PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the Propagation of a Disturbance in a Fluid under Gravity

F. B. Pidduck

Proc. R. Soc. Lond. A 1910 **83**, 347-356
doi: 10.1098/rspa.1910.0023

On the Propagation of a Disturbance in a Fluid under Gravity.

By F. B. PIDDUCK, B.A., Fellow of Queen's College, Oxford.

(Communicated by Prof. A. E. H. Love, F.R.S. Received January 8,—Read February 10, 1910.)

CONTENTS.

1. *Introduction.*

In a recent paper* Lord Rayleigh has called attention to the fact of the instantaneous propagation of a limited disturbance over the surface of heavy incompressible fluid. This instantaneity occurs in spite of the fact that the velocity of any simple-harmonic gravity wave is finite, the depth being finite and constant. As the points thereby raised are of some delicacy, and are not completely settled in the paper quoted, some further remarks on the phenomenon may not be superfluous.

2. *Solution of the Cauchy-Poisson Problem for Finite Depth.*†

In his treatment of this case Lord Rayleigh used the integral form of solution. There are, however, great difficulties raised in this way on account of lack of convergence at the surface. I proceed to obtain a solution in the form of a series similar in type to the known serial solution of the problem for infinite depth.

We have $$\phi = \frac{g}{\pi}\int_0^\infty \frac{\cosh k(h-y)}{\cosh kh}\frac{\sin\sigma t}{\sigma}\cos kx\,dk,$$

where $\sigma^2 = gk\tanh kh$.

Write $$\chi = \frac{g}{\pi}\int_0^\infty \frac{\cosh k(h-y)}{\cosh kh}\frac{\sin\sigma t}{\sigma}\sin kx\,dk.$$

* Lord Rayleigh, 'Phil. Mag.' [6], vol. 18, 1909, p. 1. The paper was brought to my notice by Prof. Love, to whom I am also indebted for much helpful criticism of the present paper before communication.

† The notation used is that of Lamb, 'Hydrodynamics,' 3rd ed., pp. 344—346, 364—374, where the theory for deep water is contained. The sign of y is, however, reversed.

Expanding the sine in powers of t, we have

$$\phi+i\chi = \frac{g}{\pi}\left[t\int_0^\infty \frac{\cosh k(h-y)}{\cosh kh} e^{ikx}\,dk - \frac{gt^3}{3\,!}\int_0^\infty \frac{\cosh k(h-y)}{\cosh kh} k\tanh kh\,.\,e^{ikx}\,dk + \ldots \right.$$
$$\left. + \frac{(-)^n g^n t^{2n+1}}{(2n+1)\,!}\int_0^\infty \frac{\cosh k(h-y)}{\cosh kh} k^n \tanh^n kh\,.\,e^{ikx}\,dk + \ldots\right].$$

The integrals assume a more convenient form by writing

$$p \equiv y/2h, \qquad q \equiv x/2h, \tag{1}$$

and making the substitution $u = 2kh$. Thus, if we write

$$\theta_n \equiv \int_0^\infty \frac{\cosh(\tfrac{1}{2}u - pu)}{\cosh\tfrac{1}{2}u} u^n \tanh^n \tfrac{1}{2}u\,.\,e^{iqu}\,du, \tag{2}$$

we have

$$\phi+i\chi = \frac{1}{\pi}\left[\frac{g}{2h}t\theta_0 - \frac{1}{3\,!}\left(\frac{g}{2h}\right)^2 t^3\theta_1 + \ldots + \frac{(-)^n}{(2n+1)\,!}\left(\frac{g}{2h}\right)^{n+1} t^{2n+1}\theta_n + \ldots\right]. \tag{3}$$

Further,
$$\theta_n = \int_0^\infty \frac{e^{\frac{1}{2}u-pu}+e^{pu-\frac{1}{2}u}}{e^{\frac{1}{2}u}+e^{-\frac{1}{2}u}} u^n \left(\frac{e^{\frac{1}{2}u}-e^{-\frac{1}{2}u}}{e^{\frac{1}{2}u}+e^{-\frac{1}{2}u}}\right)^n e^{iqu}\,du$$
$$= \int_0^\infty (e^{-pu}+e^{pu-u})\,u^n(1-e^{-u})^n(1+e^{-u})^{-n-1}e^{iqu}\,du.$$

Let (n, λ) denote the coefficient of z^λ in the expansion of $(1+z)^n(1-z)^{-n-1}$ in ascending powers of z. Then

$$\theta_n = \int_0^\infty \{e^{-u(p-iq)}+e^{-u(1-p-iq)}\}\,du - (n,1)\int_0^\infty \{e^{-u(1+p-iq)}+e^{-u(2-p-iq)}\}\,u\,du$$
$$+\ldots+(-)^\lambda(n,\lambda)\int_0^\infty \{e^{-u(\lambda+p-iq)}+e^{-u(1+\lambda-p-iq)}\}\,u^n du + \ldots.$$

Now if the real part of ϵ is positive,

$$\int_0^\infty e^{-\epsilon z} z^n dz = \frac{n\,!}{\epsilon^{n+1}}.$$

Hence, if we write

$$\alpha = p+iq = \frac{1}{2h}(y+ix), \qquad \beta = p-iq = \frac{1}{2h}(y-ix), \tag{4}$$

we have

$$\frac{\theta_n}{n\,!} = \left[\frac{1}{\beta^{n+1}}+\frac{1}{(1-\alpha)^{n+1}}\right] - (n,1)\left[\frac{1}{(1+\beta)^{n+1}}+\frac{1}{(2-\alpha)^{n+1}}\right]+\ldots$$
$$+(-)^\lambda(n,\lambda)\left[\frac{1}{(\lambda+\beta)^{n+1}}+\frac{1}{(1+\lambda-\alpha)^{n+1}}\right]+\ldots. \tag{5}$$

It is convenient to introduce the quantity $\{n, \lambda\}$, which is the coefficient of z^λ in the expansion of $(1+z)^n(1-z)^{-n}$. The quantities (n, λ) and $\{n, \lambda\}$ are positive. It is easily proved that

$$\left.\begin{aligned}(n,\lambda)+(n,\lambda-1) &= (n+1,\lambda)-(n+1,\lambda-1),\\ (n,\lambda)-(n,\lambda-1) &= \{n,\lambda\}.\end{aligned}\right\} \tag{6}$$

By consideration of the explicit expressions for the quantities it is not difficult to derive the inequalities

$$\left.\begin{array}{l}(n,\lambda) < 2^{2n}\lambda^n \\ \{n,\lambda\} < 2^{2n-1}\lambda^{n-1}\end{array}\right\}, \tag{7}$$

which will be useful in what follows.

3. *Convergence of the Series.*

We have to prove that the series (5) converges, and to find an upper limit for θ_n. Regrouping the series without change of order, we write

$$\begin{aligned}\frac{\theta_n}{n!} = \frac{1}{\beta^{n+1}} - \left[\frac{(n,1)}{(1+\beta)^{n+1}} - \frac{1}{(1-\alpha)^{n+1}}\right] + \left[\frac{(n,2)}{(2+\beta)^{n+1}} - \frac{(n,1)}{(2-\alpha)^{n+1}}\right] - \dots \\ +(-)^\lambda\left[\frac{(n,\lambda)}{(\lambda+\beta)^{n+1}} - \frac{(n,\lambda-1)}{(\lambda-\alpha)^{n+1}}\right] + \dots . \end{aligned} \tag{8}$$

Substitute for (n,λ) its value $(n,\lambda-1)+\{n,\lambda\}$. Then, after some further transformations, we get

$$\begin{aligned}\frac{\theta_n}{n!} &= \frac{1}{\beta^{n+1}} - \frac{1}{(1+\beta)^{n+1}} + \frac{1}{(1-\alpha)^{n+1}} \\ &\quad - \left[\frac{\{n,1\}}{(1+\beta)^{n+1}} - (n,1)\left\{\frac{1}{(2+\beta)^{n+1}} - \frac{1}{(2-\alpha)^{n+1}}\right\}\right] \\ &\quad + \left[\frac{\{n,2\}}{(2+\beta)^{n+1}} - (n,2)\left\{\frac{1}{(3+\beta)^{n+1}} - \frac{1}{(3-\alpha)^{n+1}}\right\}\right] - \dots \\ &\quad + (-)^\lambda\left[\frac{\{n,\lambda\}}{(\lambda+\beta)^{n+1}} - (n,\lambda)\left\{\frac{1}{(\lambda+1+\beta)^{n+1}} - \frac{1}{(\lambda+1-\alpha)^{n+1}}\right\}\right] + \dots .\end{aligned}$$

Let us write

$$u_\lambda \equiv (-)^\lambda \frac{\{n,\lambda\}}{(\lambda+\beta)^{n+1}}, \qquad v_\lambda \equiv (-)^\lambda (n,\lambda)\left\{\frac{1}{(\lambda+1+\beta)^{n+1}} - \frac{1}{(\lambda+1-\alpha)^{n+1}}\right\}.$$

Then $$\frac{\theta_n}{n!} = \frac{1}{\beta^{n+1}} - \frac{1}{(1+\beta)^{n+1}} + \frac{1}{(1-\alpha)^{n+1}} + \sum_{\lambda=1}^{\infty}(u_\lambda + v_\lambda).$$

Consider the u and v series separately. The real part of $\lambda+\beta$ is positive and greater than λ, so that $|\lambda+\beta| > \lambda$. Combining this with (7), we get

$$|u_\lambda| < 2^{2n-1}/\lambda^2.$$

The u series therefore converges absolutely and uniformly, and

$$\left|\sum_{\lambda=1}^{\infty} u_\lambda\right| < 2^{2n-1}\sum_{\lambda=1}^{\infty}\frac{1}{\lambda^2} < \frac{1}{6}\pi^2 . 2^{2n-1}. \tag{9}$$

Further, $$v_\lambda = (-)^\lambda (n,\lambda)\frac{(\lambda+1+\beta)^{n+1} - (\lambda+1-\alpha)^{n+1}}{(\lambda+1+\beta)^{n+1}(\lambda+1-\alpha)^{n+1}}.$$

Since $0 \leqq p \leqq \frac{1}{2}$, it is easily proved that $|\lambda+1-\alpha|^2$ and $|\lambda+1+\beta|^2$ each exceed λ^2+M^2, where

$$M \equiv |\alpha| \equiv |\beta| \equiv \frac{1}{2h}(x^2+y^2)^{\frac{1}{2}}.$$

Again,

$$(\lambda+1+\beta)^{n+1}-(\lambda+1-\alpha)^{n+1} = (\lambda-iq+1+p)^{n+1}-(\lambda-iq+1-p)^{n+1}$$

$$= \binom{n+1}{1}(\lambda-iq)^n[(1+p)-(1-p)]+\binom{n+1}{2}(\lambda-iq)^{n-1}[(1+p)^2-(1-p)^2]+\ldots$$

$$+\binom{n+1}{r}(\lambda-iq)^{n-r+1}[(1+p)^r-(1-p)^r]+\ldots+[(1+p)^{n+1}-(1-p)^{n+1}].$$

Since $|\lambda-iq|<(\lambda^2+M^2)^{\frac{1}{2}}$, while $|1+p|$ and $|1-p|$ are less than 2, we have

$$|(\lambda+1+\beta)^{n+1}-(\lambda+1-\alpha)^{n+1}| < \binom{n+1}{1}(\lambda^2+M^2)^{n/2}\,2^2+\ldots$$

$$+\binom{n+1}{r}(\lambda^2+M^2)^{(n-r+1)/2}\,2^{r+1}+\ldots+2^{n+2}$$

$$< 2^{n+2}(\lambda^2+M^2)^{n/2}\left[\binom{n+1}{1}+\ldots+\binom{n+1}{r}+\ldots+1\right]$$

$$< 2^{2n+3}(\lambda^2+M^2)^{n/2}.$$

Making use of (13) and of the inequality

$$\left|\frac{1}{(\lambda+1+\beta)^{n+1}(\lambda+1-\alpha)^{n+1}}\right| < \frac{1}{(\lambda^2+M^2)^{n+1}},$$

we have
$$|v_\lambda| < \frac{2^{2n}\lambda^n 2^{2n+3}(\lambda^2+M^2)^{n/2}}{(\lambda^2+M^2)^{n+1}} < \frac{2^{4n+3}}{\lambda^2}.$$

Thus
$$\left|\sum_{\lambda=1}^{\infty} v_\lambda\right| < \frac{1}{6}\pi^2 . 2^{4n+3}. \tag{10}$$

The moduli of $1-\alpha$ and $1+\beta$ each exceed $\frac{1}{2}$. Hence, using (9) and (10), we find that

$$\frac{1}{n!}|\theta_n| < \frac{1}{M^{n+1}}+2^{n+2}+\frac{1}{6}\pi^2\,2^{2n+1}+\frac{1}{6}\pi^2\,2^{4n+3}.$$

Cut out the origin by a definite small semicircle, and consider the domain

$$M > \epsilon, \qquad 0 \leqq y \leqq h. \tag{11}$$

Then the inequality for θ_n shows that within the domain we can choose finite numbers a and b, independent of n, x, y, to secure that

$$|\theta_n| < ab^n . n!. \tag{12}$$

The uniformity of the convergence of the x and y derivates of θ_n is proved in the same way, and the derived quantities satisfy inequalities of the form (12).

Returning to the equation (3), we see that the n series for $\phi+i\chi$ converges absolutely and uniformly in the domain if the series

$$\sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} k^n \tag{13}$$

converges, k being a positive constant. This is clearly the case. It is easily seen that we can use term-by-term differentiation with respect to x, y, and t everywhere in the domain, even when $y = 0$, and we are in a position to verify the solution rigidly.

4. *Various Verifications.*

Taking the solution given by (3) in conjunction with either (5) or (8), it is easy to verify that the following conditions are satisfied in the domain, these being the conditions of the problem:—

$$\begin{array}{rl}
\text{(i)} & \partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2 = 0, \\
\text{(ii)} & \partial\phi/\partial y = 0 \ \text{ when } \ y = h, \\
\text{(iii)} & g\,\partial\phi/\partial y = \partial^2\phi/\partial t^2 \ \text{ when } \ y = 0, \\
\text{(iv)} & \phi = 0 \ \text{ when } \ t = 0, \\
\text{(v)} & \partial\phi/\partial t = 0 \ \text{ for } \ y = t = 0.
\end{array}$$

The equation (8) may be written

$$\frac{1}{n!}\frac{\theta_n}{(2h)^{n+1}} = \frac{1}{(y-ix)^{n+1}} - \left[\frac{(n,1)}{(y-ix+2h)^{n+1}} - \frac{1}{(-y-ix+2h)^{n+1}}\right] + \dots$$
$$+(-)^\lambda\left[\frac{(n,\lambda)}{(y-ix+2h\lambda)^{n+1}} - \frac{(n,\lambda-1)}{(-y-ix+2h\lambda)^{n+1}}\right] + \dots.$$

Let us write

$$\left.\begin{array}{rlrl}
y &= r\cos\theta, & x &= r\sin\theta, \\
y+2h\lambda &= \rho_\lambda\cos\phi_\lambda, & x &= \rho_\lambda\sin\phi_\lambda, \\
-y+2h\lambda &= \sigma_\lambda\cos\psi_\lambda, & x &= \sigma_\lambda\sin\psi_\lambda.
\end{array}\right\} \tag{14}$$

Then if ω_n is the real part of $\dfrac{1}{n!}\dfrac{\theta_n}{(2h)^{n+1}}$, we have

$$\omega_n = \frac{\cos(n+1)\theta}{r^{n+1}} - \left[\frac{(n,1)\cos(n+1)\phi_1}{\rho_1{}^{n+1}} - \frac{\cos(n+1)\psi_1}{\sigma_1{}^{n+1}}\right] + \dots$$
$$+(-)^\lambda\left[\frac{(n,\lambda)\cos(n+1)\phi_\lambda}{\rho_\lambda{}^{n+1}} - \frac{(n,\lambda-1)\cos(n+1)\psi_\lambda}{\sigma_\lambda{}^{n+1}}\right] + \dots. \tag{15}$$

The explicit expression of ϕ now follows by observing that

$$\phi = \frac{1}{\pi}\left[gt\,\omega_0 - \frac{1}{3!}g^2t^3\omega_1 + \frac{2!}{5!}g^3t^5\omega_2 - \dots + (-)^n\frac{n!}{(2n+1)!}g^{n+1}t^{2n+1}\omega_n + \dots\right]. \tag{16}$$

When h is large we have

$$\omega_n = r^{-n-1}\cos(n+1)\theta,$$

and the expression for ϕ passes over into the well-known result* for deep water.

It is also easy to derive Lord Rayleigh's expression for the first term in the surface elevation. The coefficient of t^3 in $\phi+i\chi$ is $-g^2\theta_1/24\,h^2\pi$. Thus, if J denotes the real part of $[\theta_1]_{y=0}$, the coefficient of t^2 in the surface elevation is $-g\mathrm{J}/8\,h^2\pi$.

From (8) we have

$$[\theta_1]_{y=0} = -\frac{1}{q^2}-\frac{2}{(1-iq)^2}+\frac{2}{(2-iq)^2}-\ldots+(-)^\lambda\frac{2}{(\lambda-iq)^2}+\ldots.$$

Hence
$$\mathrm{J} = -\frac{1}{q^2}-\left[\frac{1}{(1+iq)^2}+\frac{1}{(1-iq)^2}\right]+\left[\frac{1}{(2+iq)^2}+\frac{1}{(2-iq)^2}\right]-\ldots$$
$$+(-)^\lambda\left[\frac{1}{(\lambda+iq)^2}+\frac{1}{(\lambda-iq)^2}\right]+\ldots.$$

Differentiating the well-known result

$$\frac{\pi}{\sin\pi\theta} = \frac{1}{\theta}-\left[\frac{1}{1+\theta}-\frac{1}{1-\theta}\right]+\left[\frac{1}{2+\theta}-\frac{1}{2-\theta}\right]-\ldots+(-)^\lambda\left[\frac{1}{\lambda+\theta}-\frac{1}{\lambda-\theta}\right]+\ldots$$

with respect to θ, and putting $\theta = iq$, we see that

$$\mathrm{J} = -\frac{\pi^2\cosh(\pi x/2\,h)}{\sinh^2(\pi x/2\,h)}.$$

The first term in η is, therefore,

$$\frac{g\pi t^2}{8\,h^2}\frac{\cosh(\pi x/2\,h)}{\sinh^2(\pi x/2\,h)},$$

in agreement with Lord Rayleigh's result. The next term is

$$-\frac{g^2\pi t^4}{384\,h^4}\,\frac{4\,h\sinh(\pi x/h)-\pi x\{3+\cosh(\pi x/h)\}}{\sinh^3(\pi x/2\,h)}.$$

5. *The Synthesis of Harmonic Solutions.*

Leaving aside for the moment the question of the cause of the instantaneity of the propagation, let us consider the apparent discrepancy between this fact and the velocities of simple harmonic waves. Taking a one-dimensional problem in a dispersive medium, let it be required to determine a quantity ζ at place x and time t. For the sake of simplicity, take ζ as symmetrical about $x=0$ and stationary for $t=0$, and let a particular solution of the differential equation of the problem be $\zeta = \cos\sigma t\cos kx$. Here σ and k are connected by an arbitrary functional relation. Corresponding to the initial value $f(x)$ of ζ, we generalise this solution into

$$\zeta = \int_0^\infty \phi(k)\cos\sigma t\cos kx\,dk,$$

* Lamb, 'Hydrodynamics,' p. 365.

where $$f(x) = \int_0^\infty \phi(k) \cos kx \, dk.$$

By proper adjustment of $\phi(k)$ we can make $f(x)$ vanish when x exceeds a certain value. It is not, however, obvious that ζ will in general vanish for the larger values of x, if σ/k remains finite. Thus, in the absence of such proof we may conclude that the finiteness of the simple harmonic wave-velocity is not a sufficient condition for finite propagation with a wave-front. For non-dispersive media, for which $\sigma/k = c$, a constant, it is easily seen that $\zeta = \frac{1}{2}f(x+ct)+\frac{1}{2}f(x-ct)$, and there is a wave-front advancing with velocity c.

The presence or absence of a wave-front can also be investigated in more general cases. Let us start with a limited disturbance, which itself possesses some degree of discontinuity. Then, if the propagation is finite, there will be at any subsequent time an undisturbed portion of the medium and a new field. Between the two there should, therefore, be some degree of discontinuity as well. Now the solutions of wave problems corresponding to arbitrary initial conditions occur naturally in the form of infinite integrals. The test of a wave-front is, therefore, non-uniformity of the convergence of the integral. This is otherwise obvious if we reflect that discontinuity means preponderating effect of short-waved components at the point in question, *i.e.* importance of the part of the integral corresponding to large values of k.

In this way we can find out to what extent irregularities of the initial state are propagated in an advancing wave, in many cases in which an exact solution is unobtainable. In applying this method to the propagation in a heavy incompressible fluid, we meet with certain difficulties connected with the non-convergence of the integrals at the surface. We can, however, obtain rigorous results by using a non-concentrated symmetrical initial elevation of triangular form, given by

$$\eta_0 = l-x \text{ if } x < l, \quad = 0 \text{ if } x > l.$$

It is found that outside the range $-l < x < l$ the surface elevation is continuous for all values of x and t, as well as its time derivates up to $\partial^3\eta/\partial t^3$. If we had taken $\sigma = kc$ and fitted the solution to the same initial conditions, we should, of course, have found discontinuity of $\partial\eta/\partial t$ at the points $x \pm ct = 0$, $x \pm l \pm ct = 0$.

A very similar case occurs when h is small. If we simultaneously take g large in such a way as to keep $gh \equiv c^2$ finite, we find that the "remainder" of the integral becomes important near the same points. As h cannot be regarded as indefinitely small, it is, however, clear that the propagation of discontinuity can never actually be attained. The continuity of the general case is thus brought into contact with the non-dispersive equations of tidal waves.

The above remarks must not be taken to imply that instantaneity of propagation is a property of such dispersive media as occur in nature. Where it is indicated by the analysis it is probably to be traced to some imperfection in the physical assumptions upon which the analysis is based.

6. *Apparent Instantaneity of the Propagation.*

It is easy to extend the ordinary methods to treat the problem of the propagation of disturbances over incompressible fluid covering an attracting spherical core. The solution has the same features of continuity already encountered, and thus the instantaneity is not connected in any way with neglect of the curvature of the bed on which the fluid lies.

The instantaneity can therefore only be ascribed to the fact that the fluid is taken as incompressible, as surmised by Lord Rayleigh (*loc. cit.*, p. 5). In order to see the precise way in which this assumption enters it will be necessary to investigate the propagation of disturbances in a heavy compressible fluid.

Taking the pressure as a function of the density, we have

$$\int \frac{dp}{\rho} = \frac{\partial \phi}{\partial t} + gy, \qquad \frac{1}{\rho}\frac{\partial p}{\partial t} = \frac{\partial^2 \phi}{\partial t^2}.$$

The equation of continuity is

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial \phi}{\partial y}\right) = \frac{\partial \rho}{\partial t}.$$

For the sake of illustration, assume the law

$$p = \text{constant} + a^2 \rho.$$

Then for small motion we have

$$\frac{a^2}{\rho}\frac{\partial \rho}{\partial t} = \frac{\partial^2 \phi}{\partial t^2}, \tag{17}$$

and
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2\gamma \frac{\partial \phi}{\partial y} = \frac{1}{\rho}\frac{\partial \rho}{\partial t},$$

where 2γ is the value of $\rho^{-1}(\partial \rho/\partial y)$ in the state of rest, *i.e.*

$$\gamma = g/2a^2. \tag{18}$$

Getting rid of ρ, we have for the differential equation for ϕ

$$\frac{\partial^2 \phi}{\partial t^2} = a^2\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2\gamma \frac{\partial \phi}{\partial y}\right). \tag{19}$$

The boundary conditions and the expression for the surface elevation are the same as usual. Consider the case of finite depth h, and try a solution $\phi = f \sin \sigma t \cos kx$, where f depends on y only. Then from (19)

$$f = Ae^{\theta_1 y} + Be^{\theta_2 y},$$

θ_1 and θ_2 being the roots of the equation

$$\theta^2 + 2\gamma\theta - k^2 + \sigma^2/a^2 = 0.$$

The solution takes different forms, according as $\gamma^2 + k^2 - \sigma^2/a^2$ is positive or negative. If it is positive, put it equal to μ^2. Then $\theta_1 = -\gamma + \mu$ and $\theta_2 = -\gamma - \mu$. The boundary conditions give

$$A(\mu - \gamma)e^{\mu h} = B(\mu + \gamma)e^{-\mu h} = C\ (\text{say}),$$

and
$$g\{A(\mu - \gamma) - B(\mu + \gamma)\} + \sigma^2(A + B) = 0.$$

Eliminating A and B, we find that the relationship of μ, σ, and k is given by the equations

$$\sigma^2/a^2 = k^2 - \mu^2 + \gamma^2, \tag{20}$$

and
$$\mu \coth \mu h = \gamma \frac{k^2 + \mu^2 - \gamma^2}{k^2 - \mu^2 + \gamma^2}. \tag{21}$$

By proper adjustment of C we find, further, that

$$\phi = \frac{g}{e^{\gamma y}} \frac{\mu \cosh \mu(h - y) - \gamma \sinh \mu(h - y)}{\mu \cosh \mu h - \gamma \sinh \mu h} \frac{\sin \sigma t}{\sigma} \cos kx, \tag{22}$$

corresponding to
$$\eta = \cos \sigma t \cos kx.$$

The position of the admissible roots μ for a given k is seen most easily by drawing the curves

$$y = x \coth xh, \qquad y = \gamma(k^2 + x^2 - \gamma^2)/(k^2 - x^2 + \gamma^2).$$

The quantity μ may be taken as positive without loss of generality. It is seen that if $h\gamma(k^2 - \gamma^2) < k^2 + \gamma^2$, there is certainly one root μ corresponding to k, while if $h\gamma(k^2 - \gamma^2) > k^2 + \gamma^2$ there are at least two roots.

When $\gamma^2 + k^2 - \sigma^2/a^2$ is negative, put it equal to $-\nu^2$. We get a second set of solutions, which are formally obtained by writing $\mu = i\nu$. For a given k, ν is given by the equation

$$\nu \cot \nu h = \gamma(k^2 - \nu^2 - \gamma^2)/(k^2 + \nu^2 + \gamma). \tag{23}$$

It is easily seen that the roots are infinite in number.

In order to see what kinds of waves are formed in this way, let us take k large. It is found that one root μ approximates to $k - \gamma$, so that σ^2 approximates to gk. With this mode, the propagation of surface-discontinuity follows the same course as with incompressible fluids. The remaining root μ and all the ν roots approximate to quantities independent of k, and σ tends to ak. The discontinuities are propagated as by waves of expansion.

More definite information is obtained by taking k finite and h large, that is by investigating the simple waves on deep compressible fluid. It is easily proved that one root μ approximates to $k-\gamma$ for $h=\infty$, provided $k>\gamma$. We have $\sigma^2=gk$ as before, and corresponding to $\eta=\cos\sigma t\cos kx$ we have

$$\phi=ge^{-ky}\sigma^{-1}\sin\sigma t\cos kx.$$

That is, if $k>\gamma$ there is a "standing" solution of exactly the same kind as for an incompressible fluid. This solution is easily verified directly. We know that it satisfies $\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2=0$, and it is easily seen that $\partial^2\phi/\partial t^2=g\partial\phi/\partial y$. The equation (19) therefore holds.

Further, it may be shown easily that the normal Cauchy-Poisson solution

$$\phi=\frac{g}{\pi}\left(t\frac{\cos\theta}{r}-\frac{1}{3\,!}gt^3\frac{\cos 3\theta}{r^3}+\ldots+(-)^n\frac{n\,!}{(2n+1)\,!}g^nt^{2n+1}\frac{\cos(n+1)\theta}{r^{n+1}}+\ldots\right), \tag{24}$$

in which $y=r\cos\theta$, $x=r\sin\theta$, satisfies the equation (19). It also satisfies the initial conditions of zero initial velocity and zero initial elevation everywhere except at the origin of r. But it is not the solution of the problem of a concentrated initial disturbance in a compressible fluid, because it implies an initial condensation determined by the formula

$$\left(\frac{\partial\phi}{\partial t}\right)_{t=0}=\frac{g}{\pi}\frac{\cos\theta}{r}.$$

The corresponding initial pressure is given by the equation

$$a^2\log\left[p+\text{const.}\right]_{t=0}=\text{const.}+gy+(\pi r)^{-1}g\cos\theta.$$

In like manner, the formula (24) implies, in the case of an incompressible fluid, an unequilibrated initial distribution of pressure according to the formula

$$\rho^{-1}\left[p\right]_{t=0}=\text{const.}+gy+(\pi r)^{-1}g\cos\theta,$$

so that the fluid is represented as starting to move at once with finite acceleration. The solution corresponding to (24) in the case of initial impulse,* viz. :—

$$\phi=\frac{1}{\pi\rho}\left[\frac{\cos\theta}{r}-\frac{1}{2\,!}gt^2\frac{\cos 2\theta}{r^2}+\ldots\right],$$

similarly implies an initial impulsive pressure everywhere, as might be expected in an incompressible fluid.

The solution (24) is obtained by generalising one that corresponds to a single root of equation (21). It has been seen to be incomplete in the case of any actual (compressible) fluid. In any such case the complete solution must be sought by taking account of the modes of motion which correspond to the remaining root or roots of equation (21) and the roots of equation (23); and these modes are capable of transmitting pressural effects with the velocity of sound in the medium.

* Lamb, 'Hydrodynamics,' p. 369.