of x-rays. The production of diffuse reflexion by solid bodies is very easily observed; but hitherto I have no clear evidence of specular reflexion. Air unfortunately itself sets up diffusion, behaving as a semi-opaque fluid. If ordinary experiments on the reflexion of light had to be carried on in dense smoke or in milky water, a similar diffusion would interfere with specular reflexion. In one set of experiments a V-tube made of lead pipes set at right angles, and open at the bottom, was used, the x-ray source being made to shine down one limb, while a shielded photographic plate was placed at the upper end of the other. The surfaces to reflect x-rays were placed at the open lower ends at 45° to the lines of incidence and of presumed reflexion. Reflexion of a sort was indeed obtained when surfaces of metal and of glass were placed across the bottom of the tubes. But an effect was also obtained even when nothing was placed across the open bottom. It seems exceedingly doubtful whether true specular reflexion has been observed in any case.

XV. *On the Theory of Optical Images, with Special Reference to the Microscope.* By LORD RAYLEIGH, *Sec. R. S.**

THE special subject of this paper has been treated from two distinct points of view. In the work of Helmholtz † the method followed is analogous to that which had long been used in the theory of the telescope. It consists in tracing the image representative of a mathematical point in the object, the point being regarded as self-luminous. The limit to definition depends upon the fact that owing to diffraction the image thrown even by a perfect lens is not confined to a point, but distends itself over a patch or disk of light of finite diameter. Two points in the object can appear fully separated only when the representative disks are nearly clear of one another. The application to the microscope was traced by means of a somewhat extended form of Lagrange's general optical theorem, and the conclusion was reached that the smallest resolvable distance ϵ is given by

$$\epsilon=\tfrac{1}{2}\lambda/\sin\alpha, \qquad (1)$$

λ being the wave-length in the medium where the object is situated, and α the divergence-angle of the extreme ray (the semi-angular aperture) in the same medium. If λ_0 be the wave-length in vacuum,

$$\lambda=\lambda_0/\mu, \qquad (2)$$

* Communicated by the Author.

† Pogg. *Ann.* Jubelband, 1874.

μ being the refractive index of the medium ; and thus

$$\epsilon = \tfrac{1}{2}\lambda_0 / \mu \sin\alpha. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The denominator $\mu \sin\alpha$ is the quantity now well known (after Abbe) as the "numerical aperture."

The extreme value possible for α is a right angle, so that for the microscopic limit we have

$$\epsilon = \tfrac{1}{2}\lambda_0 / \mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The limit can be depressed only by a diminution in λ_0, such as photography makes possible, or by an increase in μ, the refractive index of the medium in which the object is situated.

This method, in which the object is considered point by point, seems the most straight-forward, and to a great extent it solves the problem without more ado. When the representative disks are thoroughly clear of one another, the two points in which they originate are resolved, and on the other hand, when the disks overlap the points are not distinctly separated. Open questions can relate only to intermediate cases of partial overlapping and various degrees of resolution. In these cases (as has been insisted upon by Dr. Stoney) we have to consider the relative phases of the overlapping lights before we can arrive at a complete conclusion.

If the various points of the object are self-luminous, there is no permanent phase-relation between the lights of the overlapping disks, and the resultant illumination is arrived at by simple addition of separate intensities. This is the situation of affairs in the ordinary use of a telescope, whether the object be a double star, the disk of the sun, the disk of the moon, or a terrestrial body. The distribution of light in the image of a double point, or of a double line, was especially considered in a former paper *, and we shall return to the subject later.

When, as sometimes happens in the use of the telescope, and more frequently in the use of the microscope, the overlapping lights have permanent phase-relations, these intermediate cases require a further treatment ; and this is a matter of some importance as involving the behaviour of the instrument in respect to the finest detail which it is capable of rendering. We shall see that the image of a double point under various conditions can be delineated without difficulty.

In the earliest paper by Prof. Abbe †, which somewhat

* "Investigations in Optics, with special reference to the Spectroscope." Phil. Mag. vol. viii. p. 266 (1879).

† *Archiv. f. Mikr. Anat.* vol. ix. p. 413 (1873).

preceded that of Helmholtz, similar conclusions were reached; but the demonstrations were deferred, and, indeed, they do not appear ever to have been set forth in a systematic manner. Although some of the positions then taken up, as for example that the larger features and the finer structure of a microscopic object are delineated by different processes, have since had to be abandoned *, the publication of this paper marks a great advance, and has contributed powerfully to the modern development of the microscope †. In Prof. Abbe's method of treating the matter the typical object is not a luminous *point*, but a *grating* illuminated by plane waves. Thence arise the well-known diffraction spectra, which are focussed near the back of the object-glass in its principal focal plane. If the light be homogeneous, the spectra are reduced to points, and the final image may be regarded as due to the simultaneous action of these points acting as secondary centres of light. It is argued that the complete representation of the object requires the co-operation of all the spectra. When only a few are present, the representation is imperfect; and when there is only one—for this purpose the central image counts as a spectrum—the representation wholly fails.

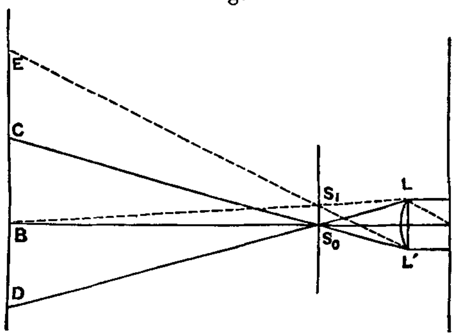
That this point of view offers great advantages, at least when the object under consideration is really a grating, is at once evident. More especially is this the case in respect of the question of the limit of resolution. It is certain that if one spectrum only be operative, the image must consist of a uniform field of light, and that no sign can appear of the real periodic structure of the object. From this consideration the resolving-power is readily deduced, and it may be convenient to recapitulate the argument for the case of perpendicular incidence. In fig. 1 AB represents the axis, A being in the plane of the object (grating) and B in the plane of the image. The various diffraction spectra are focussed by the lens LL′ in the principal focal plane, S_0 representing the central image due to rays which issue normally from the grating. After passing S_0 the rays diverge in a

* Dallenger's edition of Carpenter's 'Microscope,' p. 64, 1891.

† It would seem that the present subject, like many others, has suffered from over specialization, much that is familiar to the microscopist being almost unknown to physicists, and *vice versâ*. For myself I must confess that it is only recently, in consequence of a discussion between Mr. L. Wright and Dr. G. J. Stoney in the 'English Mechanic' (Sept., Oct., Nov., 1894; Nov. 8, Dec. 13, 1895; Jan. 17, 1896), that I have become acquainted with the distinguishing features of Prof. Abbe's work, and have learned that it was conducted upon different lines to that of Helmholtz. I am also indebted to Dr. Stoney for a demonstration of some of Abbe's experiments.

cone corresponding to the aperture of the lens and illuminate a circle CD in the plane of the image, whose centre is B. The first lateral spectrum S_1 is formed by rays diffracted from

Fig. 1.



the grating at a certain angle ; and in the critical case the region of the image illuminated by the rays diverging from S_1 just includes B. The extreme ray S_1B evidently proceeds from A, which is the image of B. The condition for the co-operation at B of the first lateral spectrum is thus that the angle of diffraction do not exceed the semi-angular aperture α. By elementary theory we know that the sine of the angle of diffraction is λ/ϵ, so that the action of the lateral spectrum requires that ϵ exceed $\lambda/\sin\alpha$. If we allow the incidence upon the grating to be oblique, the limit becomes $\frac{1}{2}\lambda/\sin\alpha$, as in (1).

We have seen that if one spectrum only illuminate B, the field shows no structure. If two spectra illuminate it with equal intensities, the field is occupied by ordinary interference bands, exactly as in the well known experiments of Fresnel. And it is important to remark that the character of these bands is always the same, both as respects the graduation of light and shade, and in the fact that they have no focus. When more than two spectra co-operate, the resulting interference phenomena are more complicated, and there is opportunity for a completer representation of the special features of the original grating *.

* These effects were strikingly illustrated in some observations upon gratings with 6000 lines to the inch, set up vertically in a dark room and illuminated by sunlight from a distant vertical slit. The object-glass of the microscrope was a quarter inch. When the original grating, divided upon glass (by Nobert), was examined in this way, the lines were well seen if the instrument was in focus, but, as usual, a comparatively slight disturbance of focus caused all structure to disappear. When, however, a photographic copy of the same glass original, made

While it is certain that the image ultimately formed may be considered to be due to the spectra focussed at S_0, S_1..., the degree of conformity of the image to the original object is another question. From some of the expositions that have been given it might be inferred that if all the spectra emitted from the grating were utilized, the image would be a complete representation of the original. By considering the case of a very fine grating, which might afford no lateral spectra at all, it is easy to see that this conclusion is incorrect, but the matter stands in need of further elucidation. Again, it is not quite clear at what point the utilization of a spectrum really begins. All the spectra which the grating is competent to furnish are focussed in the plane $S_0\,S_1$; and some of them might be supposed to operate partially even although the part of the image under examination is outside the geometrical cone defined by the aperture of the object-glass. For these and other reasons it will be seen that the spectrum theory *, valuable as it is, needs a good deal of supplementing, even when the representation of a grating under parallel light is in question.

When the object under examination is not a grating or a structure in which the pattern is repeated an indefinite number of times, but for example a double point, or when the incident light is not parallel, the spectrum theory, as hitherto developed, is inapplicable. As an extreme example of the latter case we may imagine the grating to be self-luminous. It is obvious that the problem thus presented must be within the scope of any complete theory, and equally so that here there are no spectra formed, as these require the radiations from the different

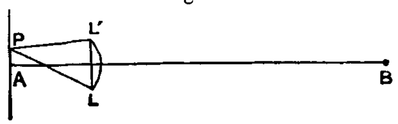
with bitumen, was substituted for it, very different effects ensued. The structure could be seen even although the object-glass were drawn back through $1\frac{1}{2}$ inch from its focussed position; and the visible lines were twice as close, as if at the rate of 12,000 to the inch. The difference between the two cases is easily explained upon Abbe's theory. A soda flame viewed through the original showed a strong central image (spectrum of zero order) and comparatively faint spectra of the first and higher orders. A similar examination of the copy revealed very brilliant spectra of the first order on both sides, and a relatively feeble central image. The case is thus approximately the same as when in Abbe's experiment all spectra except the first (on the two sides) are blocked out.

* The special theory initiated by Prof. Abbe is usually called the "diffraction theory," a nomenclature against which it is necessary to protest. Whatever may be the view taken, any theory of resolving power of optical instruments must be a diffraction theory in a certain sense, so that the name is not distinctive. Diffraction is more naturally regarded as the obstacle to fine definition, and not, as with some exponents of Prof. Abbe's theory, the machinery by which good definition is brought about.

elements of the grating to possess permanent phase-relations. It appears, therefore, to be a desideratum that the matter should be reconsidered from the older point of view, according to which the typical object is a point and not a grating. Such a treatment illustrates the important principle that the theory of resolving-power is essentially the same for all instruments. The peculiarities of the microscope arise from the fact that the divergence-angles are not limited to be small, and from the different character of the illumination usually employed; but, theoretically considered, these are differences of detail. The investigation can, without much difficulty, be extended to gratings, and the results so obtained confirm for the most part the conclusions of the spectrum theory.

It will be convenient to commence our discussion by a simple investigation of the resolving-power of an optical instrument for a self-luminous double point, such as will be applicable equally to the telescope and to the microscope. In fig. 2 AB represents the axis, A being a point of the object and B a point of the image. By the operation of the object-glass LL′ all the rays issuing from A arrive in the same phase at B. Thus if A be self-luminous, the illumination is a maximum at B, where all the secondary waves agree in phase.

Fig. 2.



B is in fact the centre of the diffraction disk which constitutes the image of A. At neighbouring points the illumination is less, in consequence of the discrepancies of phase which there enter. In like manner, if we take a neighbouring point P in the plane of the object, the waves which issue from it will arrive at B with phases no longer absolutely accordant, and the discrepancy of phase will increase as the interval AP increases. When the interval is very small, the discrepancy of phase, though mathematically existent, produces no practical effect, and the illumination at B due to P is as important as that due to A, the intensities of the two luminous centres being supposed equal. Under these conditions it is clear that A and P are not separated in the image. The question is, to what amount must the distance AP be increased in order that the difference of situation may make itself felt in the image. This is necessarily a question of degree; but it does not require detailed calculations in order to show that the discrepancy first becomes conspicuous when the phases corresponding to the various secondary waves which travel from P to B range over about a complete period. The illumination at B due to P then becomes comparatively small, indeed for some forms of aperture evanescent. The extreme discrepancy is that between the waves which travel through the outermost parts of the object-glass at L and L′; so that, if we adopt the above standard of resolution, the question is, where must P be situated in order that the relative retardation of the rays PL and PL′ may on their arrival at B amount to a wave-length (λ). In virtue of the general law that the reduced optical path is stationary in value, this retardation may be calculated without allowance for the different paths pursued on the further side of L, L′, so that its value is simply PL−PL′. Now since AP is very small, AL′−PL′ is equal to AP . sin α, where α is the semi-angular aperture L′AB. In like manner PL−AL has the same value, so that

$$\mathrm{PL}-\mathrm{PL}'=2\mathrm{AP}\,.\,\sin\alpha.$$

According to the standard adopted, the condition of resolution is therefore that AP, or ϵ, should exceed $\frac{1}{2}\lambda/\sin\alpha$, as in (1). If ϵ be less than this, the images overlap too much; while if ϵ greatly exceed the above value the images become unnecessarily separated.

In the above argument the whole space between the object and the lens is supposed to be occupied by matter of one refractive index, and λ represents the wave-length *in this medium* of the kind of light employed. If the restriction as to uniformity be violated, what we have ultimately to do with is the wave-length in the medium immediately surrounding the object.

The statement of the law of resolving-power has been made in a form appropriate to the microscope, but it admits also of immediate application to the telescope. If 2R be the diameter of the object-glass, and D the distance of the object, the angle subtended by AP is ϵ/D, and the angular resolving-power is given by

$$\frac{\lambda}{2\mathrm{D}\sin\alpha}=\frac{\lambda}{2\mathrm{R}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

the well-known formula.

This method of derivation makes it obvious that there is no essential difference of principle between the two cases, although the results are conveniently stated in different forms. In the case of the telescope we have to do with a linear measure of aperture and an angular limit of resolution, whereas in the case of the microscope the limit of resolution is linear and it is expressed in terms of angular aperture.

In the above discussion it has been supposed for the sake of simplicity that the points to be discriminated are self-luminous, or at least behave as if they were such. It is of interest to inquire how far this condition can be satisfied when the object is seen by borrowed light. We may imagine that the object takes the form of an opaque screen, perforated at two points, and illuminated by distant sources situated behind.

If the source of light be reduced to a point, so that a single train of plane waves falls upon the screen, there is a permanent phase-relation between the waves incident at the two points, and therefore also between the waves scattered from them. In this case the two points are as far as possible from behaving as if they were self luminous. If the incidence be perpendicular, the secondary waves issue in the same phase; but in the case of obliquity there is a permanent phase-difference. This difference, measured in wave-lengths, increases up to ϵ, the distance between the points, the limit being attained as the incidence becomes grazing.

When the light originates in distant independent sources, not limited to a point, there is no longer an absolutely definite phase-relationship between the secondary radiations from the two apertures; but this condition of things may be practically maintained, if the angular magnitude of the source be not too large. For example, if the source be limited to an angle θ round the normal to the screen, the maximum phase-difference measured in wave-lengths is $\epsilon \sin \theta$, so that if $\sin \theta$ be a small fraction of λ/ϵ, the finiteness of θ has but little effect. When, however, $\sin \theta$ is so great that $\epsilon \sin \theta$ becomes a considerable multiple of λ, the secondary radiations become approximately independent, and the apertures behave like self-luminous points. It is evident that even with a complete hemispherical illumination this condition can scarcely be attained when ϵ is less than λ.

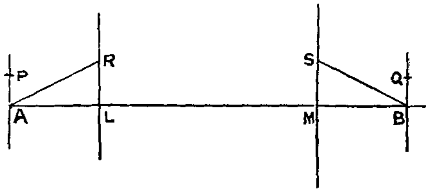
The use of a condenser allows the widely-extended source to be dispensed with. By this means an image of a distant source composed of independently radiating parts, such as a lamp-flame, may be thrown upon the object, and it might at first sight be supposed that the problem under consideration was thus completely solved in all cases, inasmuch as the two apertures correspond to different parts of the flame. But we have to remember here and everywhere that optical images are not perfect, and that to a point of the flame corresponds in the image, not a point, but a disk of finite magnitude. When this consideration is taken into account, the same limitation as before is encountered.

For what is the smallest disk into which the condenser is capable of concentrating the light received from a distant point? Fig. 2 and the former argument apply almost without modification, and they show that the radius A P of the disk has the value $\frac{1}{2}\lambda/\sin\alpha$, where α is the semi-angular aperture of the condenser. Accordingly the diameter of the disk cannot be reduced below λ; and if ϵ be less than λ the radiations from the two apertures are only partially independent of one another.

It seems fair to conclude that the function of the condenser in microscopic practice is to cause the object to behave, at any rate in some degree, as if it were self-luminous, and thus to obviate the sharply-marked interference-bands which arise when permanent and definite phase-relations are permitted to exist between the radiations which issue from various points of the object.

As we shall have occasion later to employ Lagrange's theorem, it may be well to point out how an instantaneous proof of it may be given upon the principles already applied. As before, A B (fig. 3) represents the axis of the instrument,

Fig. 3.



A and B being conjugate points. P is a point near A in the plane through A perpendicular to the axis, and Q is its image in the perpendicular plane through B. Since A and B are conjugate, the optical distance between them is the same for all paths, *e. g.* for A R S B and A L M B. And, since A P, B Q are perpendicular to the axis, the optical distance from P to Q is the same (to the first order of small quantities) as from A to B. Consequently the optical distance P R S Q is the same as A R S B. Thus, if μ, μ' be the refractive indices in the neighbourhood of A and B respectively, α and β the divergence-angles R A L, S B M for a given ray, we have

$$\mu \,.\, AP \,.\, \sin\alpha = \mu' \,.\, BQ \,.\, \sin\beta, \quad . \quad . \quad . \quad (6)$$

where AP, BQ denote the corresponding linear magnitudes of the two images. This is the theorem of Lagrange, extended by Helmholtz so as to apply to finite divergence-angles*.

We now pass on to the actual calculation of the images to be expected upon Fresnel's principles in the various cases that may arise. The origin of coordinates ($\xi=0$, $\eta=0$) in the focal plane is the geometrical image of the radiant point. If the vibration incident upon the lens be represented by $\cos(2\pi Vt/\lambda)$, where V is the velocity of light, the vibration at any point ξ, η in the focal plane is†

$$-\frac{1}{\lambda f}\iint \sin\frac{2\pi}{\lambda}\left\{Vt-f+\frac{x\xi+y\eta}{f}\right\}dx\,dy, \quad . \quad . \quad (7)$$

in which f denotes the focal length, and the integration with respect to x and y is to be extended over the aperture of the lens. If for brevity we write

$$2\pi\xi/\lambda f=p, \quad 2\pi\eta/\lambda f=q, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

(7) may be put into the form

$$-\frac{C}{\lambda f}\sin\frac{2\pi}{\lambda}(Vt-f)-\frac{S}{\lambda f}\cos\frac{2\pi}{\lambda}(Vt-f), \quad . \quad . \quad (9)$$

where

$$S=\iint \sin(px+qy)\,dx\,dy, \quad . \quad . \quad . \quad . \quad (10)$$

$$C=\iint \cos(px+qy)\,dx\,dy. \quad . \quad . \quad . \quad . \quad (11)$$

It will suffice for our present purpose to limit ourselves to the case where the aperture is symmetrical with respect to x and y. We have then $S=0$, and

$$C=\iint \cos px \cos qy\,dx\,dy, \quad . \quad . \quad . \quad . \quad (12)$$

the phase of the vibration being the same at all points of the diffraction pattern.

When the aperture is rectangular, of width a parallel to x, and of width b parallel to y, the limits of integration are from $-\frac{1}{2}a$ to $+\frac{1}{2}a$ for x, and from $-\frac{1}{2}b$ to $+\frac{1}{2}b$ for y. Thus

$$C=ab\,\frac{\sin(\pi\xi a/\lambda f)}{\pi\xi a/\lambda f}\,\frac{\sin(\pi\eta b/\lambda f)}{\pi\eta b/\lambda f}, \quad . \quad . \quad . \quad (13)$$

and by (9) the amplitude of vibration (irrespective of sign) is $C/\lambda f$. This expression gives the diffraction pattern due to a single point of the object whose geometrical image is at $\xi=0$,

* I learn from Czapski's excellent *Theorie der Optischen Instrumente* that a similar derivation of Lagrange's theorem from the principle of minimum path had already been given many years ago by Hockin (Micros. Soc. Journ. vol. iv. p. 337, 1884).

† See for example *Enc. Brit.*, "Wave Theory," p. 430 (1878).

$\eta=0$. Sometimes, as in the application to a grating, we wish to consider the image due to a uniformly luminous *line*, parallel to η, and this can always be derived by integration from the expression applicable to a point. But there is a distinction to be observed according as the radiations from the various parts of the line are independent or are subject to a fixed phase-relation. In the former case we have to deal only with the *intensity*, represented by I^2 or $C^2/\lambda^2 f^2$; and we get

$$\int_{-\infty}^{+\infty} I^2 d\eta = \frac{a^2 b}{\lambda f}\frac{\sin^2(\pi\xi a/\lambda f)}{(\pi\xi a/\lambda f)^2} \quad . \quad . \quad . \quad (14)$$

by means of the known integral

$$\int_{-\infty}^{+\infty}\frac{\sin^2 x}{x^2}dx = \int_{-\infty}^{+\infty}\frac{\sin x}{x}dx = \pi. \quad . \quad . \quad . \quad (15)$$

This gives, as a function of ξ, the intensity due to a self-luminous line whose geometrical image coincides with $\xi=0$.

Under the second head of a fixed phase-relation we need only consider the case where the radiations from the various parts of the line start in the *same* phase. We get, almost as before,

$$\frac{1}{\lambda f}\int_{-\infty}^{+\infty} C\, d\eta = a\,\frac{\sin(\pi\xi a/\lambda f)}{\pi\xi a/\lambda f} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

for the expression of the resultant amplitude corresponding to ξ.

In order to make use of these results we require a table of the values of $\sin u/u$, and of $\sin^2 u/u^2$. The following will suffice for our purposes :—

TABLE I.

$\frac{4u}{\pi}$.	$\frac{\sin u}{u}$.	$\frac{\sin^2 u}{u^2}$.	$\frac{4u}{\pi}$.	$\frac{\sin u}{u}$.	$\frac{\sin^2 u}{u^2}$.
0	+1·0000	1·0000	9	+·1000	·0100
1	·9003	·8105	10	·1273	·0162
2	·6366	·4053	11	·0818	·0067
3	·3001	·0901	12	·0000	·0000
4	·0000	·0000	13	−·0692	·0048
5	−·1801	·0324	14	−·0909	·0083
6	−·2122	·0450	15	−·0600	·0036
7	−·1286	·0165	16	·0000	·0000
8	·0000	·0000			

When we have to deal with a single point or a single line only, this table gives directly the distribution of light in the image, u being equated to $\pi\xi a/\lambda f$. The illumination first vanishes when $u=\pi$, or $\xi/f=\lambda/a$.

On a former occasion* it has been shown that a self-luminous point or line at $u=-\pi$ is barely separated from one at $u=0$. It will be of interest to consider this case under three different conditions as to phase-relationship: (i.) when the phases are the same, as will happen when the illumination is by plane waves incident perpendicularly; (ii.) when the phases are opposite; and (iii.) when the phase-difference is a quarter period, which gives the same result for the intensity as if the apertures were self-luminous. The annexed table gives the numerical values required. In cases (i.) and (iii.) the resultant amplitude is symmetrical with respect to the point $u=-\frac{1}{2}\pi$ midway between the two geometrical images; in case (ii.) the sign is reversed, but this of course has no effect upon the intensity. Graphs of the three functions are given in fig. 4, the geometrical images being at the points marked $-\pi$ and 0. It will be seen that while in case iii., relating to self-luminous points or lines, there is an approach to separation, nothing but an accurate comparison with the curve due to a single source would reveal the duplicity in case i. On the other hand, in case ii., where

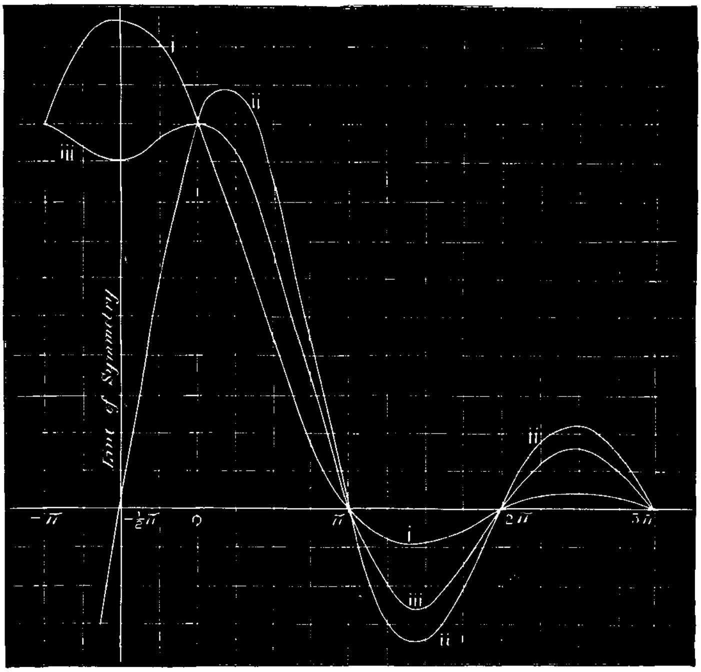
TABLE II.

$\frac{4u}{\pi}$.	$\frac{\sin u}{u}+\frac{\sin(u+\pi)}{u+\pi}$.	$\frac{\sin u}{u}-\frac{\sin(u+\pi)}{u+\pi}$.	$\sqrt{\left\{\frac{\sin^2 u}{u^2}+\frac{\sin^2(u+\pi)}{(u+\pi)^2}\right\}}$.
−4...	+1·0000	−1·0000	+1·000
−3...	+1·2004	− ·6002	+ ·949
−2...	+1·2732	·0000	+ ·900
−1...	+1·2004	+ ·6002	+ ·949
0...	+1·0000	+1·0000	+1·000
1...	+ ·7202	+1·0804	+ ·918
2...	+ ·4244	+ ·8488	+ ·671
3...	+ ·1715	+ ·4287	+ ·326
4...	·0000	·0000	·000
5...	− ·0800	− ·2801	− ·206
6...	− ·0849	− ·3395	− ·247
7...	− ·0468	− ·2105	− ·152
8...	·0000	·0000	·000
9...	+ ·0308	+ ·1693	+ ·122
10...	+ ·0364	+ ·2183	+ ·156
11...	+ ·0218	+ ·1419	+ ·101
12...	·0000	·0000	·000

* Phil. Mag. vol. viii. p. 266, 1879.

there is a phase-difference of half a period between the radiations, the separation may be regarded as complete.

Fig. 4.



In a certain sense the last conclusion remains undisturbed even when the double point is still closer, and also when the aperture is of any other symmetrical form, *e. g.* circular. For at the point of symmetry in the image, midway between the two geometrical images of the radiant points, the component amplitudes are necessarily equal in numerical value and opposite in sign, so that the resultant amplitude or illumination vanishes. For example, suppose that the aperture is rectangular and that the points or lines are twice as close as before, the geometrical images being situated at $u=-\frac{1}{2}\pi$, $u=0$. The resultant amplitude is represented by $f(u)$, where

$$f(u)=\frac{\sin u}{u}-\frac{\sin(u+\frac{1}{2}\pi)}{u+\frac{1}{2}\pi}. \quad . \quad . \quad . \quad (17)$$

The values of $f(u)$ are given in Table III. They show that the resultant vanishes at the place of symmetry $u=-\frac{1}{4}\pi$, and rises to a maximum at a point near $u=\frac{1}{2}\pi$, considerably beyond the geometrical image at $u=0$. Moreover, the value of the maximum itself is much less than before, a feature which would become more and more pronounced as the points were taken closer. At this stage the image becomes only a

TABLE III.

$\frac{4u}{\pi}$.	$f(u)$.	$\frac{4u}{\pi}$.	$f(u)$.
−1	+·00	5	−·05
0	+·36	6	−·21
1	+·60	7	−·23
2	+·64	8	−·13
3	+·48	9	+·02
4	+·21		

very incomplete representation of the object; but if the formation of a black line in the centre of the pattern be supposed to constitute resolution, then resolution occurs at all degrees of closeness*. We shall see later, from calculations conducted by the same method, that a grating of an equal degree of closeness would show no structure at all but would present a uniformly illuminated field.

* These results are easily illustrated experimentally. I have used two parallel slits, formed in films of tin-foil or of chemically deposited silver, of which one is conveniently made longer than the other. These slits are held vertically and are viewed through a small telescope, provided with a high-power eye-piece, whose horizontal aperture is restricted to a small width. The distance may first be so chosen that when backed by a neighbouring flame the double part of the slit just manifests its character by a faint shadow along the centre. If the flame is replaced by sunlight shining through a distant vertical slit, the effect depends upon the precise adjustment. When everything is in line the image is at its brightest, but there is now no sign of resolution of the double part of the slit. A very slight sideways displacement, in my case effected most conveniently by moving the telescope, brings in the half-period retardation, showing itself by a black bar down the centre. An increased displacement, leading to a relative retardation of three halves of a period, gives much the same result, complicated, however, by chromatic effects.

In conformity with theory the black bar down the image of the double slit may still be observed when the distance is increased much beyond that at which duplicity disappears under flame illumination.

For these experiments I chose the telescope, not only on account of the greater facility of manipulation which it allows, but also in order to make it clear that the theory is general, and that such effects are not limited, as is sometimes supposed, to the case of the microscope.

But before proceeding to such calculations we may deduce by Lagrange's theorem the interval ϵ in the original object corresponding to that between $u=0$ and $u=\pi$ in the image, and thence effect a comparison with a grating by means of Abbe's theory. The linear dimension (ξ) of the image corresponding to $u=\pi$ is given by $\xi=\lambda f/a$; and from Lagrange's theorem

$$\epsilon/\xi = \sin\beta/\sin\alpha, \quad . \quad . \quad . \quad . \quad (17\,a)$$

in which α is the "semi-angular aperture," and $\beta=a/2f$. Thus, corresponding to $u=\pi$,

$$\epsilon = \tfrac{1}{2}\lambda/\sin\alpha.$$

The case of a double point or line represented in fig. 4 lies therefore at the extreme limit of resolution for a grating in which the period is the interval between the double points. And if the incidence of the light upon the grating were limited to be perpendicular, the period would have to be doubled before the grating could show any structure.

When the aperture is circular, of radius R, the diffraction pattern is symmetrical about the geometrical image ($p=0$, $q=0$), and it suffices to consider points situated upon the axis of ξ for which η (and q) vanish. Thus from (12)

$$C = \iint \cos px\, dx\, dy = 2\int_{-R}^{+R} \cos px\, \sqrt{(R^2-x^2)}\, dx \quad . \quad (18)$$

This integral is the Bessel function of order unity, definable by

$$J_1(z) = \frac{z}{\pi}\int_0^{\pi} \cos(z\cos\phi)\sin^2\phi\, d\phi. \quad . \quad . \quad (19)$$

Thus, if $x=R\cos\phi$,

$$C = \pi R^2 \frac{2J_1(pR)}{pR}, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

or, if we write $u=\pi\xi\,.\,2R/\lambda f$,

$$C = \pi R^2 \frac{2J_1(u)}{u}. \quad . \quad . \quad . \quad . \quad . \quad (21)^*$$

This notation agrees with that employed for the rectangular aperture if we consider that 2R corresponds with a.

The illumination at various parts of the image of a double point may be investigated as before, especially if we limit ourselves to points which lie upon the line joining the two

* *Enc. Brit.*, "Wave Theory," p. 432.

geometrical images. The only difference in the calculations is that represented by the substitution of $2J_1$ for sine. We shall not, however, occupy space by tables and drawings such as have been given for a rectangular aperture. It may suffice to consider the three principal points in the image due to a double source whose geometrical images are situated at $u=0$ and $u=-\pi$, these being the points just mentioned and that midway between them at $u=-\frac{1}{2}\pi$. The values of the functions required are

$$2J_1(0)/0 = 1{\cdot}0000 = \sqrt{\{1{\cdot}0000\}}.$$
$$2J_1(\pi)/\pi = {\cdot}1812 = \sqrt{\{{\cdot}03283\}}.$$
$$2J_1(\tfrac{1}{2}\pi)/\tfrac{1}{2}\pi = {\cdot}7217 = \sqrt{\{{\cdot}5209\}}.$$

In the case (corresponding to i. fig. 4) where there is similarity of phase, we have at the geometrical images amplitudes 1·1812 as against 1·4434 at the point midway between. When there is opposition of phase the first becomes $\pm{\cdot}8188$, and the last zero *. When the phases differ by a quarter period, or when the sources are self-luminous (iii. fig. 4), the amplitudes at the geometrical images are $\sqrt{\{1{\cdot}0328\}}$ or 1·0163, and at the middle point $\sqrt{\{1{\cdot}0418\}}$ or 1·0207. The partial separation, indicated by the central depression in curve iii. fig. 4, is thus lost when the rectangular aperture is exchanged for a circular one of equal width. It should be borne in mind that these results do not apply to a double *line*, which in the case of a circular aperture behaves differently from a double *point*.

There is one respect in which the theory is deficient, and the deficiency is the more important the larger the angular aperture. The formula (7) from which we start assumes that a radiant point radiates equally in all directions, or at least that the radiation from it after leaving the object-glass is equally dense over the whole area of the section. In the case of telescopes, and microscopes of moderate angular aperture, this assumption can lead to no appreciable error ; but it may be otherwise when the angular aperture is very large. The radiation from an ideal centre of transverse vibrations is certainly not uniform in various directions, and indeed vanishes in that of primary vibration. If we suppose such an ideal source to be situated upon the axis of a wide-angled object-glass, we might expect the diffraction pattern to be less closely limited in that axial plane

* The zero illumination extends to *all* points upon the line of symmetry.

which includes the direction of primary vibration than in that which is perpendicular to it. The result for a double point illuminated by borrowed light would be a better degree of separation when the primary vibrations are perpendicular to to the line of junction than when they are parallel to it.

Although it is true that complications and uncertainties under this head are not without influence upon the theory of the microscopic limit, it is not to be supposed that any considerable variation from that laid down by Abbe and Helmholtz is admissible. Indeed, in the case of a grating the theory of Abbe is still adequate, so far as the limit of resolution is concerned; for, as Dr. Stoney has remarked, the irregularity of radiation in different directions tells only upon the relative brightness and not upon the angular position of the spectra. And it will remain true that there can be no resolution without the cooperation of two spectra at least.

In Table II. and fig. 4 we have considered the image of a double point or line as formed by a lens of rectangular aperture. It is now proposed to extend the calculation to the case where the series of points or lines is infinite, constituting a *row* of points or a *grating*. The intervals are supposed to be strictly equal, and also the luminous intensities. When the aperture is rectangular, the calculation is the same whether we are dealing with a row of points or with a grating, but we have to distinguish according as the various centres radiate independently, viz., as if they were self-luminous, or are connected by phase-relations. We will commence with the former case.

If the geometrical images of the various luminous points are situated at $u=0$, $u=\pm v$, $u=\pm 2v$, &c., the expressions for the intensity at any point u of the field may be written as an infinite series,

$$\mathrm{I}(u) = \frac{\sin^2 u}{u^2} + \frac{\sin^2(u+v)}{(u+v)^2} + \frac{\sin^2(u-v)}{(u-v)^2} + \frac{\sin^2(u+2v)}{(u+2v)^2} + \frac{\sin^2(u-2v)}{(u-2v)^2} + \dots \quad (22)$$

Being an even function of u and periodic in period v, (22) may be expanded by Fourier's theorem in a series of cosines. Thus

$$\mathrm{I}(u) = \mathrm{I}_0 + \mathrm{I}_1 \cos\frac{2\pi u}{v} + \dots + \mathrm{I}_r \cos\frac{2\pi r u}{v} + \dots; \quad (23)$$

and the character of the field of light will be determined when the values of the constants I_0, I_1, &c., are known. For these we have as usual

$$I_0 = \frac{1}{v}\int_0^v I(u)\,du, \quad I_r = \frac{2}{v}\int_0^v I(u)\cos\frac{2\pi r u}{v}\,du\,; \quad . \quad (24)$$

and it only remains to effect the integrations. To this end we may observe that each term in the series (22) must in reality make an equal contribution to I_r. It will come to the same thing whether, as indicated in (24), we integrate the sum of the series from 0 to v, or integrate a single term of it, *e. g.* the first, from $-\infty$ to $+\infty$. We may therefore take

$$I_0 = \frac{1}{v}\int_{-\infty}^{+\infty} \frac{\sin^2 u}{u^2}\,du = \frac{\pi}{v}\,; \quad . \quad . \quad . \quad . \quad (25)$$

$$I_r = \frac{2}{v}\int_{-\infty}^{+\infty} \frac{\sin^2 u}{u^2}\cos\frac{2\pi r u}{v}\,du. \quad . \quad . \quad . \quad (26)$$

To evaluate (26) we have

$$\int_{-\infty}^{+\infty} \frac{\sin^2 u \cos su}{u^2}\,du = \int_{-\infty}^{+\infty} \frac{1}{u}\frac{d}{du}(\sin^2 u \cos su)\,du,$$

and

$$\frac{d}{du}(\sin^2 u \cos su) = -\frac{s}{2}\sin su$$

$$+\frac{2+s}{4}\sin(2+s)u + \frac{2-s}{4}\sin(2-s)u;$$

so that by (15) (s being positive)

$$\int_{-\infty}^{+\infty} \frac{\sin^2 u \cos su}{u^2}\,du = \pi\left\{-\frac{s}{2} + \frac{2+s}{4} \pm \frac{2-s}{4}\right\},$$

the *minus* sign being taken when $2-s$ is negative.

Hence

$$I_r = \frac{2\pi}{v}\left(1 - \frac{\pi r}{v}\right), \text{ or } 0, \quad . \quad . \quad . \quad . \quad (27)$$

according as v exceeds or falls short of $r\pi$.

We may now trace the effect of altering the value of v. When v is large, a considerable number of terms in the Fourier expansion (23) are of importance, and the discontinuous character of the luminous grating or row of points is fairly well represented in the image. As v diminishes, the higher terms drop out in succession, until when v falls below 2π only I_0 and I_1 remain. From this point onwards I_1 continues to diminish until it also finally disappears when v drops below π. The field is then uniformly illuminated, showing no trace of the original structure. The case $v=\pi$ is that of fig. 4, and curve iii. shows that at a stage when an infinite series shows no structure, a *pair* of luminous points or lines of the same closeness are still in some degree separated. It will be remembered that $v=\pi$ corresponds to $\epsilon=\frac{1}{2}\lambda/\sin\alpha$, ϵ being the linear period of the original object and α the semi-angular aperture.

We will now pass on to consider the case of a grating or row of points perforated in an opaque screen and illuminated by plane waves of light. If the incidence be oblique, the phase of the radiation emitted varies by equal steps as we pass from one element to the next. But for the sake of simplicity we will commence with the case of perpendicular incidence, where the radiations from the various elements all start in the same phase. We have now to superpose amplitudes, and not as before intensities. If A be the resultant amplitude, we may write

$$\mathrm{A}(u)=\frac{\sin u}{u}+\frac{\sin(u+v)}{u+v}+\frac{\sin(u-v)}{u-v}+\ldots\ldots$$

$$=\mathrm{A}_0+\mathrm{A}_1\cos\frac{2\pi u}{v}+\ldots+\mathrm{A}_r\cos\frac{2\pi r u}{v}+\ldots\ldots \quad (28)$$

When v is very small, the infinite series identifies itself more and more nearly with the integral

$$\frac{1}{v}\int_{-\infty}^{+\infty}\frac{\sin u}{u}du, \quad \text{viz. } \frac{\pi}{v}.$$

In general we have, as in the last problem,

$$\mathrm{A}_0=\frac{1}{v}\int_{-\infty}^{+\infty}\frac{\sin u}{u}du\ ; \quad \mathrm{A}_r=\frac{2}{v}\int_{-\infty}^{+\infty}\frac{\sin u}{u}\cos\frac{2\pi r u}{v}du\ ; \quad (29)$$

so that $\mathrm{A}_0=\pi/v$. As regards A_r, writing s for $2\pi r/v$, we have

$$\mathrm{A}_r=\frac{1}{v}\int_{-\infty}^{+\infty}\frac{\sin(1+s)u+\sin(1-s)u}{u}du=\frac{\pi}{v}(1\pm 1),$$

the lower sign applying when $(1-s)$ is negative. Accordingly,

$$\mathrm{A}(u)=\frac{\pi}{v}\left\{1+2\cos\frac{2\pi u}{v}+2\cos\frac{4\pi u}{v}+\ldots\right\} \quad . \quad (30)$$

the series being continued so long as $2\pi r<v$.

If the series (30) were continued *ad infinitum,* it would represent a discontinuous distribution, limited to the points (or lines) $u=0$, $u=\pm v$, $u=\pm 2v$, &c., so that the image formed would accurately correspond to the original object. This condition of things is most nearly realised when v is very great, for then (30) includes a large number of terms. As v diminishes the higher terms drop out in succession, retaining however (in contrast with (27)) their full value up to the moment of disappearance. When v is less than 2π, the series is reduced to its constant term, so that the field becomes uniform. Under this kind of illumination, the resolving-power is only half as great as when the object is self-luminous.

These conclusions are in entire accordance with Abbe's theory. The first term of (30) represents the central image, the second term the *two* spectra of the first order, the third term the two spectra of the second order, and so on. Resolution fails at the moment when the spectra of the first order cease to cooperate, and we have already seen that this happens for the case of perpendicular incidence when $v=2\pi$. The two spectra of any given order fail at the same moment.

If the series stops after the lateral spectra of the first order,

$$A(u)=\frac{\pi}{v}\left\{1+2\cos\frac{2\pi u}{v}\right\}, \quad . \quad . \quad . \quad (31)$$

showing a maximum intensity when $u=0$, or $\frac{1}{2}v$, and zero intensity when $u=\frac{1}{3}v$, or $\frac{2}{3}v$. These bands are not the simplest kind of interference bands. The latter require the operation of two spectra only; whereas in the present case there are three—the central image and the two spectra of the first order.

We may now proceed to consider the case when the incident plane waves are inclined to the grating. The only difference is that we require now to introduce a change of phase between the image due to each element and its neighbour. The series representing the resultant amplitude at any point u may still be written

$$\frac{\sin u}{u}+\frac{\sin(u+v)}{u+v}e^{-imv}+\frac{\sin(u-v)}{u-v}e^{+imv}$$
$$+\frac{\sin(u+2v)}{u+2v}e^{-2imv}+\ \dots\dots \quad (32)$$

For perpendicular incidence $m=0$. If γ be the obliquity, ϵ the grating-interval, λ the wave-length,

$$mv/2\pi=\epsilon\sin\gamma/\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The series (32), as it stands, is not periodic with respect to u in period v, but evidently it can differ from such a periodic series only by the factor e^{imu}.

The series

$$\frac{e^{-imu}\sin u}{u}+\frac{e^{-im(u+v)}\sin(u+v)}{u+v}$$
$$+\frac{e^{-im(u-v)}\sin(u-v)}{u-v}+\frac{e^{-im(u+2v)}\sin(u+2v)}{u+2v}+\dots \quad . \quad (34)$$

is truly periodic, and may therefore be expanded by Fourier's theorem in periodic terms:

$$(34)=A_0+iB_0+(A_1+iB_1)\cos(2\pi u/v)$$
$$+(C_1+iD_1)\sin(2\pi u/v)+\dots\dots$$
$$+(A_r+iB_r)\cos(2r\pi u/v)+(C_r+iD_r)\sin(2r\pi u/v)+\dots \quad (35)$$

As before, if $s=2r\pi/v$,

$$\tfrac{1}{2}v(A_r+iB_r)=\int_{-\infty}^{+\infty}\frac{e^{-imu}\sin u\cos su}{u}du;$$

so that $B_r=0$, while

$$\tfrac{1}{2}v.\ A_r=\int_{-\infty}^{+\infty}\frac{\cos mu\sin u\cos su}{u}du. \quad . \quad . \quad (36)$$

In like manner $C_r=0$, while

$$-\tfrac{1}{2}v.\ D_r=\int_{-\infty}^{+\infty}\frac{\sin mu\sin u\sin su}{u}du. \quad . \quad . \quad (37)$$

In the case of the zero suffix

$$B_0=0,\quad vA_0=\int_{-\infty}^{+\infty}\frac{\cos mu\sin u}{u}du. \quad . \quad . \quad . \quad . \quad (38)$$

When the products of sines and cosines which occur in (36) &c. are transformed in a well known manner, the integration may be effected by (15). Thus

$$\cos mu\sin u\cos su=\tfrac{1}{4}\{\sin(1+m+s)u+\sin(1-m-s)u$$
$$+\sin(1+m-s)u+\sin(1-m+s)u\};$$

so that

$$\tfrac{1}{2}v.\ A_r=\tfrac{1}{4}\pi\{[1+m+s]+[1-m-s]+[1+m-s]$$
$$+[1-m+s]\} \quad . \quad (39)$$

where each symbol such as $[1+m+s]$ is to be replaced by ± 1, the sign being that of $(1+m+s)$. In like manner

$$-\tfrac{1}{2}v.\ D_r=\tfrac{1}{4}\pi\{[1+m-s]+[1-m+s]-[1+m+s]$$
$$-[1-m-s]\} \quad . \quad (40)$$

The rth terms of (35) are accordingly

$$\frac{\pi}{2v}\{e^{isu}([1+m+s]+[1-m-s])$$
$$+e^{-isu}([1+m-s]+[1-m+s])\};$$

or for the original series (32),

$$\frac{\pi}{2v}\{e^{i(m+s)u}([1+m+s]+[1-m-s])$$
$$+e^{i(m-s)u}([1+m-s]+[1-m+s])\} \quad . \quad (41)$$

For the term of zero order,

$$A_0 e^{imu}=\frac{\pi}{2v}e^{imu}([1+m]+[1-m]). \quad . \quad . \quad (42)$$

From (41) we see that the term in $e^{i(m+s)u}$ vanishes unless $(m+s)$ lies between ± 1, and that then it is equal to $\pi/v \, . \, e^{i(m+s)u}$; also that the term in $e^{i(m-s)u}$ vanishes unless $(m-s)$ lies between ± 1, and that it is then equal to $\pi/v \, . \, e^{i(m-s)u}$. In like manner the term in e^{imu} vanishes unless m lies between ± 1, and when it does not vanish it is equal to $\pi/v \, . \, e^{imu}$. This particular case is included in the general statement by putting $s=0$.

The image of the grating, or row of points, expressed by (32), is thus capable of representation by the sum of terms

$$\pi/v \, . \, \{e^{imu}+e^{i(m+s_1)u}+e^{i(m-s_1)u}+e^{i(m+s_2)u}+\ldots\} \quad . \quad (43)$$

where $s_1=2\pi/v$, $s_2=4\pi/v$, &c., every term being included for which the coefficient of u lies between ± 1. Each of these terms corresponds to a spectrum of Abbe's theory, and represents plane progressive waves inclined at a certain angle to the plane of the image. Each spectrum when it occurs at all contributes equally, and it goes out of operation suddenly. If but one spectrum operates, the field is of uniform brightness. If two spectra operate, we have the ordinary interference bands due to two sets of plane waves crossing one another at a small angle of obliquity *.

Any consecutive pair of spectra give the same interference bands, so far as illumination is concerned. For

$$\frac{\pi}{v}\left\{e^{iu[m+2r\pi/v]}+e^{iu[m+2(r+1)\pi/v]}\right\}=\frac{2\pi}{v}\cos\frac{\pi u}{v}e^{iu[m+2(r+\frac{1}{2})\pi/v]},$$

of which the exponential factor influences only the phase.

In (43) the critical value of v for which the rth spectrum

* *Enc. Brit.* "Wave Theory," p. 425.

disappears is given by, when we introduce the value of m from (33),

$$\frac{2\pi}{v}\left(\frac{\epsilon \sin \gamma}{\lambda} \pm r\right) = \pm 1;$$

or, since (as we have seen) $\frac{v}{2\pi} = \frac{\epsilon \sin \alpha}{\lambda}$, ... (44)

$$\epsilon(\sin \gamma \mp \sin \alpha) = \mp r\lambda. \quad . \quad . \quad . \quad . \quad . \quad (45)$$

This is the condition, according to elementary theory, in order that the rays forming the spectrum of the rth order should be inclined at the angle α, and so (fig. 2) be adjusted to travel from A to B, through the edge of the lens L.

The discussion of the theory of a rectangular aperture may here close. This case has the advantage that the calculation is the same whether the object be a row of points or a grating. A parallel treatment of other forms of aperture, *e. g.* the circular form, is not only limited to the first alternative, but applies there only to those points of the field which lie upon the line joining the geometrical images of the luminous points. Although the advantage lies with a more general method of investigation to be given presently, it may be well to consider the theory of a circular aperture as specially deduced from the formula (21) which gives the image of a single luminous centre.

If we limit ourselves to the case of parallel waves and perpendicular incidence, the infinite series to be discussed is

$$A(u) = \frac{J_1(u)}{u} + \frac{J_1(u+v)}{u+v} + \frac{J_1(u-v)}{u-v} + \frac{J_1(u+2v)}{u+2v} +, \quad . \quad (46)$$

where

$$u = \pi\xi \,.\, 2R/\lambda f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Since A is necessarily periodic in period v, we may assume

$$A(u) = A_0 + A_1 \cos(2\pi u/v) + \ldots + A_r \cos(2r\pi u/v) + \ldots; \quad . \quad (48)$$

and, as in the case of the rectangular aperture,

$$A_0 = \frac{1}{v}\int_{-\infty}^{+\infty} \frac{J_1(u)}{u}\,du, \quad A_r = \frac{2}{v}\int_{-\infty}^{+\infty} \frac{J_1(u)}{u} \cos\frac{2r\pi u}{v}\,du. \quad . \quad (49)$$

These integrals may be evaluated. If a and b be real, and a be positive *,

$$\int_0^{\infty} e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{(a^2+b^2)}}. \quad . \quad . \quad . \quad (50)$$

* Gray and Mathews' 'Bessel's Functions,' 1895, p. 72.

Multiplying by bdb and integrating from 0 to b, we find

$$\int_0^\infty \frac{J_1(bx)e^{-ax}}{x}\,dx = \frac{\sqrt{(a^2+b^2)}-a}{b}. \quad . \quad . \quad (51)$$

In this we write $b=1$, $a=is$, where s is real. Thus

$$\int_0^\infty \frac{J_1(x)\{\cos sx - i\sin sx\}}{x}\,dx = \sqrt{(1-s^2)} - is.$$

If $s^2>1$, we must write $i\sqrt{(s^2-1)}$ for $\sqrt{(1-s^2)}$. Hence, if $s<1$,

$$\int_0^\infty \frac{J_1(x)\cos sx}{x}\,dx = \sqrt{(1-s^2)}, \quad . \quad . \quad (52)$$

$$\int_0^\infty \frac{J_1(x)\sin sx}{x}\,dx = s; \quad . \quad . \quad . \quad . \quad . \quad (53)$$

while, if $s>1$,

$$\int_0^\infty \frac{J_1(x)\cos sx}{x}\,dx = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$\int_0^\infty \frac{J_1(x)\sin sx}{x}\,dx = -\sqrt{(s^2-1)} + s. \quad . \quad (55)$$

We are here concerned only with (52), (54), and we conclude that $A_0=2/v$, and that

$$A_r = \frac{4\sqrt{(1-s^2)}}{v}, \quad \text{or } 0, \quad . \quad . \quad . \quad (56)$$

according as s is less or greater than 1, viz. according as $2r\pi$ is less or greater than v.

If we compare this result with the corresponding one (30) for a rectangular aperture of equal width ($2R=a$), we see that the various terms representing the several spectra enter or disappear at the same time; but there is one important difference to be noted. In the case of the rectangular aperture the spectra enter suddenly and with their full effect, whereas in the present case there is no such discontinuity, the effect of a spectrum which has just entered being infinitely small. As will appear more clearly by another method of investigation, the discontinuity has its origin in the sudden rise of the ordinate of the rectangular aperture from zero to its full value.

In the method referred to the form of the aperture is supposed to remain symmetrical with respect to both axes, but otherwise is kept open, the integration with respect to x being postponed. Starting from (12) and considering only those points of the image for which η and q in equation (8) vanish, we have as applicable to the image of a single luminous source

$$C=\iint \cos px\, dx\, dy=2\int y \cos px\, dx \quad . \quad . \quad . \quad (57)$$

in which $2y$ denotes the whole height of the aperture at the point x. This gives the amplitude as a function of p. If there be a row of luminous points, from which start radiations in the same phase, we have an infinite series of terms, similar to (57) and derived from it by the addition to p of positive and negative integral multiples of a constant (p_1) representing the period. The sum of the series $A(p)$ is necessarily periodic, so that we may write

$$A(p)=A_0+\ldots+A_r \cos(2r\pi p/p_1)+\ldots; \quad . \quad (58)$$

and, as in previous investigations, we may take

$$A_r=\int_{-\infty}^{+\infty} C \cos sp\, dp, \quad . \quad . \quad . \quad . \quad . \quad (59)$$

s (not quite the same as before) standing for $2r\pi/p_1$, and a constant factor being omitted. To ensure convergency we will treat this as the limit of

$$\int_{-\infty}^{+\infty} e^{\pm hp}\, C \cos sp\, dp \quad . \quad . \quad . \quad . \quad . \quad (60)$$

the sign of the exponent being taken negative, and h being ultimately made to vanish. Taking first the integration with respect to p, we have

$$\int_{-\infty}^{+\infty} e^{\pm hp} \cos xp \cos sp\, dp=\frac{h}{h^2+(x+s)^2}+\frac{h}{h^2+(x-s)^2};$$

and thus

$$A_r=\int \frac{hy\, dx}{h^2+(x+s)^2}+\int \frac{hy\, dx}{h^2+(x-s)^2},$$

in which h is to be made to vanish. In the limit the integrals receive sensible contributions only from the neighbourhoods of $x=\pm s$; and since

$$\int_{-\infty}^{+\infty} \frac{du}{1+u^2}=\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

we get

$$A_r = \pi(y_{x=-s} + y_{x=+s}) = 2\pi y_{x=s}. \quad . \quad . \quad . \quad (62)$$

From (62) we see that the occurrence of the term in A_r, *i.e.* the appearance of the spectrum of the rth order, is associated with the value of a particular ordinate of the object-glass. If the ordinate be zero, *i.e.* if the abscissa exceed numerically the half-width of the object-glass, the term in question vanishes. The first appearance of it corresponds to

$$\tfrac{1}{2}a = 2r\pi/p_1 = r\lambda f/\xi_1,$$

in which a is the entire width of the object-glass and ξ_1 the linear period in the image. By (17a),

$$\frac{\lambda f}{\xi_1} = \frac{\lambda f \sin\beta}{\epsilon \sin\alpha} = \frac{\frac{1}{2}a\lambda}{\epsilon\sin\alpha};$$

so that the condition is, as before,

$$\epsilon \sin\alpha = r\lambda.$$

When A_r has appeared, its value is proportional to the ordinate at $x=s$. Thus in the case of a circular aperture ($a=2R$) we have

$$y_{x=s} = R\sqrt{\{1 - r^2\lambda^2/\epsilon^2 \sin^2\alpha\}}. \quad . \quad . \quad . \quad . \quad (63)$$

The above investigation relates to a row of luminous points emitting light of the same intensity and phase, and it is limited to those points of the image for which η (and q) vanish. If the object be a grating radiating under similar conditions, we have to retain $\cos qy$ in (12) and to make an integration with respect to q. Taking this first, and introducing a factor $e^{\pm kq}$, we have

$$\int_{-\infty}^{+\infty} e^{\pm kq} \cos qy \, dq = \frac{2k}{k^2 + y^2}. \quad . \quad . \quad . \quad . \quad (64)$$

This is now to be integrated with respect to y between the limits $-y$ and $+y$. If this range be finite, we have

$$\text{Limit}_{k=0} \int_{-y}^{+y} \frac{2k\,dy}{k^2+y^2} = 2\pi, \quad . \quad . \quad . \quad . \quad (65)$$

independent of the length of the particular ordinate. Thus

$$C_1 = \int_{-\infty}^{+\infty} C\,dq = 2\pi \int \cos px\,dx, \quad . \quad . \quad . \quad . \quad (66)$$

the integration with respect to x extending over the range for which y is finite, that is, over the width of the object-glass.

If this be 2R, we have

$$\int_{-\infty}^{+\infty} C\,dq = 4\pi/p\,.\sin pR. \quad . \quad . \quad . \quad (67)$$

From (67) we see that the image of a luminous line, all parts of which radiate in the same phase, is independent of the form of the aperture of the object-glass, being, for example, the same for a circular aperture as for a rectangular aperture of equal width. This case differs from that of a *self-luminous* line, the images of which thrown by circular and rectangular apertures are of different types *.

The comparison of (67) with (20), applicable to a circular aperture, leads to a theorem in Bessel's functions. For, when q is finite,

$$C = \pi R^2 \frac{2J_1\{\sqrt{(p^2+q^2)}R\}}{\sqrt{(p^2+q^2)}}; \quad . \quad . \quad . \quad . \quad (68)$$

so that, setting $R=1$, we get

$$\int_0^{\infty} \frac{J_1\{\sqrt{(p^2+q^2)}\}}{\sqrt{(p^2+q^2)}}\,dq = \frac{\sin p}{p}. \quad . \quad . \quad . \quad . \quad (69)\dagger$$

The application to a grating, of which all parts radiate in the same phase, proceeds as before. If, as in (58), we suppose

$$A(p) = A_0 + \ldots + A_r \cos sp + \ldots, \quad . \quad . \quad (70)$$

we have

$$A_r = \int_{-\infty}^{+\infty} C_1 \cos sp\,dp\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

from which we find that A_r is $4\pi^2$ or 0, according as the ordinate is finite or not finite at $x=s$. The various spectra enter and disappear under the same conditions as prevailed when the object was a row of points ; but now they enter discontinuously and retain constant values, instead of varying with the particular ordinate of the object-glass which corresponds to $x=s$.

We will now consider the corresponding problems when the illumination is such that each point of the row of points or of the grating radiates independently. The integration then relates to the intensity of the field as due to a single source.

By (9), (10), (11), the intensity I^2 at the point (p, q) of the field, due to a single source whose geometrical image is

* *Enc. Brit.*, "Wave Theory," p. 434.

† This may be verified by means of Neumann's formula (Gray & Matthews, 'Bessel's Functions' (70) p. 27).

situated at (0, 0) is given by

$$\lambda^2 f^2 I^2 = \left\{\iint \cos(px+qy)\,dx\,dy\right\}^2 + \left\{\iint \sin(px+qy)\,dx\,dy\right\}^2$$
$$= \iint \cos(px'+qy')\,dx'\,dy' \times \iint \cos(px+qy)\,dx\,dy$$
$$+ \iint \sin(px'+qy')\,dx'\,dy' \times \iint \sin(px+qy)\,dx\,dy$$
$$= \iiiint \cos\{p(x'-x)+q(y'-y)\}\,dx\,dy\,dx'\,dy', \quad . \quad (72)$$

the integrations with respect to x', y', as well as those with respect to x, y, being over the area of the aperture.

In the present application to sources which are periodically repeated, the term in $\cos sp$ of the Fourier expansion representing the intensity at various points of the image has a coefficient found by multiplying (72) by $\cos sp$ and integrating with respect to p from $p=-\infty$ to $p=+\infty$. If the object be a row of points, we may take $q=0$; if it be a grating, we have to integrate with respect also to q from $q=-\infty$ to $q=+\infty$.

Considering the latter case, and taking first the integrations with respect to p, q, we introduce the factors $e^{\mp hp \mp kq}$, the *plus* or *minus* being so chosen as to make the elements of the integral vanish at infinity. After the operations have been performed, h and k are to be supposed to vanish*. The integrations are performed as for (60), (64), and we get the sum of the two terms denoted by

$$\frac{2hk}{\{h^2+(x'-x\pm s)^2\}\{k^2+(y'-y)^2\}} \quad . \quad . \quad . \quad (73)$$

We have still to integrate with respect to $dx\,dy\,dx'\,dy'$. As in (65), since the range for y' always includes y,

$$\text{Limit}_{k=0} \int \frac{2k\,dy'}{k^2+(y'-y)^2} = 2\pi;$$

and we are left with

$$\iiint \frac{2\pi h\,dx\,dy\,dx'}{h^2+(x'-x\pm s)^2}. \quad . \quad . \quad . \quad . \quad . \quad (74)$$

If s were zero, the integration with respect to x' would be precisely similar; but with s finite it will be only for certain values of x that $(x'-x\pm s)$ vanishes within the range of integration. Unless this evanescence takes place, the limit when h vanishes becomes zero. The effect of the integration with respect to x' is thus to limit the range of the subsequent

* The process is that employed by Stokes in his evaluation of the integral intensity, Edin. Trans. xx. p. 317 (1853). See also *Enc. Brit.*, "Wave Theory," p. 431.

integration with respect to x. The result may be written

$$2\pi^2 \iint dx\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

upon the understanding that, while the integration for y ranges over the whole vertical aperture, that for x is limited to such values of x as bring $x \mp s$ (as well as x itself) within the range of the horizontal aperture. The coefficient of the Fourier component of the intensity involving $\cos sp$, or $\cos (2r\pi p/p_1)$, is thus proportional to a certain part of the area of the aperture. Other parts of the area are inefficient, and might be stopped off without influencing the result.

The limit to resolution, corresponding to $r=1$, depends only on the width of the aperture, and is therefore for all forms of aperture the same as for the case of the rectangular aperture already fully investigated.

If the object be a row of points instead of a row of lines, $q=0$, and there is no integration with respect to it. The process is nearly the same as above, and the result for the coefficient of the rth term in the Fourier expansion is proportional to $\int y^2\,dx$, instead of $\int y\,dx$, the integration with respect to x being over the same parts of the aperture as when the object was a grating. The application to a circular aperture would lead to an evaluation of

$$\int_{-\infty}^{+\infty} \frac{J_1{}^2(u) \cos su}{u^2}\,du.$$

XVI. *The Operation Groups of order* 8p, p *being any prime number**. *By* G. A. Miller, *Ph.D.*†

According to Sylow's theorem these groups contain $kp+1$ conjugate subgroups of order p and $b \overline{\gtrless} 1$ in the equation

$$\frac{8p}{kp+1} = bp.$$

Hence they must contain a self-conjugate subgroup of this order when $p>3$ and $p \neq 7$. We shall first consider all the possible groups that contain such a self-conjugate subgroup.

* M. Levavasseur gives an enumeration of these groups, without explaining how they were obtained, in *Comptes Rendus*, March 2, 1896. His enumeration is not quite correct. He states that there are three groups which exist only when $p-1$ is a multiple of 4 without being also a multiple of 8. We shall prove that there are only two such groups.

† Communicated by the Author.