XLIV. The highest waves in water

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damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.

ject to the law of uniformity in the assemblage, act between the points of our primary assemblage : and, if we please, also between all the points of our second assemblage ; and between all the points of the two assemblages.   Let these forces fulfil the conditions of equilibrium ; of which the principle is de-scribed in § 16 and applied to find the equations of equilibrium for the simpler case of a single homogeneous assemblage there considered.   Thus we have an incompressible elastic solid ; and, as in § 17 above, we see that there are fifteen indepen-dent coefficients in the quadratic function of the strain-components expressing the work required to produce an infinitesimal strain.   Thus we realize the result described in § 7 above.

§ 35. Suppose now each of the four tie-struts to be not infinitely resistant against change of length, and to have a given modulus of longitudinal rigidity, which, for brevity, we shall call its stiffness.   By assigning proper values to these four stiffnesses, and by supposing the tetrahedron to be freed from the two conditions making it our special tetrahedron, we have six quantities arbitrarily assignable, by which, adding these six to the former fifteen, we may give arbitrary values to each of the twenty-one coefficients in the quadratic function of the six strain-components with which we have to deal when change of bulk is allowed.   Thus, in strictest Boscovichian doctrine, we provide for twenty-one independent coefficients in Green's energy-function.   The dynamical details of the consideration of the equilibrium of two homogeneous assem-blages with mutual attraction between them, and of the extension of §§ 9–17 to the larger problem now before us, are full of purely scientific and engineering interest, but must be reserved for what I hope is a future communication.

XLIV. *The Highest Waves in Water.   By* J. H. MICHELL, *M.A., Fellow of Trinity College, Cambridge*.*

THE waves contemplated are those in which the motion is parallel to one plane and which advance without change of form.   Most of the work already done on irrotational waves applies only to those of small height and unbroken outline. Stokes† has, however, long since expressed the opinion that the height of the wave could be increased until the summit became pointed, and showed that the angle at the summit would be 120°.   The object of this communication is to make known a method of investigating such maximum waves.

The work of Prof. Stokes fills the gap between these and the infinitesimal waves.

Waves in Deep Water.

Suppose at first the water so deep that it may be considered infinitely so. Make the motion steady by moving the water forward bodily with the backward velocity V of the wave. Take rectangular axes, x horizontal, y vertically downwards, in a plane of motion, with the origin at a wave-summit. Let ϕ, ψ be the velocity and stream functions, the surface of the water being $\psi = 0$ and the bottom $\psi = \infty$.

Call the velocity q, the inclination of the wave-line to the horizontal at ϕ, θ, the curvature at the same point K.

At the summit the curvature and the velocity are zero, and the ratio of the two is finite and not far from constant along the whole wave-line, as appears from the investigation following.

Accordingly we assume, taking $\phi = 0$, $\phi = \pi$ as consecutive summits,

$$K = q\,(a_0 + a_1 \cos 2\phi + a_2 \cos 4\phi + \ldots),$$

where a_1, $a_2 \ldots$ are small compared with a_0.

Since $K = q\dfrac{d\theta}{d\phi}$,

$$\frac{d\theta}{d\phi} = a_0 + a_1 \cos 2\phi + a_2 \cos 4\phi + \ldots .$$

In order to find the connexion between $\phi + i\psi \equiv w$ and $x + iy \equiv z$ throughout the liquid, we proceed in the manner of Riemann and Schwarz, that is, we find by means of the assumed surface-condition a function of z which is real over the surface and possesses only simple *poles* in the liquid. This function can then be extended continuously in its range, throughout the plane w, its value at $(\phi, -\psi)$ being the conjugate of that at (ϕ, ψ). We then have a function throughout the plane w whose singularities are confined to simple poles, and whose form can be written down according to the principles of Cauchy.

Put

$$\log\frac{dz}{dw} \equiv U.$$

Along the surface, or $\psi = 0$,

$$U = \log e^{i\theta}/q = -\log q + i\theta,$$

and therefore, along the surface,

$$\frac{dU}{dw} = \frac{dU}{d\phi} = -\frac{d\log q}{d\phi} + i\frac{d\theta}{d\phi}\,;$$

or, from the assumed form of $\dfrac{d\theta}{d\phi}$,

$$\frac{d\mathrm{U}}{dw} - i\left(a_0 + a_1 e^{2iw} + a_2 e^{4iw} + \ldots\right) = -\frac{d\log q}{d\phi} + a_1 \sin 2\phi + \ldots .$$

The function on the left is therefore real over the surface.

At $\psi = \infty$,

$$\frac{dz}{dw} = 1/\mathrm{V},$$

and the function will be finite there and equal to $-ia_0$.

The only singular points to be considered are, then, the summits of the waves.

Suppose near the summit $w = 0$,

$$\frac{dz}{dw} = \mathrm{A}w^n,$$

and therefore

$$\frac{dw}{dz} = \mathrm{A}^{-\frac{1}{n+1}}(n+1)^{-\frac{n}{n+1}} z^{-\frac{n}{n+1}},$$

and

$$q^2 = \mathrm{A}^{-\frac{2}{n+1}}(n+1)^{-\frac{2n}{n+1}} r^{-\frac{2n}{n+1}},$$

where r is the distance from the summit. Now, since the pressure is constant over the surface we have $q^2 = 2gy$; and, comparing, we see that $n = -\frac{1}{3}$; so that

$$\frac{dz}{dw} = \mathrm{A}w^{-\frac{1}{3}}$$

near $w = 0$. From this it follows that the angle at the summit is $120°$, as was first shown by Stokes*.

Hence, also,

$$\frac{d\mathrm{U}}{dw} = -\frac{1}{3w}$$

near the summit $w = 0$; and the summits are simple infinities of the function considered.

According, then, to the principles of Cauchy, the function can only differ from the sum of the polar elements by a constant, and we have

$$\frac{d\mathrm{U}}{dw} - i\left(a_1 e^{2iw} + a_2 e^{4iw} + \ldots\right) = -\tfrac{1}{3}\Sigma \frac{1}{w - n\pi} - \frac{i}{3}$$

$$= -\tfrac{1}{3}\cot w - \frac{i}{3},$$

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* Collected Papers, vol. i. p. 227.

the constant being so chosen as to make things right at infinity.

Integrating, we have

$$U = \log(-i \sin w)^{-\frac{1}{3}} e^{-\frac{1}{3}iw} + a_0' + \frac{1}{2} a_1 e^{2iw} + \dots,$$

where a_0' is real, since U is real at $\psi = \infty$; or, writing the series in the form

$$\log A(1 + c_1 e^{2iw} + c_2 e^{4iw} + \dots),$$

we have

$$\frac{dz}{dw} = A(-i \sin w)^{-\frac{1}{3}} e^{-\frac{1}{3}iw}(1 + c_1 e^{2iw} + c_2 e^{4iw} + \dots),$$

the velocity at $\psi = \infty$, or the wave-velocity, being $2^{-\frac{1}{3}}/A$, and the real root of $(-i \sin w)^{-\frac{1}{3}}$ being taken along $\phi = 0$.

Suppose the units so chosen that $A = 1$, and hence $V = 1/2^{\frac{1}{3}}$, and the length of wave $L = \pi/V = \pi \times 2^{\frac{1}{3}}$.

It will be more convenient, at first, to invert $\dfrac{dz}{dw}$ and write

$$\frac{dw}{dz} = (-i \sin w)^{\frac{1}{3}} e^{\frac{1}{3}iw}(1 + b_1 e^{2iw} + b_2 e^{4iw} + \dots).$$

On the surface between $\phi = 0$ and $\phi = \pi$,

$$\frac{dw}{dz} = \sin^{\frac{1}{3}} \phi \, e^{\frac{1}{3}i\left(\phi - \frac{\pi}{2}\right)}(1 + b_1 e^{2i\phi} + b_2 e^{4i\phi} + \dots),$$

the real root of $\sin^{\frac{1}{3}} \phi$ being taken. The surface-condition

$$q^2 = 2gy$$

may be written

$$\frac{dq^2}{d\phi} = 2g \frac{dy}{d\phi} = 2g \frac{1}{q^2} \frac{d\phi}{dy},$$

or

$$\frac{dq^4}{d\phi} = 4g \frac{d\phi}{dy}.$$

Taking

$$\left(\frac{dw}{dz}\right)^2 = \sin^{\frac{2}{3}} \phi \, e^{\frac{2}{3}i\left(\phi - \frac{\pi}{2}\right)} \Big\{ 1 + 2b_1 e^{2i\phi} + (2b_2 + b_1{}^2) e^{4i\phi}$$
$$+ (2b_3 + 2b_1 b_2) e^{6i\phi} + \dots \Big\},$$

and multiplying by its conjugate, we get, omitting terms of order 4 in the b's,

$$q^4 = \sin^{\frac{4}{3}} \phi \Big\{ 1 + 4b_1{}^2 + (4b_1 + 8b_1 b_2 + 4b_1{}^3) \cos 2\phi$$
$$+ (4b_2 + 2b_1{}^2) \cos 4\phi + (4b_3 + 4b_1 b_2) \cos 6\phi \Big\},$$

and therefore

$$\frac{dq^4}{d\phi} = \tfrac{4}{3}\sin^{\frac{1}{3}}\phi\left\{(1 - b_1 + 4b_1{}^2 - b_1{}^3 - 2b_1b_2)\cos\phi \right.$$
$$+ (5b_1 - 2b_1{}^2 - 4b_2 + 10b_1b_2 + 5b_1{}^3)\cos 3\phi$$
$$+ (4b_1{}^2 + 8b_2 - 7b_1b_2 - 7b_3)\cos 5\phi$$
$$\left. + (11b_1b_2 + 11b_3)\cos 7\phi\right\}.$$

To put $\dfrac{d\phi}{dy}$ into a similar form we use the Fourier expansion

$$\sin\left(\overline{2r+\tfrac{1}{3}}\phi - \frac{\pi}{6}\right) = \Sigma A_{2n+1}\cos(2n+1)\phi,$$

where

$$A_{2n+1} = -\frac{6\sqrt{3}}{\pi}\frac{6r+1}{3^2(2n+1)^2 - (6r+1)^2}.$$

Since

$$\frac{d\phi}{dy} = -\sin^{\frac{1}{3}}\phi\left(\sin\overline{\tfrac{1}{3}\phi - \frac{\pi}{6}} + b_1\sin\overline{2\phi + \tfrac{1}{3}\phi - \frac{\pi}{6}} + \cdots\right),$$

we get

$$\frac{d\phi}{dy} = \frac{6\sqrt{3}}{\pi}\sin^{\frac{1}{3}}\phi\left[\tfrac{1}{8}\cos\phi + \tfrac{1}{80}\cos 3\phi + \tfrac{1}{224}\cos 5\phi + \tfrac{1}{440}\cos 7\phi + \right.$$
$$+ b_1\left(-\tfrac{7}{40}\cos\phi + \tfrac{7}{32}\cos 3\phi + \tfrac{7}{176}\cos 5\phi + \tfrac{7}{392}\cos 7\phi + \right.$$
$$+ b_2\left(-\tfrac{13}{160}\cos\phi - \tfrac{13}{88}\cos 3\phi + \tfrac{13}{56}\cos 5\phi + \tfrac{13}{272}\cos 7\phi + \right.$$
$$+ b_3\left(-\tfrac{19}{352}\cos\phi - \tfrac{19}{280}\cos 3\phi - \tfrac{19}{136}\cos 5\phi + \tfrac{19}{80}\cos 7\phi + \right.$$
$$\left. + \cdots\right].$$

Equating the coefficients of corresponding cosines and writing $18\sqrt{3}\,g/\pi = k$, the following equations are obtained for k and the b's :—

$$1 - b_1 + 4b_1{}^2 - b_1{}^3 - 2b_1b_2$$
$$= k(\cdot 125 - \cdot 175b_1 - \cdot 08125b_2 - \cdot 05398b_3),$$

$$5b_1 - 2b_1{}^2 - 4b_2 + 10b_1b_2 + 5b_1{}^3$$
$$= k(\cdot 0125 + \cdot 21875b_1 - \cdot 14772b_2 - \cdot 06786b_3),$$

$$4b_1{}^2 + 8b_2 - 7b_1b_2 - 7b_3$$
$$= k(\cdot 00446 + \cdot 039773b_1 + \cdot 232143b_2 - \cdot 13970b_3),$$

$$11b_1b_2 + 11b_3$$
$$= k(\cdot 00227 + \cdot 01785b_1 + \cdot 04779b_2 + \cdot 2375b_3).$$

Solving, we find as sufficiently close values :

$$k = 8 \cdot 25,$$
$$b_1 = \cdot 0397,$$
$$b_2 = \cdot 0094,$$
$$b_3 = \cdot 002 ;$$

and thus, as a close approximation to the value of $\dfrac{dw}{dz}$,

$$\frac{dw}{dz} = (-i \sin w)^{\frac{1}{3}} e^{\frac{1}{3} i w} (1 + \cdot 0397 e^{2iw} + \cdot 0094 e^{4iw} + \cdot 002 e^{6iw}).$$

The Depth of the Wave.

Since

$$18 \sqrt{3}\, g/\pi = k = 8 \cdot 25,$$
$$2g = 1 \cdot 66.$$

Now at $\phi = \dfrac{\pi}{2}$,

$$q^2 = (1 - b_1 + b_2 - b_3 + \ldots)^2$$
$$= (\cdot 9677)^2.$$

The depth of the wave is $q^2/2g$, and therefore the ratio of depth to length is

$$\frac{h}{L} = \frac{(\cdot 9677)^2}{1 \cdot 66 \times 3 \cdot 96} = \cdot 142 ;$$

so that the height of the wave is very nearly one seventh of its length.

As was to be expected, actual record of high waves does not come within some distance of this: Abercromby, for example, measured waves 46 feet high and 765 long (Phil. Mag. xxv. 1888).

At the Summit.

Near the summit $\phi = 0$ we have

$$\frac{dw}{dz} = \phi^{\frac{1}{3}} \left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right) (1 + b_1 + b_2 + \ldots),$$

and therefore

$$q^2 = \phi^{\frac{2}{3}} (1 + b_1 + b_2 + \ldots)^2$$

and

$$z = \tfrac{3}{2} \phi^{\frac{2}{3}} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \Big/ (1 + b_1 + b_2 + \ldots),$$

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so that, since $q^2 = 2gy$, we have

$$(1 + b_1 + b_2 + \ldots)^3 = \tfrac{3}{2}g$$
$$= 1 \cdot 245,$$

and $1 + b_1 + b_2 + \ldots = 1 \cdot 0758,$

while $b_1 + b_2 + b_3 = \cdot 0511.$

Near the summit, the series for q^2, every term having the same sign, will not be well represented by the first three terms. This is of little consequence since the sum of the terms is known at the summit. Elsewhere the first three terms give a good approximation, as appears below.

Form of the Wave.

The coordinates of any point ϕ on the surface are got from the two equations :—

$$\frac{dx}{d\phi} = \sin^{-\frac{1}{3}} \phi \left[\cos\left(\tfrac{1}{3}\phi - \frac{\pi}{6} \right) - \cdot 04 \cos\left(2\phi - \tfrac{1}{3}\phi + \frac{\pi}{6} \right) \right.$$
$$\left. - \cdot 008 \cos\left(4\phi - \tfrac{1}{3}\phi + \frac{\pi}{6} \right) - \cdot 001 \cos\left(6\phi - \tfrac{1}{3}\phi + \frac{\pi}{6} \right) \right],$$

$$\frac{dy}{d\phi} = \sin^{-\frac{1}{3}} \phi \left[\sin\left(\frac{\pi}{6} - \tfrac{1}{3}\phi \right) - \cdot 04 \sin\left(2\phi - \tfrac{1}{3}\phi + \frac{\pi}{6} \right) \right.$$
$$\left. - \cdot 008 \sin\left(4\phi - \tfrac{1}{3}\phi + \frac{\pi}{6} \right) - \cdot 001 \sin\left(6\phi - \tfrac{1}{3}\phi + \frac{\pi}{6} \right) \right].$$

The first terms do not integrate in convenient form and are best calculated by means of the expansions

$$\sin^{-\frac{1}{3}} \phi \cos \tfrac{1}{3} \phi = \phi^{-\frac{1}{3}}(1 + \cdot 0008\phi^4 + \ldots)$$
$$\sin^{-\frac{1}{3}} \phi \sin \tfrac{1}{3} \phi = \tfrac{1}{3}\phi^{\frac{2}{3}}(1 + \cdot 037 \phi^2 + \cdot 0023\phi^4 + \ldots).$$

The other terms integrate simply.

The following table gives the values of x, y, and q^2 for the specified values of ϕ :—

ϕ.	x.	y.	q^2.	q^2/y.
$\dfrac{\pi}{50}$	$\cdot 196$	$\cdot 110$	$\cdot 174$	$1 \cdot 58$
$\dfrac{\pi}{20}$	$\cdot 366$	$\cdot 197$	$\cdot 318$	$1 \cdot 61$
$\dfrac{\pi}{10}$	$\cdot 591$	$\cdot 298$	$\cdot 489$	$1 \cdot 64$
$\dfrac{\pi}{5}$	$\cdot 974$	$\cdot 428$	$\cdot 707$	$1 \cdot 65$
$\dfrac{3\pi}{10}$	$1 \cdot 321$	$\cdot 507$	$\cdot 837$	$1 \cdot 65$
$\dfrac{2\pi}{5}$	$1 \cdot 654$	$\cdot 552$	$\cdot 912$	$1 \cdot 65$
$\dfrac{\pi}{2}$	$1 \cdot 979$	$\cdot 567$	$\cdot 936$	$1 \cdot 65$

The form of the wave is shown in the diagram. The error in q^2/y near the summit means an error of about 1·5 per cent.

in x, y, and q, due, as before stated, to the slower convergency of the series for dw/dz at that point. This does not affect the determination of the height of the wave.

Velocity of the Wave.

The length of the wave is 3·96 and $2g = 1·66$, so that, in general units, if L is the length of the wave and V its velocity,

$$V^2 = ·191\, g\text{L}.$$

For the infinitesimal wave,

$$V^2 = g\text{L}/2\pi.$$

The ratio of the velocities of the highest wave and the infinitesimal wave is therefore 1·2. The wave observed by Abercromby (*loc. cit.*) of length 765 feet and height 46 feet had a velocity of 47 miles per hour ; the highest wave of the same length would be nearly 100 feet high and would have a velocity of 47 miles an hour very nearly.

Water of Finite Depth.

Here the poles of the function $d\text{U}/dw$ have to be indefinitely reflected in $\psi = \text{K}'$, the bottom of the water, and $\psi = 0$, the surface, in order to make U real or θ constant at the bottom. The singular points in the whole field are therefore at

$$w = 2n\text{K} + 2n'i\text{K}',$$

$w = 2n\text{K}$ being the wave summits.

We are thus led to the form

$$\frac{dw}{dz} = \text{H}^{\frac{1}{2}}(w)e^{\frac{\pi i}{6\text{K}}(w-\text{K})}\left\{1 + a_1 \cos\frac{\pi}{\text{K}}(w - i\text{K}') + \ldots\right\},$$

where H is Jacobi's function so denoted. This equation is susceptible of the same treatment as the simpler case.

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