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# VIII. On the question of the stability of the flow of fluids 

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to the parabola which is being constructed. The curves so Jrawn may be used for the production of templates for lenses or mirrors, and they could be drawn small and then magnified either by photography or by a pantagraph arrangement.

This instrument is a combination of well known link movements, but I do not think they have ever before been applied to the production of a parabolic curve from a single straight line motion.

The instrument may be so constructed that any play between the sliding pins and the slot may be avoided by pressing the handle down towards the lower side of the slot, which thus becomes a ruler. In the event of any machine being constructed on this principle, gravity itself might make this pressure.

20 Bartholomew Villas, Kentish Town, N.W., February 2, 1892.

## VIII. On the Question of the Stability of the Flow of Fluids. By Lord Rayletge, Sec. R.S.*

$I^{1}$T is well known that while Sir G. Stokes's theory of viscous flow gives a completely satisfactory account of what is observed in the case of capillary tubes, no theory at present exists to explain the complete change in the laws of flow which supervenes when the tubes are of larger diameter and the velocities not very small. Prof. Osborne Reynolds $\dagger$ has applied the theory of dynamical similarity to this question, and has shown both by theory and experiment that the change in the law of resistance occurs when $c \rho w / \mu$ has a certain value, where $c$ is a linear parameter such as the diameter of the tube, $w$ is the velocity, $\mu$ the coefficient of friction, and $\rho$ the density. The conclusion is perhaps most easily reached by applying the method of dimensions to the expression for the ratio ( P ) of the difference of pressures at two points along the length of the tube to the distance between the points. The dimensions of this ratio are those of a force divided by a volume; and if we assume that it may be expressed in terms of $\nu \ddagger$ (equal to $\mu / \rho), c, \rho$, and $w$ in the form

$$
c^{x} \nu^{y} \rho^{z} w^{n},
$$

[^0]$\dagger$ Phil. Trans. elxxiv. p. 935 (1883).
$\ddagger$ Of which the dimensions are 2 in space and -1 in time.
we have the three relations
\[

$$
\begin{aligned}
-2 & =x+2 y-3 z+n \\
-2 & =-y-n \\
1 & =z
\end{aligned}
$$
\]

so that

$$
x=n-3, \quad y=2-n, \quad z=1
$$

and

$$
\begin{equation*}
\mathrm{P} \propto v^{2} c^{-3} \rho \cdot(c w / v)^{n} \tag{1}
\end{equation*}
$$

Since $n$ is here indeterminate, all we can infer from dynamical similarity is that

$$
\begin{equation*}
\mathrm{P}=\nu^{2} c^{-3} \rho f(c w / \nu) \text {, . . . . . . } \tag{2}
\end{equation*}
$$

where $f$ is an arbitrary function.
For capillary tubes and moderate velocities $P$ varies as the first power of $v$, so that in (1) $n=1$. In this case

$$
\begin{equation*}
\mathrm{P}=\mathrm{A} \nu c^{-2} \rho w, \tag{3}
\end{equation*}
$$

A being an arbitrary constant. When, on the other hand, $c w / v$ is great, experiment shows that $n=2$ nearly. If this law be exact, (1) gives

$$
\begin{equation*}
\mathrm{P}=\mathrm{B} c^{-1} \rho w^{2} \tag{4}
\end{equation*}
$$

independent of $\nu$. The second power of the velocity and independence of viscosity are thus inseparably connected.

In the above theory no account is taken of any variation in the walls of the tubes. Either they must be perfectly smooth, or else the irregularities must be in proportion to the diameters. Under this limitation (2) would appear to hold good, at least if there be no finite slip at the walls.

The proportionality to $\rho$, expressed in (4), has probably not been tested experimentally. Neither is there any complete theoretical deduction of (4). But a comparison with Torricelli's law of efflux is significant. The resistance is the same as if it were necessary to renew continually the velocity of the liquid at intervals which are proportional to the diameters of the pipes.

The connexion between the alteration in the law of resistance and the transition from regularly stratified to eddying motion has been successfully traced by Reynolds. The question is, Why do eddies arise and take possession? From the description and drawings given by Reynolds it is natural to suppose that in the absence of viscosity the stratified motion would be unstable, and that it is stable in small tubes and at low velocities only in consequence of the steadying effect of viscosity then acting at an advantage. It was with this idea
that (at an earlier date*) I attempted an investigation of the stability of stratified flow in two dimensions, fully expecting to find it unstable. The result, however, was to show that in the absence of viscosity the stratified flow between two parallel walls was not unstable, provided that the law of flow were such that the curve representing the velocities in the various strata was of one curvature throughout, a condition satisfied in the case in question. To be more precise, it was proved that if the deviation from the regularly stratified motion were, as a function of the time, proportional to $e^{\text {int }}$, then $n$ could have no imaginary part.

On the other hand, if the condition as to the curvature of the velocity curve be violated, $n$ may acquire an imaginary part, and the resulting disturbance of the steady motion is exponentially unstable, as was shown by several examples in the paper referred to, and in a later one $\dagger$ in which the subject was further pursued.

We are thus confronted with a difficulty. For if the investigation in question can be applied to a fluid of infinitely small viscosity, how are we to explain the observed instability which occurs with moderate viscosities? It seems very unlikely that the first effect of increasing viscosity should be to introduce an instability not previously existent, while, as observation shows, a large viscosity makes for stability.

Several suggestions towards an explanation of the discrepancy present themselves. In the first place, irregularities in the walls, not included in the theoretical investigation, may play an essential part. Again, according to the view of Lord Kelvin, the theoretical stability for infinitely small disturbances at all viscosities may not extend beyond very narrow limits ; so that in practice and under finite disturbances the motion would be unstable, unless the viscosity exceeded a certain value. Two other suggestions which occurred to me at the time of writing my first paper as perhaps pointing to an explanation may now be mentioned. It is possible that there may be an essential difference between the motion in two dimensions to which the calculations related, and that in a tube of circular section on which observations are made. And, secondly, it is possible that, after all, the investigation in which viscosity is altogether ignored is inapplicable to the limiting case of a viscous fluid when the viscosity is supposed infinitely small. There is more to be said in favour of this view than would at first be supposed. In the calculated

[^1]motion there is a finite slip at the walls, and this is inconsistent with even the smallest viscosity. And, further, there are kindred problems relating to the behaviour of a viscous fluid in contact with fixed walls for which it can actually be proved* that certain features of the motion which could not enter into the solutions were the viscosity ignored from the first are nevertheless independent of the magnitude of the viscosity, and therefore not to be eliminated by supposing the viscosity to be infinitely small. Another case that may be instanced is that of a large stream of viscous fluid flowing past a spherical obstacle. As Sir G. Stokes has shown, the steady motion is the same whatever be the degree of viscosity ; and yet it is entirely different from the flow of an inviscid fluid in which no rotation can be generated. Considerations such as this raise doubts as to the interpretation of much that has been written on the subject of the motion of inviscid fluids in the neighbourhood of solid obstacles.

The principal object of the present communication is to test the first of the two latter suggestions. It will appear that, as in the case of motion between parallel plane walls, so also for the case of a tube of circular section, no disturbance of the steady motion is exponentially unstable, provided viscosity be altogether ignored.

Referring the motion to cylindrical coordinates $z, r, \theta$, parallel to which the component velocities are $w, u$, $v$, we have $\dagger$

$$
\begin{gathered}
\frac{\partial u}{\partial t}-\frac{v^{2}}{r}=\frac{d \mathrm{Q}}{d r}, \quad \frac{\partial v}{\partial t}+\frac{u v}{r}=\frac{1}{r} \frac{d \mathrm{Q}}{d \theta}, \quad \frac{\partial w}{\partial t}=\frac{d \mathrm{Q}}{d z}, \\
\frac{\partial}{\partial t}=\frac{d}{d t}+u \frac{d}{d r}+\frac{v}{r} \frac{d}{d \theta}+w \frac{d}{d z},
\end{gathered}
$$

where $-\mathrm{Q}=\mathrm{V}+p / \rho$.
These are the general equations. In order to apply them to the present problem of small disturbances from a steady motion represented by

$$
u=0, v=0, w=\mathrm{W},
$$

where W is a function of $r$ only, we will regard the complete motion as expressed by $u, v, W+w$, and neglect the squares of the small quantities $u, v, w$, which express the disturbance.

[^2]Thus,

$$
\begin{gather*}
\frac{d u}{d t}+\mathrm{W} \frac{d u}{d z}=\frac{d \mathrm{Q}}{d r},  \tag{1}\\
\frac{d v}{d t}+\mathrm{W} \frac{d v}{d z}=\frac{d \mathrm{Q}}{r d \theta},  \tag{2}\\
u \frac{d \mathrm{~W}}{d r}+\frac{d w}{d t}+\mathrm{W} \frac{d w}{d z}=\frac{d \mathrm{Q}}{d z}, \tag{3}
\end{gather*}
$$

which, with the " equation of continuity,"

$$
\begin{equation*}
\frac{d(r u t)}{d r}+\frac{d v}{d \theta}+r \frac{d w}{d z}=0, \tag{4}
\end{equation*}
$$

determine the motion.
The next step is to introduce the supposition that as functions of $t, z, \theta$, the variables $u, v, v$, and Q are proportional to $e^{i(n t+k z+\theta)}$.

We get

$$
\begin{array}{r}
i(n+k \mathrm{~W}) u=\frac{d \mathrm{Q}}{d r}, \quad(n+k \mathrm{~W}) v=\frac{s}{r} \mathrm{Q}, . \\
u \frac{d \mathrm{~W}}{d r}+i(n+k \mathrm{~W}) w=i k \mathrm{Q}, . \\
\frac{d}{d r}(r u)+i s v+i k r w=0 . \tag{7}
\end{array} .
$$

From these equations three of the variables may be eliminated, so as to obtain an equation in which the fourth is isolated. The simplest result is that in which $Q$ is retained. It is

$$
\begin{equation*}
\frac{d^{2} \mathrm{Q}}{d r^{2}}+\frac{1}{r} \frac{d \mathrm{Q}}{d r}-\mathrm{Q}\left(\frac{s^{2}}{r^{2}}+k^{2}\right)-\frac{2 k}{n+k \overline{\mathrm{~W}}} \frac{d \mathrm{~W}}{d r} \frac{d \mathrm{Q}}{d r}=0 . \tag{8}
\end{equation*}
$$

But the equation in $u$ lends itself more readily to the imposition of boundary conditions. If $s=0$, that is in the case of symmetrical disturbances, the equation in $u$ is obtained at once by differentiation of (8), and substitution of $u$ from (5). After reduction it becomes

$$
\begin{align*}
(n+k W)\left\{\frac{d^{2} u}{d r^{2}}\right. & \left.+\frac{1}{r} \frac{d u}{d r}-\frac{u}{r^{2}}-k^{2} u\right\} \\
& -k u\left\{\frac{d^{2} W}{d r^{2}}-\frac{1}{r} \frac{d W}{d r}\right\}=0 . \tag{9}
\end{align*}
$$

If the undisturbed motion be that of a highly viscous fluid in a circular tube, W is of the form $\mathrm{A}+\mathrm{B} r^{2}$, and the second
part of (9) disappears. There can then be admitted no values of $n$, except such as make $n+k W=0$ for some value of $r$ included within the tube. For the equation

$$
\begin{equation*}
\frac{d^{2} u}{d r^{2}}+\frac{1}{r} \frac{d u}{d r}+\frac{u}{r^{2}}-k^{2} u=0, . . \tag{10}
\end{equation*}
$$

being that of the Bessel's function of the first order with a purely imaginary argument, admits of no solution consistent with the conditions that $u=0$ when $r$ vanishes, and also when $r$ has the finite value appropriate to the wall of the tube. But any value assumed by $-k W$ is an admissible solution for $n$. At the place where $n+k W=0$, (10) need not be satisfied, and under this exemption the required solution may be obtained consistently with the boundary conditions. It is included in the above statement that no admissible value of $n$. can include an imaginary part.

If $s$ be not zero, we have in transforming to $u$ to include also terms arising from the differentiation in (8) of $-\mathrm{Q} s^{2} / r^{2}$, that is

$$
2 s^{2} r^{-3} \mathrm{Q}-\frac{s^{2}}{r^{2}} \frac{d \mathrm{Q}}{d r}
$$

for the second of which we substitute from (5), and for the first from (8) itself. The result is

$$
\begin{align*}
(n+k W) & {\left[\frac{d^{2} u}{d r^{2}}+\frac{1}{r} \frac{d u}{d r} \frac{3 s^{2}+k^{2} r^{2}}{s^{2}+k^{2} r^{2}}\right.} \\
+ & \left.\frac{u}{r^{2}}\left\{-1-k^{2} r^{2}-s^{2}+\frac{2 s^{2}}{s^{2}+k^{2} r^{2}}\right\}\right] \\
= & k u\left\{\frac{d^{2} \mathrm{~W}}{d r^{2}}-\frac{1}{r} \frac{d \mathrm{~W}}{d r} \frac{k^{2} r^{2}-s^{2}}{k^{2} r^{2}+s^{2}}\right\} \tag{11}
\end{align*}
$$

From (11) we may fall back on the case of two dimensions by supposing $r$ to be infinite. But, in order not to lose generality, we must at the same time allow $s$ to be infinite, so that, for example, $s=k^{\prime} r$. Thus, writing $x$ for $r$, and $y$ for $r \theta$, we find for the differential equation applicable to the solution in which all the quantities are proportional to $e^{i\left(n t+k z+k^{\prime} y\right)}$,

$$
\begin{equation*}
(n+k \mathrm{~W})\left\{\frac{d^{2} u}{d x^{2}}-k^{2} u-k^{\prime 2} u\right\}=k u \frac{d^{2} \mathrm{~W}}{d x^{2}}, \tag{12}
\end{equation*}
$$

agreeing with that formerly discussed except for a slight difference of notation.

We will now consider (11) in the abbreviated form,

$$
(n+k \mathrm{~W})\left\{\frac{d^{2} u}{d r^{2}}+\frac{a}{r} \frac{d u}{d r}+b \frac{u}{r^{2}}\right\}=\mathrm{W}_{1} k u
$$

where $a$ is a positive number not less than unity ; or, again,

$$
\begin{equation*}
\frac{d}{d r}\left(r^{a} \frac{d u}{d r}\right)+b r^{a-2} u=\frac{k u r^{a} W_{1}}{n+k W^{*}} \tag{13}
\end{equation*}
$$

The question proposed for consideration is whether (13) admits of a solution with a complex value of $n$, subject to the conditions that for two values of $r$, say $r_{1}$ and $r_{2}, u$ shall vanish. This represents the flow of fluid through a channel bounded by two coaxal cylinders.

Suppose, then, that $n$ is of the form $p+i q$, and $u$ of the form $\alpha+i \beta$, where $p, q, \alpha, \beta$ are real. Separating the real and imaginary parts in (13), we get

$$
\begin{align*}
& \frac{d}{d r}\left(r^{a} \frac{d \alpha}{d r}\right)+b r^{a-2} \alpha=\frac{k r^{a} W_{1}}{(p+k W)^{2}+q^{2}}\{(p+k \mathrm{~W}) \alpha+q \beta\}  \tag{14}\\
& \frac{d}{d r}\left(r^{a} \frac{d \beta}{d r}\right)+b r^{\alpha-2} \beta=\frac{k r^{a} W_{1}}{(p+k W)^{2}+q^{2}}\{(p+k \mathrm{~W}) \beta-q \alpha\} \tag{15}
\end{align*}
$$

and thence

$$
\begin{equation*}
\beta \frac{d}{d r}\left(r \frac{d \alpha}{d r}\right)-\alpha \frac{d}{d r}\left(r^{a} \frac{d \beta}{d r}\right)=\frac{k r^{a} \mathrm{~W}_{1}\left(\alpha^{2}+\beta^{2}\right) \cdot q}{(p+k W)^{2}+q^{2}} \tag{16}
\end{equation*}
$$

We now integrate this equation with respect to $r$ over the space between the walls, viz., from $r_{1}$ to $r_{2}$. The integral of the left-hand member is

$$
\begin{equation*}
\beta r^{a} \frac{d \alpha}{d r}-\alpha r^{a} \frac{d \beta}{d r}, . \quad . \quad . \quad . \tag{17}
\end{equation*}
$$

and this vanishes at both limits, $\beta$ and $\alpha$ being there zero. The integral of the right-hand member of (16) is accordingly zero, from which it follows that if $\mathrm{W}_{1}$ be of one sign throughout, $q$ must vanish-that is to say, no complex value of $n$ is admissible.

The general value of $W_{1}$, viz.,

$$
\begin{equation*}
\frac{d^{2} \mathrm{~W}}{d r^{2}}-\frac{1}{r} \frac{d W}{d r} \frac{k^{2} r^{2}-s^{2}}{k^{2} r^{2}+s^{2}}, \quad . \quad . \quad . \tag{18}
\end{equation*}
$$

reduces in the case of two dimensions to $d^{2} W / d r^{2}$, or, as we may then write it, $d^{2} \mathrm{~W} / d x^{2}$. Instability, at any rate of the

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full-blown exponential sort, is thus excluded, provided $d^{2} \mathrm{~W} / d x^{2}$ is of one sign throughout the entire region of flow limited by the two parallel plane walls.

Commenting upon this argument, Lord Kelvin * remarks that the disturbing infinity, which arises in (13) when $n$ has a value such that $n+k W$ vanishes at some point in the field of motion, "vitiates the seeming proof of stability." Perhaps I went too far in asserting that the motion was thoroughly stable ; but it is to be observed that if $n$ be complex, there is no "disturbing infinity." The argument, therefore, does not fail, regarded as one for excluding complex values of $n$. What happens when $n$ has a real value such that $n+k W$ vanishes at an interior point, is a subject for further examination.

The condition for two dimensions that $d^{2} \mathrm{~W} / d x^{2}$ is of one sign throughout is satisfied for a law of flow such as that of a viscous fluid, and we shall see that the corresponding condition for (17) in the more general problem is also satisfied in the case of the steady flow of a viscous fluid between cylindrical walls at $r_{1}$ and $r_{2}$. The most general form of $W$ for steady motion symmetrical about the axis is $\dagger$

$$
\begin{equation*}
\mathrm{W}=\mathrm{A} r^{2}+\mathrm{B} \log r+\mathrm{C}, \tag{19}
\end{equation*}
$$

in which the constants $\mathrm{A}, \mathrm{B}, \mathrm{C}$ are related by the conditions

$$
\begin{aligned}
& 0=\mathrm{A} r_{1}{ }^{2}+\mathrm{B} \log r_{1}+\mathrm{C}, \\
& 0=\mathrm{A} r_{2}^{2}+\mathrm{B} \log r_{2}+\mathrm{C} .
\end{aligned}
$$

From the last two equations we derive

$$
\begin{equation*}
\mathrm{A}\left(r_{2}^{2}-r_{1}^{2}\right)+\mathrm{B} \log r_{2} / r_{1}=0, \tag{20}
\end{equation*}
$$

so that $A$ and $B$ have opposite signs. Introducing the value of W from (18), we obtain as the special form here applicable

$$
\frac{4 s^{2} \mathrm{~A}-2 k^{2} \mathrm{~B}}{k^{2} r^{2}+s^{2}}
$$

which is thus of one sign throughout the range. A small disturbance from the steady motion expressed by (19) is therefore not exponentially unstable.

The result now obtained is applicable however small may be the inner radius $r_{1}$ of the annular channel. But the extension to the case of the ordinary pipe of unobstructed circular section may be thought precarious, when it is remembered that provision must be made for a possible finite value of $u$ wheu $r=0$. But although $a$ and $\beta$ may be finite at the lower

[^3]limit, the annulment of (17) is secured by the factor $r^{a}$; so that complex values of $n$ are still excluded, provided $W$ be of unchangeable sign. In the present case the B of (19) vanishes, and we have
$$
\frac{d^{2} \mathrm{~W}}{d r^{2}}=2 \mathrm{~A}, \quad \frac{1}{r} \frac{d \mathrm{~W}}{d r}=2 \mathrm{~A} ;
$$
so that (18) gives
$$
\mathrm{W}_{1}=\frac{4 s^{2} \mathrm{~A}}{k^{2} r^{2}+s^{2}},
$$
satisfying the prescribed condition.
The difficulty in reconciling calculation and experiment is accordingly not to be explained by any peculiarity of the twodimensional motion to which calculation was first applied. It may indeed be argued that the instabilities excluded are only those of the exponential type, and that there may remain others on the borderland of the form $t \cos t$, \&c. But if the above calculations are really applicable to the limiting case of a viscous fluid when the viscosity is infinitely small, we should naturally expect to find that the smallest sensible viscosity would convert the feebly unstable disturbance into one distinctly stable, and if so the difficulty remains. Speculations on such a subject in advance of definite arguments are not worth much; but the impression upon my mind is that the motions calculated above for an absolutely inviscid liquid may be found inapplicable to a viscid liquid of vanishing viscosity, and that a more complete treatment might even yet indicate instability, perhaps of a local character, in the immediate neighbourhood of the walls, when the viscosity is very small.

It is on the basis of such a complete treatment, in which the terms representing viscosity in the general equations are retained, that Lord Kelvin $\dagger$ arrives at the conclusion that the flow of viscous fluid between two parallel walls is fully stable for infinitesimal disturbances, however small the amount of the viscosity may be. Naturally, it is with diffidence that I hesitate to follow so great an authority, but I must confess that the argument does not appear to me demonstrative. No attempt is made to determine whether in free disturbances of the type $e^{\text {int }}$ (in his notation $e^{i \omega t}$ ) the imaginary part of $n$ is finite, and if so whether it is positive or negative. If I rightly understand it, the process consists in an investigation of forced vibrations of arbitrary (real) frequency, and the conclusion depends upon a tacit assumption that if these forced

[^4]F 2
vibrations can be expressed in a periodic form the steady motion from which they are deviations cannot be unstable. A very simple case suffices to prove that such a principle could not be admitted. The equation to the motion of the bob of a pendulum situated near the highest point of its orbit is

$$
\begin{equation*}
\frac{d^{2} x}{d t^{2}}-m^{2} x=\mathbf{X} \tag{21}
\end{equation*}
$$

where $X$ is an impressed force. If $X=\cos p t$, the corresponding part of $x$ is

$$
\begin{equation*}
x=-\frac{\cos p t}{p^{2}+m^{2}} ; \quad . \quad . \quad . \quad . \tag{22}
\end{equation*}
$$

but this gives no indication of the inherent instability of the situation expressed by the free "vibrations,"

$$
\begin{equation*}
x=\mathrm{A} e^{m t}+\mathrm{B} e^{-m t} \tag{23}
\end{equation*}
$$

As a preliminary to a more complete investigation, it may be worth while to indicate the solution of the problem for the two-dimensional motion of viscons liquid between two parallel planes, in the relatively very simple case where there is no foundation of steady motion. The equation, given in Lord Kelvin's paper, for the motion of type $e^{i(n t+k z)}$ is

$$
\begin{equation*}
i \mu\left(\frac{d^{4} u}{d x^{4}}-2 k^{2} \frac{d^{2} w}{d x^{2}}+k^{4} u\right)+n\left(\frac{d^{2} w}{d x^{2}}-k^{2} u\right)=0 \tag{24}
\end{equation*}
$$

The boundary conditions, say at $x= \pm a$, are that $u,(v)$, and $w$ shall then vanish, or by (7) that

$$
u=0, \quad d u / d x=0
$$

The following would then be the proof from the differential equation that for all the admissible values of $n, p$ is zero and $q$ is positive.

Writing as before, $u=a+i \beta$, and separating the real and imaginary parts, we find

$$
\begin{align*}
-\mu\left(\frac{d^{2}}{d x^{2}}-k^{2}\right)^{2} \beta+p\left(\frac{d^{2} a}{d x^{2}}-k^{2} a\right)-q\left(\frac{d^{2} \beta}{d x^{2}}-k^{2} \beta\right)=0  \tag{25}\\
\mu\left(\frac{d^{2}}{d x^{2}}-k^{2}\right)^{2} a+p\left(\frac{d^{2} \beta}{d x^{2}}-k^{2} \beta\right)+q\left(\frac{d^{2} a}{d x^{2}}-k^{2} a\right)=0 . \tag{26}
\end{align*}
$$

Multiply (25), (26) by $a, \beta$ respectively, add and integrate with respect to $x$ over the range of the motion. The coefficient of $q$ is

$$
\int\left\{\beta \frac{d^{2} a}{d x^{2}}-a \frac{d^{2} \beta}{d x^{2}}\right\} d x
$$

and this is equal to zero in virtue of the conditions at the limits. In like manner the coefficient of $\mu$ is zero, as appears on successive integrations by parts. The coefficient of $p$ is

$$
-\int\left\{\left(\frac{d a}{d x}\right)^{2}+\left(\frac{d \beta}{d x}\right)^{2}+k^{2} a^{2}+k^{2} \beta^{2}\right\} d x ;
$$

so that $p=0$.
Again, multiply (25) by $\beta$, (26) by $a$, and subtract. On integration as before the coefficient of $q$ is

$$
\int\left\{\left(\frac{d a}{d x}\right)^{2}+\left(\frac{d \beta}{d x}\right)^{2}+k^{2} a^{2}+k^{2} \beta^{2}\right\} d x
$$

and that of $\mu$ is

$$
-\int\left\{\left(\frac{d^{2} a}{d x^{2}}\right)^{2}+\left(\frac{d^{2} \beta}{d x^{2}}\right)^{2}+2 k^{2}\left(\frac{d a}{d x}\right)^{2}+2 k^{2}\left(\frac{d \beta}{d x}\right)^{2}+k^{4} a^{2}+k^{4} \beta^{2}\right\} d x
$$

Hence $q$ has the same sign as $\mu$, that is to say, $q$ is positive. That $n$ in $e^{i n t}$ is a pure positive imaginary is no more than might have been inferred from general principles, seeing that the problem is one of the small motions about equilibrium of a system devoid of potential energy.

Since (24) is an equation with constant coefficients, the normal functions in this case are readily expressed. Writing it in the form

$$
\begin{equation*}
\left\{\frac{d}{d x^{2}}-k^{2}-\frac{i n}{\mu}\right\}\left\{\frac{d^{2}}{d x^{2}}-k^{2}\right\} u=0 \tag{27}
\end{equation*}
$$

we see that the four types of solution are

$$
e^{k x}, \quad e^{-k x}, \quad e^{i k^{\prime} x}, \quad e^{-i k^{\prime} x},
$$

where

$$
\begin{equation*}
-k^{\prime 2}=k^{2}+i n / \mu ; \tag{28}
\end{equation*}
$$

or, if we take advantage of what has just been proved,

$$
\begin{equation*}
k^{\prime 2}=q / \mu-k^{2}, \quad . \quad . \quad . \quad . \quad . \tag{29}
\end{equation*}
$$

where $q$ and $\mu$ are positive. It will be seen that the odd and even parts of the solution may be treated separately. Thus, for the first,

$$
\begin{equation*}
u=\mathrm{A} \sinh k x+\mathrm{B} \sin k^{\prime} x \tag{30}
\end{equation*}
$$

and the conditions to be satisfied at $x= \pm a$ give

$$
\left.\begin{array}{l}
0=\mathrm{A} \sinh k a+\mathrm{B} \sin k^{\prime} a  \tag{31}\\
0=k \mathrm{~A} \cosh k a+k^{\prime} \mathrm{B} \cos k^{\prime} a
\end{array}\right\} ;
$$

so that the equation for $k^{\prime}$ is

$$
\begin{equation*}
\frac{\tan k^{\prime} a}{k^{\prime} a}=\frac{\tanh k a}{k a} \tag{32}
\end{equation*}
$$

Again, for the solution involving the even functions, $u=\mathrm{C} \cosh k x+\mathrm{D} \cos k^{\prime} x,$.
where

$$
\begin{equation*}
\frac{\cot k^{\prime} a}{k^{\prime} a}=-\frac{\operatorname{coth} k a}{k a} \tag{33}
\end{equation*}
$$

Equations (32), (34) give an infinite number of real values for $k^{l}$, and when these are known $q$, and $\ddot{n}$, follow from (29).

The most persistent motion (for which $q$ is smallest) corresponds to a small value of $k$, and to the even functions of (33). In this case from (34)

$$
k^{\prime} a=\pi, 2 \pi, 3 \pi, \& c .
$$

the first of which gives as the smallest value of $q$

$$
\begin{equation*}
q=\mu \pi^{2} / a^{2} \tag{35}
\end{equation*}
$$

The corresponding form for $u$ is

$$
\begin{equation*}
u=e^{i k z-q t}(1+\cos (\pi x / a)) \tag{36}
\end{equation*}
$$

This type of motion is represented by the arrows in the following diagram :-


On the other hand the smallest value of $q$ under the head of the odd functions is

$$
\begin{equation*}
q=\mu \pi^{2}(1 \cdot 4303)^{2} / a^{2} \tag{37}
\end{equation*}
$$

and the motion is of the type


Terling Place, Witham.
IX. Rotating Elastic Solid Cylinders of Elliptic Section. By C. Chree, M.A., Fellow of King's College, Cambridge*.

Part I.-The Short Elliptic Cylinder or Disk.

IN the 'Quarterly Journal of . . . Mathematics,' vol. xxiii. pp. 16-33, I considered various cases of isotropic elastic solids rotating with uniform angular velocity about an axis * Communicated by the Author.


[^0]:    * Communicated by the Author.

[^1]:    * Proc. Math. Soc. February 12, 1880.
    $\dagger$ Ibid. November 1887.

[^2]:    * "On the Circulation of Air in Kundt's Tubes," Phil. Trans. November 1883.
    $\dagger$ Basset's 'Hydrodynamics,' § 470.

[^3]:    * Phil. Mag. Aug. 1887, p. 275.
    $\dagger$ Basset's 'Hydrodynamics,' § 514.

[^4]:    * Phil. Mag. Aug. and Sept. 1887.

