XXXIX. *On the Electromagnetic Effects due to the Motion of Electrification through a Dielectric.* By OLIVER HEAVISIDE *.

1. THE following paper consists of, First, a short discussion of the theory of the *slow* motion of an electric charge through a dielectric, having for object the possible correction of previously published results. Secondly, a discussion of the theory of the electromagnetic effects due to motion of a charge at any speed, with the development of the complete solution in finite form when the motion is steady and rectilinear. Thirdly, a few simple illustrations of the last when the charge is distributed.

Given a steady electric field in a dielectric, due to electrification. It is sufficient to consider a charge q at a point, as we may readily extend results later. If this charge be shifted from one position to another, the displacement varies. In accordance, therefore, with Maxwell's inimitable theory of a dielectric, there is electric current produced. Its time-integral, which is the total change in the displacement, admits of no question; but it is by no means an elementary matter to settle its rate of change in general, or the electric current. But should the speed of the moving charge be only a very small fraction of that of the propagation of disturbances, or that of light, it is clear that the accommodation of the displacement to the new positions which are assumed by the charge during its motion is practically instantaneous in its neighbourhood, so that we may imagine the charge to carry about its stationary field of force rigidly attached to it. This fixation of the displacement at any moment definitely fixes the displacement-current. We at once find, however, that to close the current requires us to regard the moving charge itself as a current-element, of moment equal to the charge multiplied by its velocity; understanding by moment, in the case of a distributed current, the product of current-density and volume. The necessity of regarding the moving charge as an element of the "true current" may be also concluded by simply considering that when a charge q is conveyed *into* any region, an equal displacement simultaneously leaves it through its boundary.

Knowing the electric current, the magnetic force to correspond becomes definitely known if the distribution of inductivity be given; and when this is constant everywhere, as we shall suppose now and later, the magnetic force is simply

* Communicated by the Author.

the vector of no divergence whose curl is 4π times the electric current; or the vector-potential of the curl of the current; or the curl of the vector-potential of the current, &c. &c. Thus, as found by J. J. Thomson *, the magnetic field of a charge moving at a speed which is a small fraction of that of light is that which is commonly ascribed to a current-element itself. I think it, however, preferable to regard the magnetic field as the primary object of attention; or else to regard the complete system of closed current derived from it by taking its curl as the unit, forming what we may term a rational current-element, inasmuch as it is not a mere mathematical abstraction, but is a complete dynamical system involving definite forces and energy.

2. Let the axis of z be the line of motion of the charge q at the speed u; then the lines of magnetic force $\mathbf{H}$ are circles centred upon the axis, in planes perpendicular to it, and its tensor H at distance r from the charge, the line $\mathbf{r}$ making an angle θ with the axis, is given by

$$\mathrm{H}=\frac{q}{r^2}u\sin\theta=c\mathrm{E}u\nu, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\nu=\sin\theta$, E the intensity of the radial electric force, c the permittivity such that $\mu_0 cv^2=1$, if μ_0 is the other specific quality of the medium, its inductivity, and v is the speed of propagation.

Since, under the circumstance supposed of u/v being very small, the alteration in the electric field is insensible, and the lines of $\mathbf{E}$ are radial, we may terminate the fields represented by (1) at any distance $r=a$ from the origin. We then obtain the solution in the case of a charge q upon the surface of a conducting sphere of radius a, moving at speed u. This realization of the problem makes the electric and magnetic energies finite. Whilst, however, agreeing with J. J. Thomson in the fundamentals, I have been unable to corroborate some of his details; and since some of his results have been recently repeated by him in another place †, it may be desirable to state the changes I propose, before proceeding to the case of a charge moving at any speed.

3. First, as regards the magnetic energy, say T. This is the space-summation $\Sigma\mu_0\mathrm{H}^2/8\pi$; or, by (1) ‡,

$$\mathrm{T}=\frac{\mu_0 q^2u^2}{8\pi}\iiint\frac{\nu^2}{r^2}dr\,d\mu\,d\phi=\frac{\mu_0 q^2u^2}{3a}. \quad . \quad . \quad . \quad (2)$$

* Phil. Mag. April 1881.

† 'Applications of Dynamics to Physics and Chemistry,' chap. iv. pp. 31 to 37.

‡ 'Electrician,' Jan. 24, 1885, p. 220.

The limits are such as include all space outside the sphere $r=a$. The coefficient $\frac{1}{3}$ replaces $\frac{2}{15}$.

4. Next, as regards the mutual magnetic energy M of the moving charge and any external magnetic field. This is the space-summation $\Sigma\mu_0\mathbf{H}_0\mathbf{H}/4\pi$, if $\mathbf{H}_0$ is the external field; and, by a well-known transformation, it is equivalent to $\Sigma\mathbf{A}_0\boldsymbol{\Gamma}$, if $\mathbf{A}_0$ is any vector whose curl is $\mu_0\mathbf{H}_0$, whilst $\boldsymbol{\Gamma}$ is the current-density of the moving system. Further, if we choose $\mathbf{A}_0$ to have no divergence, the polar part of $\boldsymbol{\Gamma}$ will contribute nothing to the summation, so that we are reduced to the volume-integral of the scalar product of the divergenceless $\mathbf{A}_0$ of the one system and the density of the convection-current in the other. Or, in the present case, with a single moving charge at a point, we have simply the scalar product $\mathbf{A}_0\mathbf{u}q$ to represent the mutual magnetic energy; or

$$\mathrm{M}=\mathbf{A}_0\mathbf{u}q, \quad \ldots \ldots \quad (3)$$

which is double J. J. Thomson's result.

5. When, therefore, we derive from (3) the mechanical force on the moving charge due to the external magnetic field, we obtain simply Maxwell's "electromagnetic force" on a current-element, the vector product of the moment of the current and the induction of the external field; or, if $\mathbf{F}$ is this mechanical force,

$$\mathbf{F}=\mu_0 q\mathrm{V}\mathbf{u}\mathbf{H}_0, \quad \ldots \ldots \quad (4)$$

which is also double J. J. Thomson's result. Notice that in the application of the "electromagnetic force" formula, it is the moment of the convection-current that occurs. This is not the same as the moment of the true current, which varies according to circumstances; for instance, in the case of a small dielectric sphere uniformly electrified throughout its volume, the moment of the true current would be only $\frac{2}{3}$ of that of the convection-current.

The application of Lagrange's equation of motion to (3) also gives the force on q due to the electric field so far as it can depend on M; that is, a force

$$-q\frac{d\mathbf{A}_0}{dt},$$

where the time-variation due to all causes must be reckoned, except that due to the motion of q itself, which is allowed for in (4). And besides this, there may be electric force not derivable from $\mathbf{A}_0$, viz.

$$-q\nabla\Psi_0,$$

where Ψ_0 is the scalar potential companion to $\mathbf{A}_0$.

6. Now if the external field be that of another moving charge, we shall obtain the mutual magnetic energy from (3) by letting $\mathbf{A}_0$ be the vector-potential of the current in the second moving system, constructed so as to have no divergence. Now the vector-potential of the convection-current $q\mathbf{u}$ is simply $q\mathbf{u}/r$; this is sufficient to obtain the magnetic force by curling; but if used to calculate the mutual energy, the space-summation would have to include every element of current in the other system. To make the vector-potential divergenceless, and so be able to abolish this work, we must add on to $q\mathbf{u}/r$ the vector-potential of the displacement current to correspond. Now the complete current may be considered to consist of a linear element $q\mathbf{u}$ having two poles; a radial current outward from the + pole in which the current-density is $qu/4\pi r_1^2$; and a radial current inward to the − pole, in which the current-density is $-qu/4\pi r_2^2$; where r_1 and r_2 are the distances of any point from the poles. The vector-potentials of these currents are also radial, and their tensors are $\frac{1}{2}qu$ and $-\frac{1}{2}qu$. We have now merely to find their resultant when the linear element is indefinitely shortened, add on to the former $q\mathbf{u}/r$, and multiply by μ_0, to obtain the complete divergenceless vector-potential of $q\mathbf{u}$, viz.:—

$$\mathbf{A}=\mu_0\frac{q}{r}\left(\mathbf{u}-\tfrac{1}{2}u\nabla\frac{dr}{ds}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where r is the distance from q to the point P when $\mathbf{A}$ is reckoned, and the differentiation is to s the axis of the convection-current. Both it and the space-variation are taken at P. The tensor of $\mathbf{u}$ is u. Though different and simpler in form (apart from the use of vectors) this vector-potential is, I believe, really the same as the one used by J. J. Thomson. From it we at once find, by the method described in § 4, the mutual energy of a pair of point-charges q_1 and q_2 moving at velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ to be

$$\mathrm{M}=\frac{\mu_0 q_1 q_2}{r}\left(\mathbf{u}_1\mathbf{u}_2-\tfrac{1}{2}u_1u_2\frac{d^2r}{ds_1ds_2}\right), \quad . \quad . \quad . \quad . \quad (6)$$

when at distance r apart. Both axial differentiations are to be effected at one end of the line $\mathbf{r}$.

As an alternative form, let ϵ be the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$, and let the differentiation to s_1 be at ds_1, that to s_2 at ds_2, as in the German investigations relating to current-elements; then *

$$\mathrm{M}=\frac{\mu_0 q_1 q_2 u_1 u_2}{r}\left(\cos\epsilon+\tfrac{1}{2}\frac{d^2r}{ds_1ds_2}\right). \quad . \quad . \quad . \quad (7)$$

* 'Electrician,' Dec. 28, 1888, p. 230.

Another form, to render its meaning plainer. Let λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2 be the direction-cosines of the elements referred to rectangular axes, with the x-axis, to which λ_1 and λ_2 refer, chosen as the line joining the elements. Then*

$$\mathrm{M}=\frac{\mu_0 q_1 q_2 u_1 u_2}{2r}(2\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2). \quad . \quad . \quad (8)$$

J. J. Thomson's estimate is †

$$\mathrm{M}=\tfrac{1}{3}\mu_0 q_1 q_2 u_1 u_2 \frac{\cos\epsilon}{r} . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Comparing this with (8) we see that there is a notable difference.

7. The mutual energy being different, the forces on the charges, as derived by J. J. Thomson by the use of Lagrange's equations, will be different. When the speeds are constant, we shall have simply the before-described vector product (4) for the "electromagnetic force;" or

$$\mathbf{F}_1=\mu_0 q_1 \mathrm{V}\mathbf{u}_1\mathbf{H}_2, \quad \mathbf{F}_2=\mu_0 q_2 \mathrm{V}\mathbf{u}_2\mathbf{H}_1 . \quad . \quad . \quad (10)$$

if $\mathbf{F}_1$ is the electromagnetic force on the first and $\mathbf{F}_2$ that on the second element, whilst $\mathbf{H}_1$ and $\mathbf{H}_2$ are the magnetic forces. Similar changes are needed in the other parts of the complete mechanical forces.

It may be remarked that (if my calculations are correct) equation (7) or its equivalents expresses the mutual energy of any two rational current-elements [see § 1] in a medium of uniform inductivity, of moments $q_1 u_1$ and $q_2 u_2$, whether the currents be of displacement, or conduction, or convection, or all mixed, it being in fact the mutual energy of a pair of definite magnetic fields. But, since the hypothesis of instantaneous action is expressly involved in the above, the application of (7) is of a limited nature.

8. Now leaving behind altogether the subject of current-elements, in the investigation of which one is liable to be led away from physical considerations and become involved in mere exercises in differential coefficients, and coming to the question of the electromagnetic effects of a charge moving in any way, I have been agreeably surprised to find that my solution in the case of steady rectilinear motion, originally an infinite series of corrections, easily reduces to a very simple and interesting finite form, provided u be not greater than v. Only when $u > v$ is there any difficulty. We must first settle

* 'Electrician,' Jan 24, 1885, p. 221.

† 'Applications of Dynamics to Physics and Chemistry,' chap. iv.; and Phil. Mag. April 1881.

upon what basis to work. First the Faraday-law (p standing for d/dt),

$$-\text{curl } \mathbf{E} = \mu_0 p\mathbf{H}, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

requires no change when there is moving electrification. But the analogous law of Maxwell, which I understand to be really a *definition* of electric current in terms of magnetic force, (or a doctrine), requires modification if the true current is to be

$$\mathbf{C} + p\mathbf{D} + \rho\mathbf{u}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

viz. the sum of conduction-current, displacement-current, and convection-current $\rho\mathbf{u}$, where ρ is the volume-density of electrification. The addition of the term $\rho\mathbf{u}$ was, I believe, proposed by G. F. Fitzgerald *.

[This was not meant exactly for a new proposal, being in fact after Rowland's experiments; besides which, Maxwell was well acquainted with the idea of a convection-current. But what is very strange is that Maxwell, who insisted so strongly upon his doctrine of the *quasi*-incompressibility of electricity, never formulated the convection-current in his treatise. Now Prof. Fitzgerald pointed out that if Maxwell, in his equation of mechanical force,

$$\mathbf{F} = \mathrm{V}\mathbf{CB} - e\nabla\Psi - m\nabla\Omega,$$

had written $\mathbf{E}$ for $-\nabla\Psi$, as it is obvious he should have done, then the inclusion of convection-current in the true current would have followed naturally. (Here $\mathbf{C}$ is the true current, $\mathbf{B}$ the induction, e the density of electrification, m that of imaginary magnetic matter, Ψ the electrostatic and Ω the magnetic potential, and $\mathbf{E}$ the real electric force.)

Now to this remark I have to add that it is as unjustifiable to derive $\mathbf{H}$ from Ω as $\mathbf{E}$ from Ψ; that is, in general, the magnetic force is not the slope of a scalar potential; so, for $-\nabla\Omega$ we should write $\mathbf{H}$, the real magnetic force.

But this is not all. There is possibly a fourth term in $\mathbf{F}$, expressed by $4\pi\mathrm{V}\mathbf{DG}$, where $\mathbf{D}$ is the displacement and $\mathbf{G}$ the magnetic current; I have termed this force the "magneto-electric force," because it is the analogue of Maxwell's "electromagnetic force" $\mathrm{V}\mathbf{CB}$. Perhaps the simplest way of deriving it is from Maxwell's electric stress, which was the method I followed †.

Thus, in a homogeneous nonconducting dielectric free from electrification and magnetization, the mechanical force is the sum of the "electromagnetic" and the "magnetoelec-

* Brit. Assoc., Southport, 1883.

† "El. Mag. Ind. and its Prop." xxii. 'Electrician,' Jan. 15, 1886, p. 187.

tric," and is given by

$$\mathbf{F}=\frac{1}{v^2}\frac{d\mathbf{W}}{dt},$$

where $\mathbf{W}=\mathrm{V}\mathbf{EH}/4\pi$ is the transfer-of-energy vector.

It must, however, be confessed that the real distribution of the stresses, and therefore of the forces, is open to question. And when æther is the medium, the mechanical force in it, as for instance in a light-wave, or in a wave sent along a telegraph-circuit, is not easily to be interpreted.]

The companion to (11) in a nonconducting dielectric is now

$$\text{curl}\ \mathbf{H}=cp\mathbf{E}+4\pi\rho\mathbf{u}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Eliminate $\mathbf{E}$ between (11) and (13), remembering that $\mathbf{H}$ has no divergence, because μ_0 is constant, and we get

$$(p^2/v^2-\nabla^2)\mathbf{H}=\text{curl}\ 4\pi\rho\mathbf{u}, \quad . \quad . \quad . \quad . \quad (14)$$

the characteristic of $\mathbf{H}$. Here $\nabla^2=d^2/dx^2+\ldots$, as usual.

Comparing (14) with the characteristic of $\mathbf{H}$ when there is impressed force e instead of electrification ρ, which is

$$(p^2/v^2-\nabla^2)\mathbf{H}=\text{curl}\ cp\mathbf{e},$$

we see that $\rho\mathbf{u}$ becomes $cp\mathbf{e}/4\pi$. We may therefore regard convection-current as *impressed* electric current. From this comparison also, we may see that an infinite plane sheet of electrification of uniform density cannot produce magnetic force by motion perpendicular to its plane. Also we see that the sources of disturbances when ρ is moved are the places where $\rho\mathbf{u}$ has curl; for example, a dielectric sphere uniformly filled with electrification (which is imaginable), when moved, starts the magnetic force solely upon its boundary.

The presence of "curl" on the right side tells us, as a matter of mathematical simplicity, to make $\mathbf{H}$/curl the variable. Let

$$\mathbf{H}=\text{curl}\ \mathbf{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and calculate $\mathbf{A}$, which may be any vector satisfying (15). Its characteristic is

$$(p^2/v^2-\nabla^2)\mathbf{A}=4\pi\rho\mathbf{u}. \quad . \quad . \quad . \quad . \quad (16)$$

The divergence of $\mathbf{A}$ is of no moment, and it is only vexatious complication to introduce Ψ. The time-rate of decrease of $\mathbf{A}$ is not the real distribution of electric force, which has to be found by the additional datum

$$\text{div}\ c\mathbf{E}=4\pi\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where $\mathbf{E}$ is the real force.

9. "Symbolically" expressed, the solution of (16) is

$$\mathbf{A}=\frac{4\pi\rho\mathbf{u}}{p^2/v^2-\nabla^2}=\frac{-4\pi\rho\mathbf{u}/\nabla^2}{1-p^2/v^2\nabla^2}. \quad . \quad . \quad . \quad . \quad (18)$$

Here the numerator of the fraction to the right is the vector-potential of the convection-current. Calling it $\mathbf{A}_0$, we have

$$\mathbf{A}_0=\frac{4\pi\rho\mathbf{u}}{-\nabla^2}=\Sigma\frac{\rho\mathbf{u}}{r}. \quad . \quad . \qquad . \quad . \quad (19)$$

Inserting in (18) and expanding, we have

$$\mathbf{A}=\{1+(p/v\nabla)^2+(p/v\nabla)^4+\ldots\}\mathbf{A}_0. \quad . \quad . \quad (20)$$

Given then $\rho\mathbf{u}$ as a function of position and time, $\mathbf{A}_0$ is known by (19), and (20) finds $\mathbf{A}$, whilst (15) finds $\mathbf{H}$.

10. When the motion of the electrification is all in one direction, say parallel to the z-axis, $\mathbf{u}$, $\mathbf{A}_0$, and $\mathbf{A}$ are all parallel to this axis, so that we need only consider their tensors. When there is simply one charge q at a point, we have

$$\mathrm{A}_0=qu/r,$$

and (20) becomes

$$\mathrm{A}=q\{1+(p/v\nabla)^2+(p/v\nabla)^4+\ldots\}(u/r) \quad . \quad (21)$$

at distance r from q. When the motion is steady, and the whole electromagnetic field is ultimately steady with respect to the moving charge, we shall have, taking it as origin,

$$p=-u(d/dz)=-u\mathrm{D}$$

for brevity; so that

$$\mathrm{A}=qu\{1+(u\mathrm{D}/v\nabla)^2+(u\mathrm{D}/v\nabla)^4+\ldots\}r^{-1}. \quad . \quad (22)$$

Now the property

$$\nabla^2 r^{n+2}=(n+2)(n+3)r^n \quad . \quad . \quad . \quad . \quad (23)$$

brings (22) to

$$\mathrm{A}=qu\left\{\frac{1}{r}+\frac{u^2}{v^2}\mathrm{D}^2\frac{r}{2\,!}+\frac{u^4}{v^4}\mathrm{D}^4\frac{r^3}{4\,!}+\ldots\right\}; \quad . \quad (24)$$

and the property

$$\mathrm{D}^{2n}r^{2n-1}=1^2.3^2.5^2\ldots(2n-1)^2\nu^{2n}/r, \quad . \quad . \quad (25)$$

where $\nu=\sin\theta$, θ being the angle between $\mathbf{r}$ and the axis, brings (24) to

$$\mathrm{A}=\frac{qu}{r}\left\{1+\frac{u^2}{v^2}\frac{\nu^2}{2}\left(1+\frac{u^2}{v^2}\frac{3}{4}\nu^2\left(1+\frac{u^2}{v^2}\frac{5}{6}\nu^2\left(1+\ldots\right.\right.\right.\right\}; \quad (26)$$

which, by the Binomial Theorem, is the same as

$$\mathrm{A}=(qu/r)\{1-u^2\nu^2/v^2\}^{-\frac{1}{2}}, \quad . \quad , \quad . \quad . \quad (27)$$

the required solution.

11. To derive H, the tensor of the circular $\mathbf{H}$, let $r\nu=h$, the distance from the axis. Then, by (15),

$$\mathrm{H}=-\frac{d\mathrm{A}}{dh}=-\nu\frac{d\mathrm{A}}{dr}+\frac{\mu\nu}{r}\frac{d\mathrm{A}}{d\mu}=\frac{qu\nu}{r^2}\left(1+\mu\frac{d}{d\mu}\right)\left(1-\frac{u^2}{v^2}\nu^2\right)^{-\frac{1}{2}}, \, . \, (28)$$

by (27), if $\mu=\cos\theta$. Performing the differentiation, and also getting out E the tensor of the electric force, we have the final result that the electromagnetic field is fully given by*

$$c\mathrm{E}=\frac{q}{r^2}\cdot\frac{1-u^2/v^2}{(1-u^2\nu^2/v^2)^{\frac{3}{2}}},\qquad \mathrm{H}=c\mathrm{E}u\nu, \quad . \quad . \quad (29)$$

with the additional information that **E** is radial and **H** circular.

Now, as regards Ψ, if we bring it in, we have only got to take it out again. When the speed is very slow we may regard the electric field as given by $-\nabla\Psi$ *plus* a small correcting vector, which we may call the electric force of inertia. But to show the *physical* inanity of Ψ, go to the other extreme, and let u nearly equal v. It is now the electric force of inertia (supposed) that equals $+\nabla\Psi$ nearly (except about the equatorial plane), and its sole utility or function is to cancel the other $-\nabla\Psi$ of the (supposed) electrostatic field. It is surely impossible to attach any physical meaning to Ψ and to propagate it, for we require two Ψ's, one to cancel the other, and both propagated infinitely rapidly.

As the speed increases, the electromagnetic field concentrates itself more and more about the equatorial plane, $\theta=\frac{1}{2}\pi$. To give an idea of the accumulation, let $u^2/v^2=\cdot 99$. Then $c\mathrm{E}$ is $\cdot 01$ of the normal value q/r^2 at the pole, and 10 times the normal value at the equator. The latitude where the value is normal is given by

$$\nu=(v/u)\left[1-(1-u^2/v^2)^{\frac{2}{3}}\right]^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (30)$$

12. When $u=v$, the solution (29) becomes a plane electromagnetic wave, E and H being zero everywhere except in the equatorial plane. As, however, the values of E and H are infinite, distribute the charge along a straight line moving in its own line, and let the linear-density be q. The solution is then †

$$\mathrm{H}=\mathrm{E}cv=2qv/r \quad . \quad . \quad . \quad . \quad . \quad (31)$$

at distance r from the line, between the two planes through the ends of the line perpendicular to it, and zero elsewhere.

To further realize, let the field terminate internally at $r=a$, giving a cylindrical-surface distribution of electrification, and terminate the tubes of displacement externally upon a coaxial cylindrical surface; we then produce a real electromagnetic plane wave with electrification, and of finite energy. We have supposed the electrification to be carried through the dielectric at speed v, to keep up with the wave, which would of course

* 'Electrician,' Dec. 7, 1888, p. 148.
† Ibid. Nov. 23, 1888, p. 84.

break up if the charge were stopped. But if perfectly-conducting surfaces be given on which to terminate the displacement, the natural motion of the wave will itself carry the electrification along them. In fact we now have the rudimentary telegraph-circuit, with no allowance made for absorption of energy in the wires, and the consequent distortion. If the conductors be not coaxial, we only alter the distribution of the displacement and induction, without affecting the propagation without distortion *.

If we now make the medium conduct electrically, and likewise magnetically, with equal rates of subsidence, we shall have the same solutions, with a time-factor $\epsilon^{-\rho t}$ producing ultimate subsidence to zero; and, with only the real electric conductivity in the medium the wave is running through, it will approximately cancel the distortion produced by the resistance of the wires the wave is passing over when this resistance has a certain value †. We should notice, however, that it could not do so perfectly, even if the magnetic retardation in the wires due to diffusion were zero; because in the case of the unreal magnetic conductivity its correcting influence is where it is wanted to be, in the body of the wave; whereas in the case of the wires, their resistance, correcting the distortion due to the external conductivity, is outside the wave; so that we virtually assume instantaneous propagation laterally from the wires of *their* correcting influence in the elementary theory of propagation along a telegraph-circuit which is symbolized by the equations

$$-\frac{dV}{dz}=(R+Lp)C, \quad -\frac{dC}{dz}=(K+Sp)V, \quad . \quad (32)$$

where R, L, K, and S are the resistance, inductance, leakage-conductance, and permittance per unit length of circuit, C the current, and V what I, for convenience, term the potential-difference, but which I have expressly disclaimed ‡ to represent the electrostatic difference of potential, and have shown to represent the transverse E.M.F. or line-integral of the electric force across the circuit from wire to wire, including the electric force of inertia. Now in case of great distortion, as in a long submarine cable, this V approximates towards the electrostatic potential-difference, which it is in Sir W. Thomson's diffusion theory; but in case of little distortion, as

* 'Electrician,' Jan. 10, 1885. Also "Self-Induction of Wires," part iv. Phil. Mag. Nov. 1886.

† "Electromagnetic Waves," § 6, Phil. Mag. Feb. 1888. 'Electrician,' June 1887.

‡ "Self-Induction of Wires," part ii. Phil. Mag. Sept. 1886.

in telephony through circuits of low resistance and large inductance, there may be a wide difference between my V and that of the electrostatic force. Consider, for instance, the extreme case of an isolated plane-wave disturbance with no spreading-out of the tubes of displacement. At the boundaries of the disturbance the difference between V and the electrostatic difference of potential is great.

But it is worth noticing, as a rather remarkable circumstance, that when we derive the system (32) by elementary considerations, viz. by extending the diffusion-system by the addition of the E.M.F. of inertia and leakage-current, we apparently as a matter of course take V to mean the same as in the diffusion-system. The resulting equations are correct, and yet the assumption is certainly wrong. The true way appears to be that given by me in the paper last referred to, by considering the line-integral of electric force in a closed curve. We cannot, indeed, make a separation of the electric force of inertia from $-\Delta\Psi$ without some assumption, though the former is quite definite when the latter is suitably defined. But, and this is the really important matter, it would be in the highest degree inconvenient, and lead to much complication and some confusion, to split V into two components, in other words, to bring in Ψ and **A**.

In thus running down Ψ, I am by no means forgetful of its utility in other cases. But it has perhaps been greatly misused. The clearest course to pursue appears to me to invariably make **E** and **H** the primary objects of attention, and only use potentials when they naturally suggest themselves as labour-saving appliances.

13. Returning to the solutions (29), the following are the special tests of their accuracy. Let E_1 and E_2 be the z and h components of **E**. Then, by (11) and (13), with the special meaning assumed by p, we have

$$\left.\begin{aligned} &\frac{1}{h}\frac{d}{dh}h\mathrm{H} = -cu\frac{d\mathrm{E}_1}{dz}, \\ &-\frac{d\mathrm{H}}{dz} = -cu\frac{d\mathrm{E}_2}{dz}, \quad \text{or } \mathrm{H} = cu\mathrm{E}_2, \\ &\frac{d\mathrm{E}_1}{dh} - \frac{d\mathrm{E}_2}{dz} = -\mu_0 u\frac{d\mathrm{H}}{dz}, \quad \text{or } \frac{d\mathrm{E}_1}{dh} = \left(1-\frac{u^2}{v^2}\right)\frac{d\mathrm{E}_2}{dz}. \end{aligned}\right\} \quad (33)$$

In addition to satisfying these equations, the displacement outward through any spherical surface centred at the charge may be verified to be q; this completes the test of the accuracy of (29).

But (33) are not limited to the case of a single point-charge, being true outside the electrification when there is symmetry with respect to the z-axis, and the electrification is all moving parallel to it at speed u.

When $u=v$, $E_1=0$, and $E_2=E=\mu v H$, so that we reduce to

$$\frac{1}{h}\frac{d}{dh}hH=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

outside the electrification. Thus, if the electrification is on the axis of z, we have

$$E/\mu v=H=2qv/r, \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

differing from (31) only in that q, the linear density, may be any function of z.

14. If, in the solutions (29), we terminate the fields internally at $r=a$, the perpendicularity of **E** and the tangentiality of **H** to the surface show that (29) represents the solutions in the case of a perfectly conducting sphere of radius a, moving steadily along the z-axis at the speed u, and possessing a total charge q. The energy is now finite. Let U be the total electric and T the total magnetic energy. By space-integration of the squares of E and H we find that they are given by

$$U=\frac{q^2}{2ca}\cdot\frac{1-u^2/v^2}{4}\left[1+\frac{\frac{3}{2}}{1-u^2/v^2}+\frac{\frac{3}{2}\tan^{-1}\dfrac{u/v}{(1-u^2/v^2)^{\frac{1}{2}}}}{(u/v)(1-u^2/v^2)^{\frac{3}{2}}}\right], \quad (36)$$

$$T=\frac{q^2}{2ca}\cdot\frac{1-u^2/v^2}{4}\left[1+\frac{2u^2/v^2-\frac{1}{2}}{1-u^2/v^2}+\frac{(2u^2/v^2-\frac{1}{2})\tan^{-1}\dfrac{u/v}{(1-u^2/v^2)^{\frac{1}{2}}}}{(u/v)(1-u^2/v^2)^{\frac{3}{2}}}\right],$$

in which $u<v$. When $u=v$, with accumulation of the charge at the equator of the sphere, we have infinite values, and it appears to be only possible to have finite values by making a zone at the equator cylindrical instead of spherical. The expression for T in (37) looks quite wrong; but it correctly reduces to that of equation (2) when u/v is infinitely small.

15. The question now suggests itself, What is the state of things when $u>v$? It is clear, in the first place, that there can be no disturbance at all in front of the moving charge (at a point, for simplicity). Next, considering that the spherical waves emitted by the charge in its motion along the z-axis travel at speed v, the locus of their fronts is a conical surface whose apex is at the charge itself, whose axis is that of z, and whose semiangle θ is given by

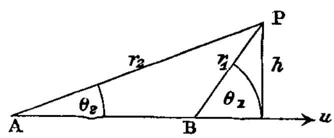
$$\sin\theta=v/u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The whole displacement, of amount q, should therefore lie within this cone. And since the moving charge is a convection-current qu, the displacement-current should be towards the apex in the axial portion of the cone, and change sign at some unknown distance, so as to be away from the apex either in the outer part of the cone or else upon its boundary. The pulling back of the charge by the electric stress would require the continued application of impressed force to keep up the motion, and its activity would be accounted for by the continuous addition made to the energy in the cone; for the transfer of energy on its boundary is perpendicularly outward, and the field at the apex is being continuously renewed.

The above general reasoning seems plausible enough, but I cannot find any solution to correspond that will satisfy all the necessary conditions. It is clear that (29) will not do when $u > v$. Nor is it of any use to change the sign of the quantity under the radical, when needed, to make real. It is suggested that whilst there should be a definite solution, there cannot be one representing a *steady* condition of **E** and **H** with respect to the moving charge. As regards physical possibility, in connexion with the structure of the æther, that is not in question.

16. Let us now derive from (29), or from (27), the results in some cases of distributed electrification, in steady rectilinear motion. The integrations to be effected being all of an elementary character, it is not necessary to give the working.

First, let a straight line AB be charged to linear density q, and be in motion at speed u in its own line from left to right. Then at P we shall have



$$A = qu \log\left(\frac{r_1}{r_2} \cdot \frac{\mu_1 + (1-\nu_1^2 u^2/v^2)^{\frac{1}{2}}}{\mu_2 + (1-\nu_2^2 u^2/v^2)^{\frac{1}{2}}}\right), \quad . \quad . \quad (39)$$

from which $H = -dA/dh$ gives

$$H = qu\left(1 - \frac{u^2}{v^2}\right)\left[\frac{\nu_1}{r_1(1-\nu_1^2 u^2/v^2) + r_1\mu_1(1-\nu_1^2 u^2/v^2)^{\frac{1}{2}}} - \text{same function of } r_2, \mu_2, \nu_2\right], \quad . \quad (40)$$

where $\mu = \cos\theta$, $\nu = \sin\theta$.

When P is vertically over B, and A is at an infinite distance, we shall find

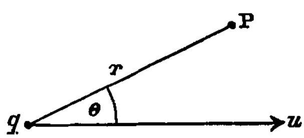
$$H = qu/h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

which is one half the value due to an infinitely long (both ways) straight current of strength qu. The notable thing is the independence of the ratio u/v.

But if $u=v$ in (40), the result is zero, unless $\nu_1=1$, when we have the result (41). But if P be still further to the left, we shall have to add to (41) the solution due to the electrification which is ahead of P. So when the line is infinitely long both ways, we have double the result in (41), with independence of u/v again.

But should q be a function of z, we do not have independence of u/v except in the already considered case of $u=v$, with plane waves, and no component of electric force parallel to the line of motion.

17. Next, let the electrified line be in steady motion perpendicularly to its length. Let q be the linear density (constant), the z-axis that of the motion, the x-axis coincident with the electrified line and that of y upward on the paper. Then the A at P will be



$$A=\frac{qu}{(1-u^2/v^2)^{\frac{1}{2}}}\log\frac{x_1+\{x_1^2+y^2+z^2(1-u^2/v^2)^{-1}\}^{\frac{1}{2}}}{x_2+\{x_2^2+y^2+z^2(1-u^2/v^2)^{-1}\}^{\frac{1}{2}}}; \quad (42)$$

where y and z belong to P, and x_1, x_2 are the limiting values of x in the charged line. From this derive the solution in the case of an infinitely long line. It is

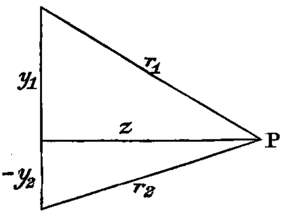
$$cE=\frac{2q}{r}\cdot\frac{(1-u^2/v^2)^{\frac{1}{2}}}{1-\nu^2u^2/v^2}, \quad H=cEu\nu, \quad . \quad . \quad . \quad . \quad (43)$$

where $\nu=\sin\theta$; understanding that **E** is radial, or along qP in the figure, and **H** rectilinear, parallel to the charged line.

Terminating the fields internally at $r=a$, we have the case of a perfectly conducting cylinder of radius a, charged with q per unit of length, moving transversely. When $u=v$ there is disappearance of E and H everywhere except in the plane $\theta=\frac{1}{2}\pi$, as in the case of the sphere, and consequent infinite values. It is the curvature that permits this to occur, *i. e.* producing infinite values; of course it is the self-induction that is the cause of the conversion to a plane wave, here and in the other cases. There is some similarity between (43) and (29). In fact, (43) is the bidimensional equivalent of (29).

18. Coming next to a plane distribution of electrification,

let q be the surface density, and the plane be moving perpendicularly to itself. Let it be of finite breadth and of infinite length, so that we may calculate H from (43). The result at P is



$$\mathrm{H}=\frac{qu}{(1-u^2/v^2)^{\frac{1}{2}}}\log\frac{r_1^{\,2}-y_1^{\,2}u^2/v^2}{r_2^{\,2}-y_2^{\,2}u^2/v^2}. \quad . \quad . \quad . \quad (44)$$

When P is equidistant from the edges, H is zero. There is therefore no H anywhere due to the motion of an infinitely large uniformly charged plane perpendicularly to itself. The displacement-current is the negative of the convection-current and at the same place, viz. the moving plane, so there is no true current.

Calculating E_1, the z-component of **E**, z being measured from left to right, we find

$$c\mathrm{E}_1=2q\left\{\tan^{-1}\frac{y_1}{z}\left(1-\frac{u^2}{v^2}\right)^{\frac{1}{2}}-\tan^{-1}\frac{y_2}{z}\left(1-u^2\right)^{\frac{1}{2}}\right\}. \quad (45)$$

The component parallel to the plane is H/cu. Thus, when the plane is infinite, this component vanishes with H, and we are left with

$$c\mathrm{E}_1=c\mathrm{E}=2\pi q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

the same as if the plane were at rest.

19. Lastly, let the charged plane be moving in its own plane. Refer to the first figure, in which let AB now be the trace of the plane when of finite breadth. We shall find that

$$\mathrm{H}=2qu\left[\tan^{-1}\frac{z}{h(1-u^2/v^2)^{\frac{1}{2}}}\right]_{z_1}^{z_2} \quad . \quad . \quad . \quad (47)$$

z_1 and z_2 being the extreme values of z, which is measured parallel to the breadth of the plane.

Therefore, when the plane extends infinitely both ways, we have

$$\mathrm{H}=2\pi qu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

above the plane, and its negative below it. This differs from the previous case of vanishing displacement-current. There is H, and the convection-current is not now cancelled by co-existent displacement-current.

The existence of displacement-current, or changing displacement, was the basis of the conclusion that moving electrification constitutes a part of the true current. Now in the problem (48) the displacement-current has gone, so that the existence of H appears to rest merely upon the assumption that moving electrification is true current. But if the plane be not infinite, though large, we shall have (48) nearly true near it, and away from the edges; whilst the displacement-current will be strong near the edges and almost nil where (48) is nearly true.

But in some cases of rotating electrification, there need be no displacement anywhere, except during the setting up of the final state. This brings us to the rather curious question whether there is any difference between the magnetic field of a convection-current produced by the rotation of electrification upon a good nonconductor and upon a good conductor respectively, other than that due to diffusion in the conductor. For in the case of a perfect conductor, it is easy to imagine that the electrification could be at rest, and the moved conductor merely slip past it. Perhaps Professor Rowland's forthcoming experiments on convection-currents may cast some light upon this matter.

December 27, 1888.

XL. *The Rotation of the Plane of Polarization of Light by the Discharge of a Leyden Jar. By* Dr. Oliver Lodge*.

THE current produced by the discharge of a Leyden jar is so violent while it lasts, that those phenomena which depend upon the value of a current independently of its duration are well excited by it. Such are the induction of currents, the production of magnetism, and the rotation of the plane of polarization.

Nothing is easier than to wind a quantity of thin gutta-percha-covered wire round a piece of heavy glass, and to witness the bright flashing of a dark field between polarizer and analyser whenver a large Leyden jar is sparked through the coil, the source of light being a paraffin-lamp or gas-flame. The suddenness of the effect suggests, of course erroneously, that it is an illumination caused by the light of the spark which one is looking at.

The fact that the discharge is oscillatory, and that the restoration of light in the dark field is oscillatory too, is proved by the fact that an adjustment of the analyser to one side or the other of complete darkness has just the same effect on the result. It is proved also by the fact that a biquartz

* Communicated by the Physical Society: read March 9, 1889.