THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FIFTH SERIES.]

AUGUST 1887.

XVII. *On the Maintenance of Vibrations by Forces of Double Frequency, and on the Propagation of Waves through a Medium endowed with a Periodic Structure.* By Lord Rayleigh, *Sec. R. S., Professor of Natural Philosophy in the Royal Institution**.

THE nature of the question to be first considered may be best explained by a paragraph from a former paper †, in which the subject was briefly treated. "There is also another kind of maintained vibration which, from one point of view, may be regarded as forced, inasmuch as the period is imposed from without, but which differs from the kind just referred to (ordinary forced vibrations) in that the imposed periodic variations do not tend directly to displace the body from its configuration of equilibrium. Probably the best-known example of this kind of action is that form of Melde's experiment in which a fine string is maintained in transverse vibration by connecting one of its extremities with the vibrating prong of a massive tuning-fork, *the direction of motion of the point of attachment being parallel to the length of the string*‡. The effect of the motion is to render the tension of the string periodically variable; and at first sight there is nothing to cause the string to depart from its equilibrium condition of straightness. It is known, however, that under these circumstances the equilibrium position may become unstable, and that the string may settle down into a state of permanent and

* Communicated by the Author.

† "On Maintained Vibrations." Phil. Mag. April 1883, p. 229.

‡ "When the direction of motion is transverse, the case falls under the head of ordinary forced vibrations."

vigorous vibration *whose period is the double of that of the point of attachment*"*. Other examples of acoustical interest are mentioned in the paper.

My attention was recalled to the subject by Mr. Glaisher's Address to the Astronomical Society†, in which he gives an interesting account of the treatment of mathematically similar questions in the Lunar Theory by Mr. Hill ‡ and by Prof. Adams§. The analysis of Mr. Hill is in many respects incomparably more complete than that which I had attempted; but his devotion to the Lunar Theory leads the author to pass by many points of great interest which arise when his results are applied to other physical questions.

By a suitable choice of the unit of time, the equation of motion of the vibrating body may be put into the form

$$\frac{d^2w}{dt^2}+2k\frac{dw}{dt}+(\Theta_0+2\Theta_1\cos 2t)w=0; \quad . \quad . \quad . \qquad (1)$$

where k is a positive quantity, which may usually be treated as small, representing the dissipative forces. $(\Theta_0+2\Theta_1\cos 2pt)$ represents the coefficient of restitution, which is here regarded as subject to a small imposed periodic variation of period π. Thus Θ_0 is positive, and Θ_1 is to be treated as relatively small.

The equation to which Mr. Hill's researches relate is in one respect less general than (1), and in another more general. It omits the dissipative term proportional to k; but, on the other hand, as the Lunar Theory demands, it includes terms proportional to $\cos 4t$, $\cos 6t$, &c. Thus

$$\frac{d^2w}{dt^2}+(\Theta_0+2\Theta_1\cos 2t+2\Theta_2\cos 4t+\ldots)w=0; \quad . \qquad (2)$$

or

$$\frac{d^2w}{dt^2}+\Theta w=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

where

$$\Theta=\Sigma_n\Theta_n e^{2int}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

n being any integer, and i representing $\sqrt{(-1)}$. In the present investigation $\Theta_{-n}=\Theta_n$.

* "See Tyndall's 'Sound,' 3rd. ed. ch. iii. § 7, where will also be found a general explanation of the mode of action."

† Monthly Notices, Feb. 1887.

‡ "On the Part of the Motion of the Lunar Perigee which is a Function of the Mean Motions of the Sun and Moon," *Acta Mathematica*, **8**:1, 1886. Mr. Hill's work was first published in 1877.

§ "On the Motion of the Moon's Node, in the case when the orbits of the Sun and Moon are supposed to have no Eccentricities, and when their Mutual Inclination is supposed to be indefinitely small." Monthly Notices, Nov. 1877.

It will be convenient to give here a sketch of Mr. Hill's method and results. Remarking that when Θ_1, Θ_2, &c. vanish, the solution of (3) is

$$w = \mathrm{K}e^{ict} + \mathrm{K}'e^{-ict}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where K, K′ are arbitrary constants, and $c = \sqrt{(\Theta_0)}$, he shows that in the general case we may assume as a particular solution

$$w = \Sigma_n b_n e^{ict+2int}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the value of c being modified by the operation of Θ_1, &c., and the original term $b_0 e^{ict}$ being accompanied by subordinate terms corresponding to the positive and negative integral values of n.

The multiplication by Θ, as given in (4), does not alter the form of (6); and the result of the substitution in the differential equation (3) may be written

$$(c+2m)^2 b_m - \Sigma_n \Theta_{m-n} b_n = 0, \quad . \quad . \quad . \quad . \quad (7)$$

which holds for all integral values of m, positive and negative. These conditions determine the ratios of all the coefficients b_n to one of them, *e. g.*, b_0, which may then be regarded as the arbitrary constant. They also determine c, the main subject of quest. Mr. Hill writes

$$[n] = (c+2n)^2 - \Theta_0; \quad . \quad . \quad . \quad . \quad . \quad (8)$$

so that the equations take the form

$$\left.\begin{array}{l} \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \ldots + [-2]b_{-2} - \Theta_1\, b_{-1} - \Theta_2\, b_0 - \Theta_3\, b_1 - \Theta_4\, b_2 - \ldots = 0, \\ \ldots - \Theta_1\, b_{-2} + [-1]b_{-1} - \Theta_1\, b_0 - \Theta_2\, b_1 - \Theta_3\, b_2 - \ldots = 0, \\ \ldots - \Theta_2\, b_{-2} - \Theta_1\, b_{-1} + [0]b_0 - \Theta_1\, b_1 - \Theta_2\, b_2 - \ldots = 0, \\ \ldots - \Theta_3\, b_{-2} - \Theta_2\, b_{-1} - \Theta_1\, b_0 + [-1]b_1 - \Theta_1\, b_2 - \ldots = 0, \\ \ldots - \Theta_4\, b_{-2} - \Theta_3\, b_{-1} - \Theta_2\, b_0 - \Theta_1\, b_1 + [2]b_2 - \ldots = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} . \quad (9)$$

The determinant formed by eliminating the b's from these equations is denoted by $\mathfrak{D}(c)$; so that the equation from which c is to be found is

$$\mathfrak{D}(c) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The infinite series of values of c determined by (10) cannot give independent solutions of (3),—a differential equation of the second order only. It is evident, in fact, that the system of equations by which c is determined is not altered if we replace c by $c+2\nu$, where ν is any positive or negative integer. Neither is any change incurred by the substitution of $-c$ for c. "It follows that if (10) is satisfied by a root $c = c_0$, it will also have, as roots, all the quantities contained in the expression

$$\pm c_0 + 2n,$$

where n is any positive or negative integer or zero. And these are all the roots the equation admits of; for each of the expressions denoted by $[n]$ is of two dimensions in c, and may be regarded as introducing into the equation the two roots $2n+c_0$ and $2n-c_0$. Consequently the roots are either all real or all imaginary; and it is impossible that the equation should have any equal root unless all the roots are integral."

On these grounds Mr. Hill concludes that $\mathfrak{D}(c)$ must be such that

$$\mathfrak{D}(c)=\mathrm{A}\left[\cos(\pi c)-\cos(\pi c_0)\right] \quad . \quad . \quad . \quad (11)$$

identically, where A is some constant independent of c; whence on putting $c=0$,

$$\mathfrak{D}(0)=\mathrm{A}[1-\cos(\pi c_0)], \quad . \quad . \quad . \quad . \quad (12)$$

in which, if we please, c_0 may be replaced by c. The value of A may now be determined by comparison with the particular case $\Theta_1=0$, $\Theta_2=0$, &c., for which of course $c=\sqrt{\Theta_0}$. Thus if $\mathfrak{D}'(0)$ denote the special form then assumed, *i. e.* the simple product of the diagonal constituents,

$$\mathfrak{D}'(0)=\mathrm{A}[1-\cos(\pi\sqrt{\Theta_0})], \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and

$$\frac{1-\cos(\pi c)}{1-\cos(\pi\sqrt{\Theta_0})}=\frac{\sin^2(\frac{1}{2}\pi c)}{\sin^2(\frac{1}{2}\pi\sqrt{\Theta_0})}=\frac{\mathfrak{D}(0)}{\mathfrak{D}'(0)}. \quad . \quad (14)$$

The fraction $\mathfrak{D}(0)\div\mathfrak{D}'(0)$ is denoted by $\square(0)$. It is the determinant formed from the original one by dividing each row by the constituent in the diagonal, so as to reduce all the diagonal constituents to unity, and by making c vanish. Thus

$$\frac{1-\cos(\pi c)}{1-\cos(\pi\sqrt{\Theta_0})}=\square(0), \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where

$$\square(0)=\left|\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\ldots+1 & -\frac{\Theta_1}{4^2-\Theta_0} & -\frac{\Theta_2}{4^2-\Theta_0} & -\frac{\Theta_3}{4^2-\Theta_0} & -\frac{\Theta_4}{4^2-\Theta_0} & \ldots \\
\ldots-\frac{\Theta_1}{2^2-\Theta_0} & +1 & -\frac{\Theta_1}{2^2-\Theta_0} & -\frac{\Theta_2}{2^2-\Theta_0} & -\frac{\Theta_3}{2^2-\Theta_0} & \ldots \\
\ldots-\frac{\Theta_2}{0^2-\Theta_0} & -\frac{\Theta_1}{0^2-\Theta_0} & +1 & -\frac{\Theta_1}{0^2-\Theta_0} & -\frac{\Theta_2}{0^2-\Theta_0} & \ldots \\
\ldots-\frac{\Theta_3}{2^2-\Theta_0} & -\frac{\Theta_2}{2^2-\Theta_0} & -\frac{\Theta_1}{2^2-\Theta_0} & +1 & -\frac{\Theta_1}{2^2-\Theta_0} & \ldots \\
\ldots-\frac{\Theta_4}{4^2-\Theta_0} & -\frac{\Theta_3}{4^2-\Theta_0} & -\frac{\Theta_2}{4^2-\Theta_0} & -\frac{\Theta_1}{4^2-\Theta_0} & +1 & \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right| \quad (16)$$

The value of $\square(0)$ is calculated for the purposes of the Lunar Theory to a high order of approximation. It will here suffice to give the part which depends upon the squares of Θ_1, Θ_2, &c. Thus

$$\square(0)=1+\frac{\pi\cot(\frac{1}{2}\pi\sqrt{\Theta_0})}{4\sqrt{\Theta_0}}\left[\frac{\Theta_1^2}{1-\Theta_0}+\frac{\Theta_2^2}{4-\Theta_0}+\frac{\Theta_3^2}{9-\Theta_0}+\dots\right]. \quad (17)$$

Another determinant, $\nabla(0)$, is employed by Mr. Hill, the relation of which to $\square(0)$ is expressed by

$$\nabla(0)=2\sin^2(\tfrac{1}{2}\pi\sqrt{\Theta_0})\,.\,\square(0); \quad . \quad . \quad . \quad (18)$$

so that the general solution for c may be written

$$\cos(\pi c)=1-\nabla(0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Mr. Hill observes that the reality of c requires that $1-\nabla(0)$ should lie between -1 and $+1$. In the Lunar Theory this condition is satisfied; but in the application to Acoustics the case of an imaginary c is the one of greater interest, for the vibrations then tend to increase indefinitely.

Cos (πc) being itself always real, let us suppose that πc is complex, so that

$$c=\alpha+i\beta,$$

where α and β are real. Thus

$$\cos\pi c=\cos\pi\alpha\cos i\pi\beta-\sin\pi\alpha\sin i\pi\beta;$$

and the reality of $\cos\pi c$ requires either (1) that $\beta=0$, or (2) that $\alpha=n$, n being an integer. In the first case c is real. In the second

$$\cos\pi c=\pm\cos i\pi\beta=1-\nabla(0), \quad . \quad . \quad . \quad (20)$$

which gives but one (real) value of β. If $1-\nabla(0)$ be positive,

$$c=\pm i\beta+2n; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

but if $1-\nabla(0)$ be negative,

$$\cos\pi c=-\cos i\pi\beta,$$

whence

$$c=\pm i\beta+2n+1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The latter is the case with which we have to do when Θ_0, and therefore c, is nearly equal to unity; and the conclusion that when c is complex, the real part is independent of Θ_1, Θ_2, &c. is of importance. The complete value of w may then be written

$$w=e^{\beta t}\Sigma b_n e^{it(1+2n)}+e^{-\beta t}\Sigma b'_n e^{it(1+2n)}, \quad . \quad . \quad . \quad (23)$$

the ratios of b_n and also of b_n' being determined by (9). After the lapse of a sufficient time, the second set of terms in $e^{-\beta t}$ become insignificant.

In the application of greatest acoustical interest Θ_0 (and c) are nearly equal to unity; so that the free vibrations are performed with a frequency about the half of that introduced by Θ_1. In this case the leading equations in (9) are those which involve the small quantities $[0]$ and $[-1]$; but for the sake of symmetry, it is advisable to retain also the equation containing $[1]$. If we now neglect Θ_2, as well as the b's whose suffix is numerically greater than unity, we find

$$\frac{b_{-1}}{\Theta_1[1]}=\frac{b_0}{[1][-1]}=\frac{b_1}{\Theta_1[-1]}, \quad . \quad . \quad . \quad (24)$$

and

$$[0][1][-1]-\Theta_1^2\{[1]+[-1]\}=0. \quad . \quad (25)$$

For the sake of distinctness it will be well to repeat here that

$$[0]=c^2-\Theta_0, \quad [-1]=(c-2)^2-\Theta_0, \quad [1]=(c+2)^2-\Theta^0.$$

Substituting these values in (25), Mr. Hill obtains

$$(c^2-\Theta_0)\{(c^2+4-\Theta_0)^2-16c^2\}-2\Theta_1^2\{c^2+4-\Theta_0\}=0,$$

and neglecting the cube of $(c^2-\Theta_0)$, as well as its product with Θ_1^2,

$$(c^2-\Theta_0)^2+2(\Theta_0-1)(c^2-\Theta_0)+\Theta_1^2=0;$$

and from this again

$$c^2=1+\sqrt{\{(\Theta_0-1)^2-\Theta_1^2\}}. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

It appears, therefore, that c is real or imaginary according as $(\Theta_0-1)^2$ is greater or less than Θ_1^2. In the problem of the Moon's apse, treated by Mr. Hill,

$$\Theta_0=1{\cdot}1588439, \qquad \Theta_1=-0{\cdot}0570440;$$

and in the corresponding problem of the node, investigated by Prof. Adams,

$$\Theta_0=1{\cdot}17804{,}44973{,}149,$$
$$\Theta_1=0{\cdot}01261{,}68354{,}6.$$

In both these cases the value of c is real, though of course not to be accurately determined by (26).

Mr. Hill's results are not immediately applicable to the acoustical problem embodied in (1), in consequence of the omission of k, representing the dissipation to which all actual vibrations are subject. The inclusion of this term leads, however, merely to the substitution for $(c+2n)^2-\Theta_0$ in (8) of

$$(c+2n)^2-2ik(c+2n)-\Theta_0;$$

so that the whole operation of k is represented if we write

$(c-ik)$ in place of c, and (Θ_0-k^2) in place of Θ_0. Accordingly

$$\cos\pi(c-ik)=1-\nabla'(0), \quad . \quad . \quad . \quad . \quad (27)$$

$\nabla'(0)$ differing from $\nabla(0)$ only by the substitution of Θ_0-k^2 for Θ_0.

If $1-\nabla'(0)$ lies between ± 1, $(c-ik)$ is real, so that

$$c=ik\pm\alpha+2n. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

In this case both solutions are affected with the factor e^{-kt}, indicating that whatever the initial circumstances may be, the motion dies away.

It may be otherwise when $1-\nabla'(0)$ lies beyond the limits ± 1. In the case of most importance, when Θ_0 is nearly equal to unity, $1-\nabla'(0)$ is algebraically less than -1. If

$$\cos i\pi\beta=-1+\nabla'(0), \quad . \quad . \quad . \quad . \quad . \quad (29)$$

we may write

$$c=1+i(k\pm\beta)+2n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Here again both motions die down unless β is numerically greater than k, in which case one motion dies down, while the other increases without limit. The critical relation may be written

$$\cos(i\pi k)=-1+\nabla'(0). \quad . \quad . \quad . \quad . \quad . \quad (31)$$

From (30) we see that, whatever may be the value of k, the vibrations (considered apart from the rise or subsidence indicated by the exponential factors) have the same frequency as if k, as well as Θ_1, Θ_2, &c. vanished.

Before leaving the general theory it may be worth while to point out that Mr. Hill's method may be applied when the coefficients of d^2w/dt^2 and dw/dt, as well as of w, are subject to given periodic variations. We may write

$$\Phi\frac{d^2w}{dt^2}+\Psi\frac{dw}{dt}+\Theta w=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where

$$\Phi=\Sigma\Phi_n e^{2int}, \quad \Psi=\Sigma\Psi_n e^{2int}, \quad \Theta=\Sigma\Theta_n e^{2int}. \quad (33)$$

Assuming, as before,

$$w=\Sigma_n b_n e^{ict+2int}, \quad . \quad . \quad . \quad . \quad . \quad (34)$$

we obtain, on substitution, as the coefficient of $e^{ict+2imt}$,

$$-\Sigma_n b_n(c+2n)^2\Phi_{m-n}+i\Sigma_n b_n(c+2n)\Psi_{m-n}+\Sigma b_n\Theta_{m-n},$$

which is to be equated to zero. The equation for c may still be written

$$\mathfrak{D}(c)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where

$$\mathfrak{D}(c)=\left|\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \ldots[-2,0], & [-1,-1], & [0,-2], & [1,-3], & [2,-4], & \ldots \\ \ldots[-2,1], & [-1,0], & [0,-1], & [1,-2], & [2,-3], & \ldots \\ \ldots[-2,2], & [-1,1], & [0,0], & [1,-1], & [2,-2], & \ldots \\ \ldots[-2,3], & [-1,2], & [0,1], & [1,0], & [2,-1], & \ldots \\ \ldots[-2,4], & [-1,3], & [0,2], & [1,1], & [2,0], & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right| \quad . . \ (36)$$

and

$$[n,r]=(c+2n)^2\Phi_r-i(c+2n)\Psi_r-\Theta_r. \quad . \quad . \quad (37)$$

By similar reasoning to that employed by Mr. Hill we may show that

$$\mathfrak{D}(c)=\mathrm{A}\,(\cos\pi c-\cos\pi c_0)$$
$$+\mathrm{B}\,(\sin\pi c-\sin\pi c_0)\ldots,$$

where A and B are constants independent of c; and, further, that

$$\mathfrak{D}(0)=\mathrm{A}(1-\cos\pi c)-\mathrm{B}\sin\pi c. \quad . \quad . \quad . \quad (38)$$

If all the quantities Φ_r, Ψ_r, Θ_r vanish except Φ_0, Ψ_0, Θ_0, $\mathfrak{D}(0)$ reduces to the diagonal row simply, say $\mathfrak{D}'(0)$. Let c_1, c_2 be the roots of

$$\Phi_0\frac{d^2w}{dt^2}+\Psi_0\frac{dw}{dt}+\Theta_0 w=0, \quad . \quad . \quad . \quad . \quad (39)$$

then

$$\mathfrak{D}'(0)=\mathrm{A}\,(1-\cos\pi c_1)-\mathrm{B}\sin\pi c_1,$$
$$=\mathrm{A}\,(1-\cos\pi c_2)-\mathrm{B}\sin\pi c_2\,;$$

so that the equation for c may be written

$$\left|\begin{array}{lll} \mathfrak{D}\,(0), & 1-\cos\pi c, & \sin\pi c, \\ \mathfrak{D}'(0), & 1-\cos\pi c_1, & \sin\pi c_1, \\ \mathfrak{D}'(0), & 1-\cos\pi c_2, & \sin\pi c_2, \end{array}\right|=0. \quad . \quad . \quad (40)$$

In this equation $\mathfrak{D}(0)\div\mathfrak{D}'(0)$ is the determinant derived from $\mathfrak{D}(0)$ by dividing each row so as to make the diagonal constituent unity.

If $\ldots\Psi_{-1}, \Psi_0, \Psi_1\ldots$ vanish (even though $\ldots\Phi_{-1}, \Phi_0, \Phi_1\ldots$ remain finite), $\mathfrak{D}(c)$ is an even function of c, and the coefficient B vanishes in (38). In this case we have simply

$$\frac{1-\cos\pi c}{1-\cos\pi\sqrt{\Theta_0}}=\frac{\mathfrak{D}(0)}{\mathfrak{D}'(0)},$$

exactly as when Φ_1, Φ_{-1}, Φ_2, $\Phi_{-2}\ldots$ vanish.

Reverting to (24), we have as the approximate particular solution, when there is no dissipation,

$$w=\frac{e^{(c-2)it}}{(c-2)^2-\Theta_0}+\frac{e^{cit}}{\Theta_1}+\frac{e^{(c+2)it}}{(c+2)^2-\Theta_0}. \quad . \quad . \quad (41)$$

If c be real, the solution may be completed by the addition of a second, found from (41) by changing the sign of c. Each of these solutions is affected with an arbitrary constant multiplier. The realized general solution may be written

$$w=\frac{\mathrm{R}\cos(c-2)t+\mathrm{S}\sin(c-2)t}{(c-2)^2-\Theta_3}$$

$$+\frac{\mathrm{R}\cos ct+\mathrm{S}\sin ct}{\Theta_1}+\frac{\mathrm{R}\cos(c+2)t+\mathrm{S}\sin(c+2)t}{(c+2)^2-\Theta_0}, \quad . \quad . (42)$$

from which the last term may usually be omitted, in consequence of the relative magnitude of its denominator. In this solution c is determined by (26).

When c^2 is imaginary, we take

$$4s^2=\Theta_1^2-(\Theta_0-1)^2; \quad . \quad . \quad . \quad . \quad . \quad (43)$$

so that

$$c^2=1+2is, \quad c=1+is, \quad c-2=-1+is.$$

The particular solution may be written

$$w=e^{-st}\{\Theta_1e^{-it}+(1-\Theta_0-2is)e^{it}\}; \quad . \quad . \quad . \quad (44)$$

or, in virtue of (43),

$$w=e^{-st}\{(1-\Theta_0+\Theta_1)\cos t+2s\sin t\}; \quad . \quad . \quad (45)$$

or, again,

$$w=e^{-st}\{\sqrt{(\Theta_1+1-\Theta_0)}\,.\cos t+\sqrt{(\Theta_1-1+\Theta_0)}\,.\sin t\}. \quad . \quad (46)$$

The general solution is

$$\left.\begin{aligned} w=&\mathrm{R}e^{-st}\{(1-\Theta_0+\Theta_1)\cos t+2s\sin t\}\\ &+\mathrm{S}e^{st}\ \{(1-\Theta_0+\Theta_1)\cos t-2s\sin t\}\end{aligned}\right\}, \quad . \quad . \quad (47)$$

R, S being arbitrary multipliers.

One or two particular cases may be noticed. If $\Theta_0=1$, $2s=\Theta_1$, and

$$\left.\begin{aligned} w=&\mathrm{R}'e^{-st}\{\cos t+\sin t\}\\ &+\mathrm{S}'e^{st}\ \{\cos t-\sin t\}\end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Again, suppose that

$$\Theta_1^2=(\Theta_0-1)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

so that s vanishes, giving the transition between the real and imaginary values of c. Of the two terms in (46), one or other preponderates indefinitely in the two alternatives. Thus, if $\Theta_1=1-\Theta_0$, the solution reduces to $\cos t$; but if $\Theta_1=-1+\Theta_0$, it reduces to $\sin t$. The apparent loss of generality by the merging of the two solutions may be repaired in the usual way by supposing s infinitely small.

When there are dissipative forces, we are to replace c by $(c-ik)$, and Θ by (Θ_0-k^2); but when k is small the latter substitution may be neglected. Thus, from (26),

$$c=1+ik+\tfrac{1}{2}\sqrt{\{(\Theta_0-1)^2-\Theta_1^{\,2}\}}. \quad . \quad . \quad . \quad (50)$$

Interest here attaches principally to the case where the radical is imaginary; otherwise the motion necessarily dies down. If, as before,

$$4s^2=\Theta_1^{\,2}-(\Theta_0-1)^2, \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$c=1+ik+is, \quad c-2=-1+ik+is, \quad . \quad . \quad (52)$$

and

$$w=\frac{e^{(c-2)it}}{(c-ik-2)^2-\Theta_0}+\frac{e^{c\,it}}{\Theta_1},$$

or

$$w=e^{-(k+s)t}\{\Theta_1 e^{-it}+(1-\Theta_0-2is)e^{it}\},$$

or

$$w=e^{-(k+s)t}\{(1-\Theta_0+\Theta_1)\cos t+2s\sin t\}. \quad . \quad . \quad (53)$$

This solution corresponds to a motion which dies away.

The second solution (found by changing the sign of s) is

$$w=e^{(s-k)t}\{(1-\Theta_0+\Theta_1)\cos t-2s\sin t\}. \quad . \quad . \quad (54)$$

The motion dies away or increases without limit according as s is less or greater than k.

The only case in which the motion is periodic is when $s=k$, or

$$4k^2=\Theta_1^{\,2}-(\Theta_0-1)^2; \quad . \quad . \quad . \quad . \quad . \quad (55)$$

and then

$$w=(1-\Theta_0-\Theta_1)\cos t-2k\sin t. \quad . \quad . \quad . \quad (56)$$

These results, under a different notation, were given in my former paper*.

If $\Theta_0=1$, we have by (51), $2s=\Theta$; and from (53), (54),

$$w=\mathrm{R}e^{-(k+s)t}\{\cos t+\sin t\}+\mathrm{S}e^{-(k-s)t}\{\cos t-\sin t\}.. \quad . \quad (57)$$

* In consequence of an error of sign, the result for a second approximation there stated is incorrect.

In the former paper some examples were given drawn from ordinary mechanics and acoustics. To these may be added the case of a stretched wire, whose tension is rendered periodically variable by the passage through it of an intermittent electric current. It is probable that an illustration might be arranged in which the vibrations are themselves electrical. Θ_0 would then represent the stiffness of a condenser, Ψ_0 resistance, and Φ_0 self-induction. The most practicable way of introducing the periodic term would be by rendering the self-induction variable with the time (Φ_1). This could be effected by the rotation of a coil forming part of the circuit.

The discrimination of the real and imaginary values of c is of so much importance, that it is desirable to pursue the approximation beyond the point attained in (26). From (11) we find

$$\frac{\mathfrak{D}(1)}{\mathfrak{D}'(1)}=\frac{1+\cos(\pi c)}{1+\cos(\pi\sqrt{\Theta_0})}; \quad . \quad . \quad . \quad (58)$$

from which, or directly, we see that if $c=1$, corresponding to the transition case between real and imaginary values,

$$\mathfrak{D}(1)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

If, as we shall now suppose, Θ_2, Θ_3... vanish, (59) may be written in the form

$$\begin{vmatrix} . & . & . & . & . & . \\ \ldots 1, & a_2, & 1, & 0, & 0, & 0 \ldots \\ \ldots 0, & 1, & a_1, & 1, & 0, & 0 \ldots \\ \ldots 0, & 0, & 1, & a_1, & 1, & 0 \ldots \\ \ldots 0, & 0, & 0, & 1, & a_2, & 1 \ldots \\ . & . & . & . & . & . \end{vmatrix}=0, \quad . \quad (60)$$

where

$$a_1=\frac{\Theta_0-1}{\Theta_1}, \quad a_2=\frac{\Theta_0-9}{\Theta_1}, \quad a_3=\frac{\Theta_0-25}{\Theta_1}, \quad . \quad (61)$$

The first approximation, equivalent to (26), is found by considering merely the central determinant of the second order involving only a_1; thus,

$$a_1^2-1=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

The second approximation is

$$a_2^2\left\{\left(a_1-\frac{1}{a_2}\right)^2-1\right\}=0. \quad . \quad . \quad . \quad . \quad (63)$$

The third is

$$a_3^2\left\{a_2-\frac{1}{a_3}\right\}^2\left\{\left(a_1-\frac{1}{a_2-\frac{1}{a_3}}\right)^2-1\right\}=0, \quad . \quad . \quad . \quad (64)$$

and so on. The equation (60) is thus equivalent to

$$a_1-\frac{1}{a_2-}\,\frac{1}{a_3-}\,\frac{1}{a_4-}\ldots=\pm1\,; \quad . \quad . \quad (65)$$

and the successive approximations are

$$N_1=\pm D_1, \quad N_2=\pm D_2, \quad . \quad . \quad . \quad . \quad (66)$$

where

$$\frac{N_1}{D_1}, \ \frac{N_2}{D_2}, \ldots$$

are the corresponding convergents to the infinite continued fraction*.

In terms of Θ_0, Θ_1, the second approximation to the equation discriminating the real and imaginary values of c is

$$(\Theta_0-1)(\Theta_0-9)-\Theta_1^2=\pm\Theta_1(\Theta_0-9). \quad . \quad . \quad (67)$$

One of the most interesting applications of the foregoing analysis is to the case of a laminated medium in which the mechanical properties are periodic functions of one of the coordinates. I was led to the consideration of this problem in connexion with the theory of the colours of thin plates. It is known that old superficially decomposed glass presents reflected tints much brighter, and transmitted tints much purer, than any of which a single transparent film is capable. The laminated structure was proved by Brewster; and it is easy to see how the effect may be produced by the occurrence of nearly similar laminæ at nearly equal intervals. Perhaps the simplest case of the kind that can be suggested is that of a stretched string, periodically loaded, and propagating transverse vibrations. We may imagine similar small loads to be disposed at equal intervals. If, then, the wave-length of a train of progressive waves be approximately equal to the *double* interval between the loads, the partial reflexions from the various loads will all concur in phase, and the result must be a powerful aggregate reflexion, even though the effect of an individual load may be insignificant.

* The relations of determinants of this kind to continued fractions has been studied by Muir (Edinb. Proc. vol. viii.).

The general equation of vibration for a stretched string of periodic density is

$$\left(\rho_0+\rho_1\cos\frac{2\pi x}{l}+\rho_1'\sin\frac{2\pi x}{l}+\rho_2\cos\frac{4\pi x}{l}\right.$$
$$\left.+\rho_2'\sin\frac{4\pi x}{l}+\ldots\right)\frac{d^2w}{dt^2}=\mathrm{T}\frac{d^2w}{dx^2}, \quad . \quad . \quad . \quad (68)$$

l being the distance in which the density is periodic. We shall suppose that $\rho_1', \rho_2', \ldots$ vanish, so that the sines disappear, a supposition which involves no loss of generality when we restrict ourselves to a simple harmonic variation of density. If we now assume that $w \propto e^{ipt}$, or $\propto \cos pt$, we obtain

$$\frac{d^2w}{d\xi^2}+(\Theta_0+2\Theta_1\cos 2\xi+2\Theta_2\cos 4\xi+\ldots)w=0, \quad . \quad . \quad (69)$$

where $\xi=\pi x/l$, and

$$\Theta_0=\frac{p^2l^2\rho_0}{\pi^2\mathrm{T}}, \quad 2\Theta_1=\frac{p^2l^2\rho_1}{\pi^2\mathrm{T}}, \text{ \&c.}; \quad . \quad . \quad . \quad (70)$$

and this is of the form of Mr. Hill's equation (2).

When c is real, we may employ the approximate solutions (41), (44). The latter (with ξ written for t) gives, when multiplied by $\cos pt$ or $\sin pt$, the stationary vibrations of the system. From (41) we get

$$w=\frac{\cos[pt+(c-2)\xi]}{(c-2)^2-\Theta_0}+\frac{\cos[pt+c\xi]}{\Theta_1}, \quad . \quad . \quad (71)$$

in which, if $c=1$ nearly, the two terms represent waves progressing with nearly equal velocities in the two directions. Neither term gains permanently in relative importance as x is increased or diminished indefinitely.

It is otherwise when the relation of Θ_0 to Θ_1 is such that c is imaginary. By (44) the solution for w, assumed to be proportional to e^{ipt}, now takes the form

$$\left.\begin{aligned} w=\mathrm{R}e^{-s\xi}&\{\Theta_1e^{i(pt-\xi)}+(1-\Theta_0-2is)e^{i(pt+\xi)}\},\\ +\mathrm{S}e^{s\xi}&\{\Theta_1e^{i(pt-\xi)}+(1-\Theta_0+2is)e^{i(pt+\xi)}\}.\end{aligned}\right\} \quad (72)$$

Whatever may be the relative values of R and S, the first solution preponderates when x is large and negative, and the second preponderates when x is large and positive. In either extreme case the motion is composed of two progressive waves moving in opposite directions, *whose amplitudes are equal in virtue of* (43).

The meaning of this is that a wave travelling in either direction is ultimately totally reflected. For example, we may so choose the values of R and S that at the origin of x there is a wave (of given strength) in the positive direction only, and we may imagine that it here passes into a uniform medium, and so is propagated on indefinitely without change. But, in order to maintain this state of things, we have to suppose on the negative side the coexistence of positive and negative waves, which at sufficient distances from the origin are of nearly equal and ever-increasing amplitudes. In order therefore that a small wave may emerge at $x=0$, we have to cause intense waves to be incident upon a face of the medium corresponding to a large negative x, of which nearly the whole are reflected.

It is important to observe that the ultimate totality of reflexion does not require a special adjustment between the frequency of the waves and the linear period of the lamination. The condition that c should be imaginary is merely that Θ_1 should numerically exceed $(1-\Theta_0)$. If λ be the wave-length of the vibration corresponding to e^{ipt} and to density ρ_0,

$$\frac{p^2\rho_0}{\pi^2 \mathrm{T}}=\frac{4}{\lambda^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

and thus the limits between real and imaginary values of c are given by

$$\frac{\lambda^2}{4l^2}-1=\pm\frac{\rho_1}{2\rho_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

If ρ_1 exceeds these limits a train of waves is ultimately totally reflected, in spite of the finite difference between $\frac{1}{2}\lambda$ and l*.

* A detailed experimental examination of various cases in which a laminated structure leads to a powerful but highly selected reflexion would be of value. The most frequent examples are met with in the organic world. It has occurred to me that Becquerel's reproduction of the spectrum in natural colours upon silver plates may perhaps be explicable in this manner. The various parts of the film of subchloride of silver with which the metal is coated may be conceived to be subjected, during exposure, to *stationary* luminous waves of nearly definite wave-length, the effect of which might be to impress upon the substance a periodic structure recurring at intervals equal to *half* the wave-length of the light; just as a sensitive flame exposed to stationary sonorous waves is influenced at the loops but not at the nodes (Phil. Mag. March 1879, p. 153). In this way the operation of any kind of light would be to produce just such a modification of the film as would cause it to reflect copiously that particular kind of light. I abstain at present from developing this suggestion, in the hope of soon finding an opportunity of making myself experimentally acquainted with the subject.

In conclusion, it may be worth while to point out the application to such a problem as the stationary vibrations of a string of variable density fixed at two points. A distribution of density,

$$\rho_0+\rho_1 \cos\frac{2\pi x}{l}+\rho_2 \cos\frac{4\pi x}{l}+\ldots \quad . \quad . \quad . \quad (75)$$

is symmetrical with respect to the points $x=0$ and $x=\frac{1}{2}l$, and between those limits is arbitrary. It is therefore possible for a string of this density to vibrate with the points in question undisturbed, and the law of displacement will be

$$w=\cos pt\left\{\mathrm{A}_1 \sin\frac{2\pi x}{l}+\mathrm{A}_2 \sin\frac{4\pi x}{l}+\mathrm{A}_3 \sin\frac{6\pi x}{l}+\ldots\right\}. \quad (76)$$

When, therefore, the problem is attacked by the method of Mr. Hill, the value of c obtained by the solution of (69) must be equal to 2. By (15) this requires

$$\square(0)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

This equation gives a relation between the quantities Θ_0, Θ_1, $\Theta_2, \ldots$; and this again, by (70), determines p, or the frequency $(p/2\pi)$ of vibration.

Since $\Theta_0=4$ nearly, the most important term in (17) is that involving Θ_2^2. The first approximation to (77) gives

$$\Theta_0=4+\Theta_2;$$

whence, by (70),

$$\left(\frac{2\pi}{p}\right)^2=\frac{l^2(\rho_0-\frac{1}{2}\rho_2)}{\mathrm{T}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

To this order of approximation the solution may be obtained with far greater readiness by the method given in my work on Sound*; but it is probable that, if the solution were required in a case where the variation of density is very considerable, advantage might be taken of Mr. Hill's determinant $\square(0)$. There are doubtless other physical problems to which a similar remark would be applicable.

Terling Place, Witham,
June 19, 1887.

* 'Theory of Sound,' vol. i. § 140. In comparing the results, it must be borne in mind that the length of the string in (78) is denoted by $\frac{1}{2}l$.