or the Equation* $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3}$ &c. $= 0$.

Suppose that any Root of the Equation $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$, as p becomes imaginary, and the two Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2}$ &c $= 0$, which by Theorem III. ought to be its *Limits*, cannot both be real Quantities; for it is manifest from the Demonstration of Theorem I. that if they are real Quantities, then being substituted for x in $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. the Quantities that result must have contrary Signs, and consequently the Root p, whereof they are *Limits*, must be a real Root; which

which is againſt the Suppoſition. The ſame is true of the Equation $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3}$ &c. $= 0$, for the ſame Reaſon.

COR. The biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, will have two imaginary Roots, if two Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ be imaginary; or if two Roots of the Equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ be imaginary. But two Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ muſt be imaginary, when two Roots of the Quadratick $6x^2 - 3Ax + B = 0$, or of the Quadratick $3Ax^2 - 4Bx + 3C = 0$ are imaginary, becauſe the Roots of theſe quadratick Equations are the *Limits* of the Roots of that Cubick, by the third Theorem; and for the ſame reaſon two Roots of the Cubick Equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ muſt be imaginary, when the Roots of the quadratick $3Ax^2 - 4Bx + 3C = 0$, or of the quadratick $Bx^2 - 3Cx + 6D = 0$ are impoſſible. Therefore two Roots of the Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ muſt be imaginary when the Roots of any one of theſe three quadratick Equations $6x^2 - 3Ax + B = 0$, $3Ax^2 - 4Bx + 3C = 0$, $Bx^2 - 3Cx + 6D = 0$ become imaginary; that is, when $\frac{3}{8}A^2$ is leſs than B, $\frac{4}{9}B^2$ leſs than AC, or $\frac{3}{8}C^2$ leſs than BD.

COR. II. By proceeding in the ſame manner, you may deduce from any Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$, as many quadratick Equations as there are Terms excepting the firſt and laſt whoſe Roots muſt be all real Quantities, if the

 pro-

propofed Equation has no imaginary Roots. The Quadratick deduced from the three firft Terms $x^n - Ax^{n-1} + Bx^{n-2}$ will manifeftly have this Form, $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}$ &c. $\times x^2 - \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}$ &c. $\times Ax + \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5}$ &c. $\times B = 0$, continuing the Factors in each till you have as many as there are Units in $n-2$. Then dividing the Equation by all the Factors $n-2$, $n-3$ &c. which are found in each Coefficient, the Equation will become $n \times \overline{n-1} \times x^2 - \overline{n-1} \times 2Ax + 2 \times 1 \times B = 0$, whofe Roots will be imaginary by Prop. i. when $n \times \overline{n-1} \times 2 \times 4B$ exceeds $\overline{n-1}|^2 \times 4A^2$, or when B exceeds $\frac{n-1}{2n}A^2$, fo that the propofed Equation muft have fome imaginary Roots when B exceeds $\frac{n-1}{2n}A^2$; as we demonftrated after another Manner in the v^th^ Propofition. The Quadratick Equation deduced in the fame Manner from the three firft Terms of the Equation $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3}$ &c. $= 0$, will have this Form $\overline{n-1} \times \overline{n-2} \times \overline{n-3}$ &c. $\times Ax^2 - \overline{n-2} \times \overline{n-3} \times \overline{n-4}$ &c. $\times 2Bx + \overline{n-3} \times \overline{n-4} \times \overline{n-5}$ &c. $\times 3C =$ 0; which by dividing by the Factors common to all the Terms, is reduced to $\overline{n-1} \times \overline{n-2} \times Ax^2 - \overline{n-2} \times 4Bx + 6C = 0$, whofe Roots muft be imaginary when $\frac{2}{3} \times \frac{n-2}{n-1} \times B^2$ is lefs than AC; and therefore in that cafe fome Roots of the propofed Equation muft be imaginary.

Cor. III. In general, let $Dx^{n-r+1} - Ex^{n-r} + Fx^{n-r-1}$ be any three Terms of the Equation, $x^n -$

Ax

$Ax^{n-1} + Bx^{n-2}$ &c. $= 0$, that immediately follow one another, multiply the Terms of this Equation first by the Progression $n, n-1, n-2$, &c. then by the Progression $n-1, n-2, n-3$, &c. then by $n-2, n-3, n-4$, &c. till you have multiplied by as many Progressions as there are Units in $n-r-1$: Then multiply the Terms of the Equation that arises, as often by the Progression 0, 1, 2, 3 &c. as there are Units in $r-1$, and you will at length arrive at a Quadratick of this Form,

$\overline{n-r+1} \times \overline{n-r} \times \overline{n-r-1} \times \overline{n-r-2}$ &c. $\times \overline{r-1} \times \overline{r-2} \times \overline{r-3} \times \overline{r-4}$ &c. Dx^2

$- \overline{n-r} \times \overline{n-r-1} \times \overline{n-r-2} \times \overline{n-r-3}$ &c. $\times r \times \overline{r-1} \times \overline{r-2} \times \overline{r-3}$ &c. $\times Ex$

$+ \overline{n-r-1} \times \overline{n-r-2} \times \overline{n-r-3} \times \overline{n-r-4}$ &c. $\times \overline{r+1} \times r \times \overline{r-1} \times \overline{r-2}$ &c. $\times F = 0$,

and dividing by the Factors $n-r-1, n-r-2$, &c. and $r-1, r-2$ &c. which are found in each Coefficient, this Equation will be reduced to $\overline{n-r+1} \times \overline{n-r} \times 2 \times 1 \times Dx^2 - \overline{n-r} \times 2 \times r \times 2Ex + 2 \times 1 \times \overline{r+1} \times rF = 0$, whose Roots must be imaginary (by Prop i.) when $\frac{n-r}{n-r+1} \times \frac{r}{r+1} \times E^2$ is less than D F. From which it is manifest that if you divide each Term of this Series of Fractions $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \frac{n-3}{4}$, &c. $\frac{n-r+1}{r}, \frac{n-r}{r+1}$ by that which preceeds it, and place the Quotients above the Terms of the Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $=$

&c. $= 0$, beginning with the ſecond: Then if the Square of any Term multiplied by the Fraction over it be found leſs than the Product of the adjacent Terms, ſome of the Roots of that Equation muſt be imaginary Quantities. There remain many things that might be added on this Subject, but I am afraid you will think I have ſaid as much of it as it deſerves; and therefore I ſhall only add the Demonſtration of ſome Algebraick Rules and Theorems that are very eaſily deduced from the xi[th] Lemma.

I. The Rule for diſcovering when two or more Roots of an Equation are equal, immediately follows from that Lemma, Suppoſe that two Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$ are equal, and two Values of y (which is equal always to $x-e$) will be equal. Suppoſe that e is equal to one of thoſe two equal Values of x; and two Values of y will vaniſh, and conſequently y^2 muſt enter each of the Terms of the Equation of y; and therefore in this Caſe the firſt and ſecond Term of the Equation of y in Lemma xi[th] muſt vaniſh, that is, $e^n - Ae^{n-1} + Be^{n-2} - Ce^{n-3}$ &c. $= 0$ and $ne^{n-1} - \overline{n-1} \times Ae^{n-2} + \overline{n-2} \times Be^{n-3} - \overline{n-3} \times Ce^{n-4}$ &c. $= 0$ at the ſame time; and conſequently theſe two Equations muſt have one Root common, which muſt be one of thoſe Values of x that were ſuppoſed equal to each other. It is manifeſt therefore that when two Values of x are equal in the Equation $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$, one of them muſt be a Root of the Equation $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$.

If three Values of x be ſuppoſed equal amongſt themſelves and to e, then three Values of y $(= x - e)$ will vaniſh, and the firſt three Terms of the Equation of y in in Lemma xi. will vanish, and therefore $n \times \overline{n-1} \times e^{n-2} - \overline{n-1} \times \overline{n-2} \times Ae^{n-3} + \overline{n-2} \times \overline{n-3} \times Be^{n-4}$ &c $= 0$; and one of the equal Values of x will be a Root of this last Equation, and two of them will be Roots of the Equation $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$. In general, it appears that if the Equation $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$ have as many Roots equal amongst themselves as there are Units in S, then shall as many of those be Roots of the Equation $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$ as there are Units in $S-1$; as many of them shall be Roots of the Equation $n \times \overline{n-1} \times x^{n-2} - \overline{n-1} \times \overline{n-2} \times Ax^{n-3} + \overline{n-2} \times \overline{n-3} \times Bx^{n-4}$ &c. $= 0$, as there are Units in $S-2$; and so on.

II. The general Rule which Sir *Isaac Newton* has given in the *Article de limitibus Equationum* for finding a *Limit* greater than any of the Values of x immediately follows from the xi[th] Lemma; for it is manifest that if e be such a Quantity as substituted in all the Coefficients of the Equation of y, *viz.* in $e^n - Ae^{n-1} + Be^{n-2}$ &c. $ne^{n-1} - \overline{n-1} \times Ae^{n-2} + \overline{n-2} \times Be^{n-3}$ &c. $n \times \frac{n-1}{2} \times e^{n-2} - \overline{n-1} \times \frac{n-2}{2} \times Ae^{n-3} + \overline{n-2} \times \frac{n-3}{2} \times Be^{n-4}$ &c. gives the Quantities that result all positive; then there being no Changes of the Signs of the Equation of y in this case, all its Values must be negative; and since y is always equal to $x - e$ it follows that e must be a greater Quantity than any of the Values of x; that is, it must be a

Limit greater than any of the Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$.

III. From this xi[th] Lemma some important Theorems in the Method of *Series*, and of *Fluxions*, and the Resolution of Equations are demonstrated with great Facility; it is obvious that the Coefficient of the second Term of the Equation of y in that Lemma is the *Fluxion* of the first Term divided by the *Fluxion* of e; the Coefficient of the third Term is the second *Fluxion* of that first Term divided by $2\dot{e}^2$, supposing e to flow uniformly. The third Term is the third *Fluxion* of the first Term divided by $2 \times 3\dot{e}^3$; and so on. Therefore supposing $e^n - Ae^{n-1} + Be^{n-2}$ &c. $= c$, the Equation for determining y will be $c + \frac{\dot{c}}{\dot{e}}y + \frac{\ddot{c}}{1 \times 2\dot{e}^2}y^2$

$+ \frac{\dddot{c}}{1 \times 2 \times 3\dot{e}^3}y^3$ &c. $= 0$; and hence, when e is near the true Value of x, Theorems may be deduced for approximating to y, and consequently to x, which is supposed equal to $y + e$.

IV. Let AP ($= x$) be the Absciss and PM ($= z$) the Ordinate of any Curve BLM; and suppose any other Absciss AK $= e$ and Ordinate KL $= c$, then shall $z\ (= \text{PM}) = c \mp \frac{\dot{c}}{\dot{e}}y + \frac{\ddot{c}}{2\dot{e}^2}y^2 \mp \frac{\dddot{c}}{2 \times 3\dot{e}^3}y^3$

$+ \frac{\ddddot{c}}{2 \times 3 \times 4\dot{e}^4}y^4$ &c.

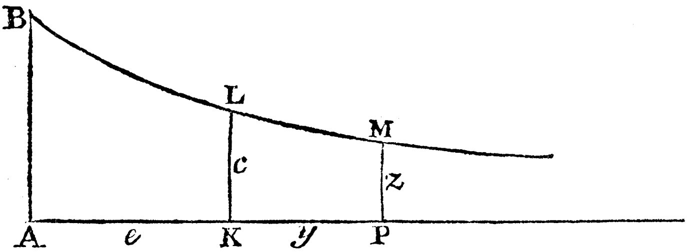
For let z be supposed equal to any Series consisting of given Quantities, and the Powers of x, as to $Ax^n + Bx^r + Cx^s$ &c. and substituting $e \mp y$ for x, we shall find after the manner of the xi[th] Lemma,

$$z = Ae^{n} \mp nAe^{n-1}y + n\times\frac{n-1}{2}\times Ae^{n-2}y^{2} \text{ \&c.}$$

$$+ Be^{r} \mp rBe^{r-1}y + r\times\frac{r-1}{2}\times Be^{r-2}\times y^{2}\text{\&c.}$$

$$+ Ce^{s} \mp sCe^{s-1}y + s\times\frac{s-1}{2}\times Ce^{s-2}\,y^{2} \text{ \&c.}$$

&c. &c. &c.



But when $x = e$ then $z = c = Ae^{n} + Be^{r} + Ce^{s}$ &c. $\dot{c} = nAe^{n-1}\dot{e} + rBe^{r-1}\dot{e} + sCe^{s-1}\dot{e}$ &c. $\ddot{c} = n\times\overline{n-1}\times Ae^{n-2}\dot{e}^{2} + r\times\overline{r-1}\times Be^{r-2}\dot{e}^{2} + s\times\overline{s-1}\times Ce^{s-2}\dot{e}^{2}$ &c. and therefore $z = c \mp \frac{\dot{c}}{\dot{e}}y + \frac{\ddot{c}}{2\dot{e}^{2}}y^{2} \mp \frac{\dddot{c}}{2\times 3\dot{e}^{3}}y^{3}$ &c. After the same manner you will find that $c = z \pm \frac{\dot{z}}{\dot{x}}y + \frac{\ddot{z}}{2\dot{x}^{2}}y^{2} \pm \frac{\dddot{z}}{2\times 3\dot{x}^{3}}y^{3}$ &c. for $c = Ae^{n} + Be^{r} + Ce^{s}$ &c. $=$ $A\times\overline{x\pm y}|^{n} + B\times\overline{x\pm y}|^{r} + C\times\overline{x\pm y}|^{s}$ &c. $= z \pm \frac{\dot{z}}{\dot{x}}y$

$+\frac{\ddot{z}}{2x^2}y^2$ &c. The Area KLMP is equal to the Fluent of $z\dot{y}$ or of $c\dot{y}$, but

$$c\dot{y} = z\dot{y} \pm \frac{\dot{z}}{x}y\dot{y} + \frac{\ddot{z}}{2x^2}y^2\dot{y} \pm \frac{\dddot{z}}{2\times 3\dot{x}^2}y^3\dot{y} \text{ \&c.}$$

and $$z\dot{y} = c\dot{y} \mp \frac{\dot{c}}{\dot{e}}y\dot{y} + \frac{\ddot{c}}{2\dot{e}^2}y^2\dot{y} \mp \frac{\dddot{c}}{2\times 3\dot{e}^2}y^3\dot{y} \text{ \&c.}$$

And consequently by finding the Fluents

$$\text{KLMP} = cy \mp \frac{\dot{c}}{2\dot{e}}y^2 + \frac{\ddot{c}}{2\times 3\dot{e}^2}y^3 \mp \frac{\dddot{c}}{2\times 3\times 4\dot{e}^3}y^4 \text{ \&c.}$$

$$\text{or KLMP} = zy \pm \frac{\dot{z}}{2\dot{x}}y^2 + \frac{\ddot{z}}{2\times 3\dot{x}^2}y^3 \pm \frac{\dddot{z}}{2\times 3\times 4\dot{x}^3}y^4 \text{ \&c.}$$

This laſt is the Theorem publiſhed by the learned Mr. *Bernouilli* in the *Acta Lipſiæ* 1694. It is now high Time to conclude this long Letter; I beg you may accept of it as a Proof of that Reſpect and Eſteem with which

I am,

SIR,

Your moſt Obedient,

Moſt Humble Servant,

Colin Mac Laurin.